Solution to Assignment 3

2012/2013 Semester I MA4264 Game Theory

Tutor: Xiang Sun*

October 25, 2012

1. Question 5, in Tutorial set 5;
2. Question 4, in Tutorial set 6;
3. Question 4, in Tutorial set 7.

Solution for Question 1 Calculate firm i's price and profit in the collusion, Bertrand competition, and deviation from punishment cases, respectively:

- Cooperative price and profit: In the collusion, the price is $p_i^c = \frac{a+c}{2}$, and profit is $\pi_i^c = \frac{(a-c)^2}{8}$;
- Non-cooperative price and profit: In the Bertrand competition, price is $p_i^m = c$, and profit is $\pi_i^m = 0$;
- Deviation price and profit: Firm j's price is $p_j^c = \frac{a+c}{2}$, Firm $i \neq j$ can increases its profit by choosing a price $p_i^d < \frac{a+c}{2}$, but as close as possible to $\frac{a+c}{2}$, and profit is almost equal to monopoly profit $\pi_i^d = \frac{(a-c)^2}{4}$.

For each i, consider the following trigger strategy T_i for Firm i:

- In the first stage, choose price p_i^c.
- In the t-th stage, choose p_i^c if Firm j chooses price p_j^c in each of the $t-1$ previous stages; otherwise, choose price p_i^m.

For any i, assume that Firm $j \neq i$ chooses the trigger strategy T_j. We want to find the condition which guarantees the trigger strategy T_i to be Firm i's best response.

- If Firm i does not choose the trigger strategy, then we consider the following two cases:
  - If Firm i always chooses the cooperative production p_i^c in every stage game (it is a strategy for Firm i, but not the trigger strategy), then the payoff is as same as the payoff when it chooses trigger strategy.

*E-mail: xiangsun@nus.edu.sg. Suggestion and comments are always welcome.

  - If Firm i deviates in some stage and the profit maxizer is p_i^d. Without loss of generality, we assume that the t-th stage is the first stage when Firm i deviates, then it can get at most π_i^d at this stage.
    From the $(t+1)$-th stage on, Firm $j \neq i$ will choose non-cooperative price p_j^m. Thus Firm i will receive at most $\pi_m^i = 0$ in each of the subsequent stages, and the t-th stage's present value of its payoff from the t-th stage onwards is at most
    $$\pi_i^d.$$
    It is easy to understand when looking at the following table, where $*$ means we do not know exactly the action of Firm i at that stage.

Stage	1	$\cdots$	$t-1$	t	$t+1$	$t+2$	$t+3$	$\cdots$
Firm $j \neq i$	p_j^c	$\cdots$	p_j^c	p_j^c	p_j^m	p_j^m	p_j^m	$\cdots$
Firm i	p_i^c	$\cdots$	p_i^c	p_i^d	$*$	$*$	$*$	$\cdots$
Firm i's payoff	π_i^c	$\cdots$	π_i^c	π_i^d	$\leq \pi_i^m$	$\leq \pi_i^m$	$\leq \pi_i^m$	$\cdots$

- If Firm i chooses the trigger strategy T_i, then it will receive π_i^c in each stage, and the present value of its payoff from t-th stage onwards is
  $$\pi_i^c + \delta\pi_i^c + \delta^2\pi_i^c + \cdots = \frac{\pi_i^c}{1-\delta}.$$

In order for firm i to play trigger strategy T_i, we should have
$$\frac{\pi_i^c}{1-\delta} \geq \pi_i^d,$$
that is $\delta \geq \frac{1}{2}$.

Solution for Question 2

- There are 2 players: Player I and Player II;
- Type spaces: $T_1 = \{\{1\}, \{2,3\}\}$, and $T_2 = \{\{1,3\}, \{2\}\}$;
- Action spaces: $A_1 = \{T, B\}$, and $A_2 = \{L, R\}$;
- Strategy spaces: $S_1 = \{TT, TB, BT, BB\}$, and $S_2 = \{LL, LR, RL, RR\}$.

Now we will find the best-response correspondence for each player and each associated type: let a_1 and a_2 be Player I's actions in Game 1, and Games 2 and 3, respectively, b_1 and b_2 Player II's actions in Games 1 and 3, and Game 2, respectively.

