Solution to Tutorial 10*

2011/2012 Semester I MA4264 Game Theory

Tutor: Xiang Sun†

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Exercise 1. *Player 1 has two types, intelligent or dumb, with equal probability of each type. Player 1 may choose either to drop out of high school or finish high school. If he finishes high school, player 2 must decide whether or not to hire player 1. Player 1 knows his type, but player 2 does not. If player 1 drops out, both players get zeros. If player 1 finishes high school, but is not employed by player 2, player 2 gets nothing, and player 1 gets x if intelligent, and y if dumb, where $y > x > 0$, and $1 > x$, but y may be either larger or smaller than 1. If player 1 finishes high school and is employed, player 2 gets a if player 1 is intelligent and b if player 1 is dumb, where $a > b$. Here $a > 0$ but b may be either positive or negative. Player 1 gets $1 - x$ if intelligent and $1 - y$ if dumb.*

(a) *For what values of a, b, x, y is there a perfect Bayesian equilibrium in which both types drop out?*

(b) *For what values of a, b, x, y is there a perfect Bayesian equilibrium which is separating.*

Solution. Figure 1 is the extensive-form representation of this game.

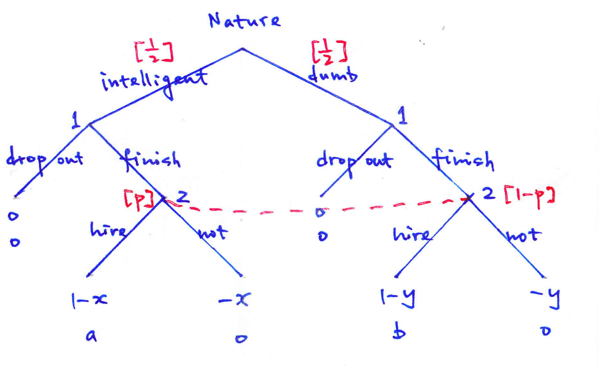

Figure 1

The normal-form representation is as follows:

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*Corrections are always welcome.

†Email: xiangsun@nus.edu.sg; Mobile: 9169 7677; Office: S17-06-14.

- $S_1 = \{dd, df, fd, ff\}$, where d and f denote "drop out" and "finish", respectively. $S_2 = \{h, n\}$, where h and n denote "hire" and "not hire", respectively.
- Payoff table:

Player 2

Player 1	h	n
dd	$0, 0$	$0, 0$
df	$\frac{1-y}{2}, \frac{b}{2}$	$-\frac{y}{2}, 0$
fd	$\frac{1-x}{2}, \frac{a}{2}$	$-\frac{x}{2}, 0$
ff	$\frac{1-x}{2} + \frac{1-y}{2}, \frac{b}{2} + \frac{a}{2}$	$-\frac{x}{2} - \frac{y}{2}, 0$

(a) Since $x < 1$, $\frac{1-x}{2} > 0$, and hence dd could not be a best response to h. Since $y > x > 0$, dd is a best response to n.

Since Player 1 chooses dd, Player 2's information set will not be reached, and hence the belief could be arbitrary by Requirement 4. To support n is Player 2's best choice given his belief, b should be nonpositive, otherwise n is strictly dominated by h.

To summarize, we need $b \leq 0$.

(b) Since $x, y > 0$, each of df and fd can not be a best response to n. Since $y > x$, df can no be a best response to h.

fd to be a best response to h if and only if $y \leq 1$. By Bayes' rule, $p = 1$, and since $a > 0$, h is a best choice given this belief.

To summarize, we need $y \geq 1$.

□

Exercise 2. *A firm and a union play the following two-period bargaining game. It is common knowledge that the firm's profit, π, is uniformly distributed between zero and one, that the union's reservation wage is w_r, and that only the firm knows the true value of π. Assume that $0 < w_r < 1/2$. Find the perfect Bayesian equilibrium of the following game:*

(a) At the beginning of period one, the union makes a wage offer to the firm, w_1.

