Solution to Assignment 4

2012/2013 Semester I MA4264 Game Theory

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1. In the Game 1, $((A, B, L), (p, 1-p))$ is a perfect Bayesian equilibrium, where p is the player 3's belief for the left decision node. What can you get about the range of p, based only on Requirement 4. Specify your reason.

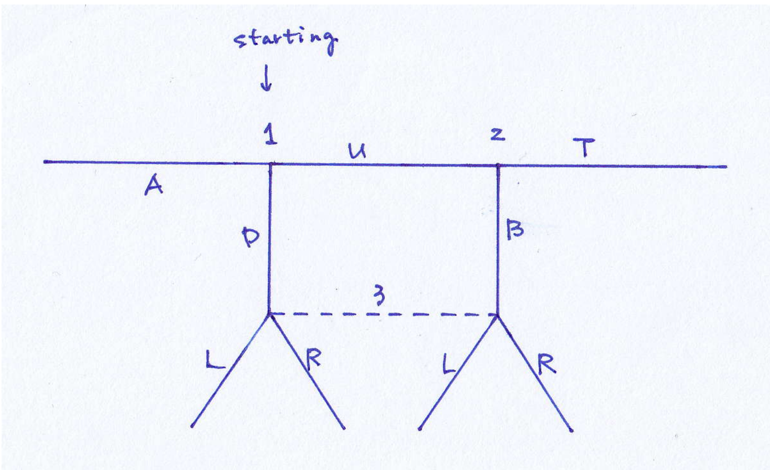

Figure 1

2. Question 1, in Tutorial set 9;

3. Question 7, in Tutorial set 10.

Solution for Question 1 p can be arbitrary.

Reason: for the information set off the equilibrium path, Bayes' law can be applied only when we can find a subgame, in which this information set is on the equilibrium path. To be more precise, consider the following example first.

Assume that $((A, L, L'), (p, 1-p))$ is a perfect Bayesian equilibrium. There is a subgame, and if we apply Bayes' law in this subgame, we will get $p = 1$.

For the game 1, there is no subgame, so we can not apply Bayes' law to get the belief, and hence p can be arbitrary.

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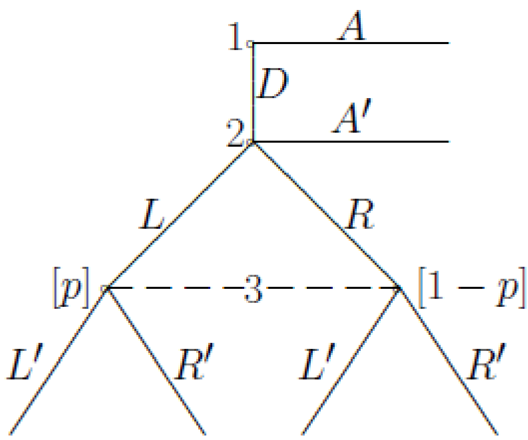

Figure 2

Solution for Question 2

(i) Developer i consider the following maximization problem

$$\max_q (1 - q - c_i)q,$$

and the maximizer is $q^* = \frac{1-c_i}{2}$. Profit is $(\frac{1-c_i}{2})^2$.

(ii) We have a first price auction where the valuation of developer i is $v_i = (1 - c_i)^2/4$. His payoff from paying bribe b_i is

$$U_i(b_i; c_i) = (v_i - b_i)\,\text{Prob}(b_j < b_i),$$

where

$$\begin{aligned}\text{Prob}(b_j < b_i) &= \text{Prob}(a_j + e_j(1 - c_j)^2 < b_i)\\ &= \text{Prob}(c_j > 1 - \sqrt{(b_i - a_j)/e_j})\\ &= 1 - \text{Prob}(c_j \le 1 - \sqrt{(b_i - a_j)/e_j})\\ &= 1 - (1 - \sqrt{(b_i - a_j)e_j}) = \sqrt{(b_i - a_j)e_j}\end{aligned}$$

Hence

$$U_i(b_i; c_i) = (v_i - b_i)\sqrt{(b_i - a_j)e_j}.$$

But maximizing $U_i(b_i; c_i)$ is the same as maximizing

$$e_j^{-1} U_i(b_i; c_i)^2 = (v_i - b_i)^2(b_i - a_j).$$

The first order condition yields

$$2(b_i - v_i)(b_i - a_j) + (b_i - v_i)^2 = 0,$$

hence

$$b_i = \frac{1}{3}v_i + \frac{2}{3}a_j = \frac{1}{12}(1 - c_i)^2 + \frac{2}{3}a_j.$$

Therefore,

$$e_i = \frac{1}{12} \text{ and } a_i = \frac{2}{3}a_j = \left(\frac{2}{3}\right)^2 a_i,$$

yielding

$$b_i = \frac{1}{12}(1 - c_i)^2.$$

Solution for Question 3 Assume that the core is nonempty, and $x = (x_1, \ldots, x_n)$ is an element in it. Since

$$\sum_{i \in N} \frac{v(N - \{i\})}{n - 1} \le \sum_{i \in N} \Big(\frac{1}{n-1} \sum_{j \neq i} x_j \Big) = \sum_{j \in N} x_j \Big(\sum_{i \neq j} \frac{1}{n-1} \Big) = \sum_{j \in N} x_j = v(N),$$

we have $\sum_{i \in N} v(N - \{i\}) \le (n - 1)v(N)$, and hence

$$v(N) \le n \cdot v(N) - \sum_{i \in N} v(N - \{i\}) = \sum_{i \in N} [v(N) - v(N - \{i\})] = \sum_{i \in N} \delta_i$$

which is a contradiction.

Remark 1. *Bondareva-Shapley Theorem: Let (N, v) be a cooperative game, where N is the set of players, and v is the characteristic function. The core of (N, v) is nonempty if and only if for every function $\lambda \colon 2^N - \{\emptyset\} \to [0, 1]$, where for any $i \in N$, $\sum_{S \ni i} \lambda(S) = 1$, the following condition holds:*

$$\sum_{S \in 2^N - \{\emptyset\}} \lambda(S) v(S) \le v(N).$$