- If Game 1 is drawn, then Player I's best-response correspondence is
  $$a_1^*(b_1) = \begin{cases} T, & \text{if } b_1 = L; \\ T, & \text{if } b_1 = R. \end{cases}$$
- If Game 1 is not drawn, then by considering the expected payoff, Player I's best-response correspondence is
  $$a_2^*(b_1 b_2) = \begin{cases} T, & \text{if } b_1 b_2 = LL; \\ T, & \text{if } b_1 b_2 = LR; \\ B, & \text{if } b_1 b_2 = RL; \\ B, & \text{if } b_1 b_2 = RR. \end{cases}$$

- If Game 2 is drawn, then Player II's best-response correspondence is

$$b_2^*(a_2) = \begin{cases} L, & \text{if } a_2 = T; \\ R, & \text{if } a_2 = B. \end{cases}$$

- If Game 2 is not drawn, then by considering the expected payoff, Player II's best-response correspondence is

$$b_1^*(a_1a_2) = \begin{cases} R, & \text{if } a_1a_2 = TT; \\ L, & \text{if } a_1a_2 = TB; \\ R, & \text{if } a_1a_2 = BT; \\ L, & \text{if } a_1a_2 = BB. \end{cases}$$

Therefore, by definition, there is no Bayesian Nash equilibrium. The reason is as follows:

- If Player I chooses TT, then Player II should choose RL; on the other hand, TT is not a best response for RL. So there is no Bayesian Nash equilibrium when Player I chooses TT.
- If Player I chooses TB, then Player II should choose LR; on the other hand, TB is not a best response for LR. So there is no Bayesian Nash equilibrium when Player I chooses TB.
- If Player I chooses BT, then Player II should choose RL; on the other hand, BT is not a best response for RL. So there is no Bayesian Nash equilibrium when Player I chooses BT.
- If Player I chooses BB, then Player II should choose LR; on the other hand, BB is not a best response for LR. So there is no Bayesian Nash equilibrium when Player I chooses BB.

Solution for Question 3

- There are two players: seller (s) and buyer (b);
- Type spaces: $T_s = [\alpha_s, \beta_s]$ and $T_b = [\alpha_b, \beta_b]$;
- Action spaces: $A_s = A_b = [0, \infty)$;
- Strategy spaces: $S_b = \{\text{function from } T_b \text{ to } A_b\}$, and $S_s = \{\text{function from } T_s \text{ to } A_s\}$;
- Payoff:

$$u_s(p_s, p_b; v_s, v_b) = \begin{cases} \frac{p_s+p_b}{2} - v_s, & p_b \geq p_s \\ 0, & p_b < p_s \end{cases},$$
$$u_b(p_s, p_b; v_s, v_b) = \begin{cases} v_b - \frac{p_s+p_b}{2}, & p_b \geq p_s \\ 0, & p_b < p_s \end{cases}.$$

Suppose (p_s^*, p_b^*) is a linear Bayesian Nash equilibrium, where

$$p_s^*(v_s) = a_s + c_s v_s, \quad p_b^*(v_b) = a_b + c_b v_b.$$

Note that a_s, c_s, a_b, c_b are to be determined. Here we should assume $c_s, c_b > 0$.

- For seller, when v_s is drawn, given buyer's strategy p_b^*, $p_s^*(v_s)$ will maximize his expected payoff