(b) The firm either accepts or rejects w_1. If the firm accepts w_1 then production occurs in both periods, so payoffs are $2w_1$ for the union and $2(\pi - w_1)$ for the firm. (There is no discounting.) If the firm rejects w_1 then there is no production in the first period, and payoffs for the first period are zero for both the firm and the union.

(c) At the beginning of the second period (assuming that the firm rejected w_1), the firm makes a wage offer to the union, w_2. (Unlike in the Sobel-Takahashi model, the union does not make this offer.)

(d) *The union either accepts or rejects w_2. If the union accepts w_2 then production occurs in the second period, so seocnd-period (and total) payoffs are w_2 for the union and $\pi - w_2$ for the firm. (Recall that first-period payoffs were zero.) If the union rejects w_2 then there is no productioin. The union then earns its alternative wage, w_r, for the second period and the firm shuts down and earns zero.*

Solution. Figure 2 is the extensive-form representation of this game.

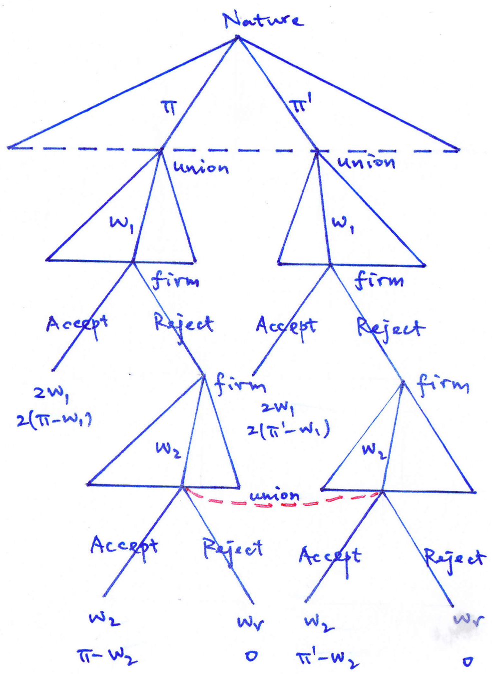

Figure 2

At first, we will apply backwards induction to find the subgame-perfect Nash equilibrium.

- If period 2, the union's best response is

$$a_u^* = \begin{cases} A, & \text{if } w_2 \geq w_r \\ R, & \text{if } w_2 < w_r \end{cases},$$

and firm's best response is

$$w_2^* = \begin{cases} w_r, & \text{if } \pi \geq w_r \\ [0, w_r), & \text{if } \pi < w_r \end{cases}.$$

Then the payoffs are as follows:

$$\pi_f = \max\{\pi - w_r, 0\} = \begin{cases} \pi - w_r, & \text{if } \pi \geq w_r \\ 0, & \text{if } \pi < w_r \end{cases}, \text{ and } \pi_u = w_r.$$

- In period 1, the firm will accept if and only if

$$2(\pi - w_1) \geq \max\{\pi - w_r, 0\}.$$

Thus, the union's payoff by offering w_1 is

$$\pi_u = \begin{cases} 2w_1, & \text{if } 2(\pi - w_1) \geq \max\{\pi - w_r, 0\} \\ w_r, & \text{if } 2(\pi - w_1) < \max\{\pi - w_r, 0\} \end{cases},$$

and union's expected payoff is

$$\mathbb{E}\pi_u = 2w_1 \,\text{Prob}\{2(\pi - w_1) \geq \max\{\pi - w_r, 0\}\} + w_r \,\text{Prob}\{2(\pi - w_1) < \max\{\pi - w_r, 0\}\}.$$

Since $2(\pi - w_1) \geq \max\{\pi - w_r, 0\}$ is equivalent to $\pi \geq 2w_1 - w_r$ and $\pi \geq w_1$,

$$\text{Prob}\{2(\pi - w_1) \geq \max\{\pi - w_r, 0\}\} = \begin{cases} 1 - w_1, & \text{if } w_1 \leq w_r \\ 1 + w_r - 2w_1, & \text{if } w_r < w_1 \leq \frac{1+w_r}{2} \\ 0, & \text{if } w_1 > \frac{1+w_r}{2} \end{cases},$$

and hence the expected payoff is

$$\mathbb{E}\pi_u = \begin{cases} 2w_1(1 - w_1) + w_r w_1, & \text{if } w_1 \leq w_r \\ 2w_1(1 + w_r - 2w_1) + w_r(2w_1 - w_r), & \text{if } w_r < w_1 \leq \frac{1+w_r}{2} \\ w_r, & \text{if } w_1 > \frac{1+w_r}{2} \end{cases}.$$

From Figure 3, the unique maximizer of $\mathbb{E}\pi_u$ is $w_1^* = \frac{w_r + 1/2}{2}$.