$$\begin{aligned}
&\mathbb{E}[u_s(p_s, p_b^*; v_s, v_b)] \\
&= \frac{1}{\beta_b - \alpha_b} \int_{p_s \le p_b^*(v_b) \le p_b^*(\beta_b)} \frac{p_s + p_b^*(v_b)}{2} - v_s \, \mathrm{d}v_b + \frac{1}{\beta_b - \alpha_b} \int_{p_b^*(\alpha_b) \le p_b^*(v_b) < p_s} 0 \, \mathrm{d}v_b \\
&= \frac{1}{\beta_b - \alpha_b} \int_{\frac{p_s - a_b}{c_b}}^{\beta_b} \frac{p_s + a_b + c_b v_b}{2} - v_s \, \mathrm{d}v_b \\
&= \frac{1}{\beta_b - \alpha_b} \left[\left(\frac{p_s + a_b}{2} - v_s \right) \left(\beta_b - \frac{p_s - a_b}{c_b} \right) + \frac{c_b}{2} \int_{\frac{p_s - a_b}{c_b}}^{\beta_b} v_b \, \mathrm{d}v_b \right] \\
&= \frac{1}{\beta_b - \alpha_b} \left[\left(\frac{p_s + a_b}{2} - v_s \right) \left(\beta_b - \frac{p_s - a_b}{c_b} \right) + \frac{c_b}{4} \left(\beta_b - \frac{p_s - a_b}{c_b} \right) \left(\beta_b + \frac{p_s - a_b}{c_b} \right) \right] \\
&= \frac{1}{\beta_b - \alpha_b} \left(\beta_b - \frac{p_s - a_b}{c_b} \right) \left[\left(\frac{p_s + a_b}{2} - v_s \right) + \frac{c_b}{4} \left(\beta_b + \frac{p_s - a_b}{c_b} \right) \right] \\
&= \frac{c_b}{\beta_b - \alpha_b} (c_b \beta_b - p_s + a_b) \left[-v_s + \frac{3}{4} p_s + \frac{1}{4} (a_b + c_b \beta_b) \right]
\end{aligned}$$

Therefore, by the first order condition,

$$p_s^*(v_s) = \frac{2}{3} v_s + \frac{1}{3} a_b + \frac{1}{3} c_b \beta_b,$$

and hence

$$c_s = \frac{2}{3}, \quad a_s = \frac{1}{3}(a_b + c_b \beta_b). \tag{1}$$

- For buyer, when v_b is drawn, given seller's strategy p_s^*, $p_b^*(v_b)$ will maximize his expected payoff

$$\begin{aligned}
&\mathbb{E}[u_b(p_s^*, p_b; v_s, v_b)] \\
&= \frac{1}{\beta_s - \alpha_s} \int_{p_s^*(\alpha_s) \le p_s^*(v_s) \le p_b} v_b - \frac{p_s^*(v_s) + p_b}{2} \, \mathrm{d}v_s + \frac{1}{\beta_s - \alpha_s} \int_{p_b < p_s^*(v_s) \le p_s^*(\beta_s)} 0 \, \mathrm{d}v_s \\
&= \frac{1}{\beta_s - \alpha_s} \int_{\alpha_s}^{\frac{p_b - a_s}{c_s}} v_b - \frac{a_s + c_s v_s + p_b}{2} \, \mathrm{d}v_s \\
&= \frac{1}{\beta_s - \alpha_s} \left[\left(v_b - \frac{a_s + p_b}{2} \right) \left(\frac{p_b - a_s}{c_s} - \alpha_s \right) - \frac{c_s}{2} \int_{\alpha_s}^{\frac{p_b - a_s}{c_s}} v_s \, \mathrm{d}v_s \right] \\
&= \frac{1}{\beta_s - \alpha_s} \left[\left(v_b - \frac{a_s + p_b}{2} \right) \left(\frac{p_b - a_s}{c_s} - \alpha_s \right) - \frac{c_s}{4} \left(\frac{p_b - a_s}{c_s} - \alpha_s \right) \left(\frac{p_b - a_s}{c_s} + \alpha_s \right) \right] \\
&= \frac{1}{\beta_s - \alpha_s} \left(\frac{p_b - a_s}{c_s} - \alpha_s \right) \left[\left(v_b - \frac{a_s + p_b}{2} \right) - \frac{c_s}{4} \left(\frac{p_b - a_s}{c_s} + \alpha_s \right) \right] \\
&= \frac{c_s}{\beta_s - \alpha_s} (p_b - a_s - c_s \alpha_s) \left[v_b - \frac{3}{4} p_b - \frac{1}{4} (a_s + c_s \alpha_s) \right]
\end{aligned}$$

Therefore, by the first order condition,

$$p_b^*(v_b) = \frac{2}{3} v_b + \frac{1}{3} a_s + \frac{1}{3} c_s \alpha_s,$$

and hence

$$c_b = \frac{2}{3}, \quad a_b = \frac{1}{3}(a_s + c_s \alpha_s). \tag{2}$$

Solving Equations (1) and (2), we will have

$$a_s = \frac{\alpha_s}{12} + \frac{\beta_b}{4}, \quad a_b = \frac{\beta_b}{12} + \frac{\alpha_s}{4}.$$