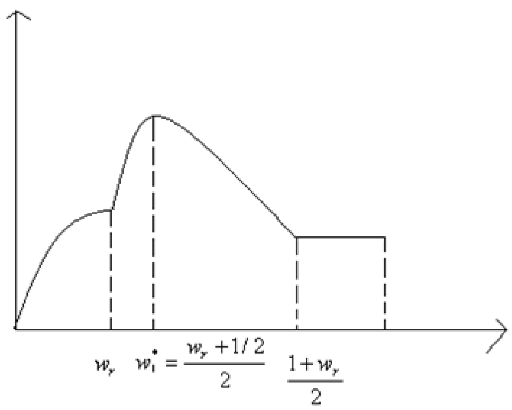

Figure 3

Therefore, the subgame-perfect Nash equilibrium is:

- In period 1,

$$w_1^* = \frac{w_r + 1/2}{2}, \text{ and } a_f^* = \begin{cases} A, & \text{if } 2(\pi - w_1) \geq \max\{\pi - w_r, 0\} \\ R, & \text{otherwise} \end{cases}.$$

- In period 2,

$$w_2^* = \begin{cases} w_r, & \text{if } \pi \geq w_r \\ [0, w_r), & \text{if } \pi < w_r \end{cases} \text{ and } a_u^* = \begin{cases} A, & \text{if } w_2 \geq w_r \\ R, & \text{if } w_2 < w_r \end{cases}.$$

Next we will find the union's belief system, such that the subgame-perfect Nash equilibrium we found above is a perfect Bayesian equilibrium. Assume the union and the firm play the subgame-perfect Nash equilibrium above.

- In period 1, the union has only one information set, and each decision node is reached. Thus, the union's belief about π should be uniformly distributed on $[0, 1]$.

- The firm accepts in period 1 if and only if $2(\pi - w_1^*) \geq \max\{\pi - w_r, 0\}$, that is

  $$\pi \geq 2w_1^* - w_r \text{ and } \pi \geq w_1^*.$$

  Since $w_r < \frac{1}{2}$, $w_1^* = \frac{2w_r+1}{4} \geq \frac{2w_r+2w_r}{4} = w_r$, and hence the firm accepts in period 1 if and only if $\pi \geq 2w_1^* - w_r = \frac{1}{2}$.

  Assume $\pi < \frac{1}{2}$, then the firm will reject in period 1, and game goes into period 2. In period 2, if the union observes that $w_2 = w_r$, then it should know that $\pi \geq w_r$, and hence its belief about π should be a uniform distribution on $[w_r, \frac{1}{2}]$. If the union observes that $w_2 < w_r$, then it should know that $\pi < w_r$, and hence its belief about π should be a uniform distribution on $[0, w_r]$.

□

Exercise 3. *A seller and a buyer is settling a price for an object. They agree to play the following game. They preset 4 numbers: x_l, x_h, L and H, with $0 \leq x_l < x_h$ and $x_l \leq L < H$. A number x is drawn from the interval $[x_l, x_h]$ uniformly. The seller reads the number x and then names a price p_s which is either L or H. The buyer observes p_s, but not x, then names a price $p_b \in [0, p_s]$. The object is traded at a price $\bar{p}$ which is set to be p_s if $x \geq (p_s + p_b)/2$ and p_b otherwise. The seller's objective is to maximize the price $\bar{p}$ and the buyer's objective is to minimize the price $\bar{p}$. Formulate the situation as a dynamic game of incomplete information. Find a perfect Bayesian equilibrium in which the seller's strategy is in the form*

$$p_s(x) = \begin{cases} L, & \text{if } x \leq \bar{x} \\ H, & \text{if } x > \bar{x} \end{cases}$$

for some $\bar{x} \in (x_l, x_h)$, if it exists. Otherwise, prove that such a PBE does not exist. (Consider cases $x_l < L$ and $x_l = L$ respectively.)

Solution. [1] Figure 4 is the extensive-form representation of this game.

[1] The solution is incomplete. You may skip this question.

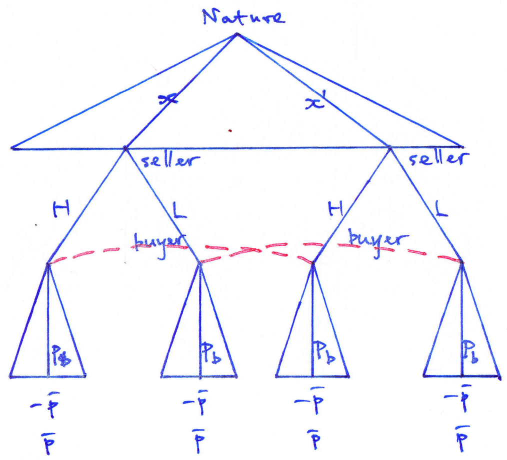

Figure 4

- Given seller's strategy

$$p_s^*(x) = \begin{cases} L, & \text{if } x \leq \bar{x} \\ H, & \text{if } x > \bar{x} \end{cases},$$

if buyer observes L, then he should believe x is uniformly distributed on $[x_l, \bar{x}]$; and if buyer observes R, then he should believe x is uniformly distributed on $[\bar{x}, x_h]$.

If buyer receives L, his expected trading price is

$$\bar{p}_L = p_b \cdot \text{Prob}\left(x < \frac{p_b + L}{2}\right) + L \cdot \text{Prob}\left(x \geq \frac{p_b + L}{2}\right).$$

We only need to consider the case $\bar{x} \geq \frac{p_b+L}{2}$: When $\bar{x}$ realizes, and seller chooses L, the upper bound of buyer's price should be $2\bar{x} - L$, that is, $p_b \leq 2\bar{x} - L$.

In this case, since buyer believes that x is uniformly distributed on $[x_l, \bar{x}]$,

$$\begin{aligned} \bar{x}_L &= p_b \cdot \frac{\frac{p_b+L}{2} - x_l}{\bar{x} - x_l} + L \cdot \frac{\bar{x} - \frac{p_b+L}{2}}{\bar{x} - x_l} \\ &= \frac{1}{\bar{x} - x_l}\left[\frac{p_b^2}{2} - x_l p_b + L\bar{x} - \frac{L^2}{2}\right] \end{aligned}$$

The unique maximizer is $p_b^* = x_l$. That is, if the message is L, then the optimal action of buyer is to pay just the lowest possible value x_l.

Similarly, if buyer receives H, his expected trading price is

$$\bar{p}_H = p_b \cdot \text{Prob}\left(x < \frac{p_b + H}{2}\right) + H \cdot \text{Prob}\left(x \geq \frac{p_b + H}{2}\right).$$

We also need to consider the case $\bar{x} \leq \frac{p_b+H}{2}$. In this case, since buyer believes that x is uniformly distributed on $[\bar{x}, x_h]$,

$$\begin{aligned}\bar{p}_H &= p_b \cdot \frac{\frac{p_b+H}{2} - \bar{x}}{x_h - \bar{x}} + H \cdot \frac{x_h - \frac{p_b+H}{2}}{x_h - \bar{x}} \\ &= \frac{1}{x_h - \bar{x}} \left[\frac{p_b^2}{2} - \bar{x} p_b + H x_h - \frac{H^2}{2} \right]\end{aligned}$$

The unique maximizer is $p_b^* = \bar{x}$. That is, if the message is H, then the optimal action of buyer is to pay just the lowest possible value $\bar{x}$.

To summarize, given seller's strategy p_s^*, buyer's best response is

$$p_b^* = \begin{cases} x_l, & \text{if } p_s^* = L \\ \bar{x}, & \text{if } p_s^* = H \end{cases}.$$

- Given p_b^*, if seller chooses L, the trading price is

$$\bar{p}_L = \begin{cases} x_l, & \text{if } x \leq \frac{x_l+L}{2} \\ L, & \text{otherwise} \end{cases};$$

if seller chooses H, the trading price is

$$\bar{p}_H = \begin{cases} \bar{x}, & \text{if } x \leq \frac{\bar{x}+H}{2} \\ H, & \text{otherwise} \end{cases}.$$

Finally, given p_b^*, p_s^* is a best response for seller if and only if

$$\begin{cases} x_l \geq \bar{x}, & \text{when } x_l \leq x \leq \frac{x_l+L}{2} \\ L \geq \bar{x}, & \text{when } \frac{x_l+L}{2} \leq x \leq \bar{x} \\ L \leq \bar{x}, & \text{when } \bar{x} \leq x \leq \frac{\bar{x}+H}{2} \end{cases}.$$

As a result, we have $\bar{x} = L$.

- Since $\bar{x} \geq x_l$, we have $L \geq x_l$.
  - No required perfect Bayesian equilibrium for $L > x_l$. (Exercise)
  - It is possible for $x_l = L$: buyer's belief is as follows: at the information set L, he should believe $x = x_l$ for sure, and at the information set H, he should believe x is uniformly distributed on $(x_l, x_h]$.

□

Exercise 4. *Suppose the set H consists of the points lying on and within a circle of radius 2, having a center at $(2, 2)$. If the threat point, d, is at $(2, 2)$, what is the Nash bargaining solution? If the threat point, d, is at $(0, 2)$, what is the Nash bargaining solution?*

Solution. $H = \{(u_1, u_2) \colon (u_1 - 2)^2 + (u_2 - 2)^2 \leq 4\}$.

(a) $d = (2, 2)$. Consider the following problem:

$$\max (u_1 - 2)(u_2 - 2) \tag{1}$$
$$\text{subject to: } (u_1 - 2)^2 + (u_2 - 2)^2 \leq 4 \tag{2}$$
$$u_1 \geq 2, \; u_2 \geq 2 \tag{3}$$

Consider (1) and (2), and apply the method of Lagrange multipliers, we will have

$$f(u_1, u_2, \lambda) = (u_1 - 2)(u_2 - 2) - \lambda[(u_1 - 2)^2 + (u_2 - 2)^2 - 4]$$
$$\frac{\partial f}{\partial u_1} = 0 \Rightarrow (u_2 - 2) = 2\lambda(u_1 - 2)$$
$$\frac{\partial f}{\partial u_2} = 0 \Rightarrow (u_1 - 2) = 2\lambda(u_2 - 2)$$
$$\frac{\partial f}{\partial \lambda} = 0 \Rightarrow (u_1 - 2)^2 + (u_2 - 2)^2 = 4$$

The solutions are: $(2 + \sqrt{2}, 2 + \sqrt{2})$ and $(2 - \sqrt{2}, 2 - \sqrt{2})$. Note that only $(2 + \sqrt{2}, 2 + \sqrt{2})$ satisfies (3). Therefore, $(2 + \sqrt{2}, 2 + \sqrt{2})$ is the unique Nash bargaining solution.

(b) $d = (0, 2)$. Consider the following problem:

$$\max (u_1 - 0)(u_2 - 2) \tag{4}$$
$$\text{subject to: } (u_1 - 2)^2 + (u_2 - 2)^2 \leq 4 \tag{5}$$
$$u_1 \geq 0, \; u_2 \geq 2 \tag{6}$$

Consider (4) and (5), and apply the method of Lagrange multipliers, we will have

$$f(u_1, u_2, \lambda) = u_1(u_2 - 2) - \lambda[(u_1 - 2)^2 + (u_2 - 2)^2 - 4]$$
$$\frac{\partial f}{\partial u_1} = 0 \Rightarrow (u_2 - 2) = 2\lambda(u_1 - 2)$$
$$\frac{\partial f}{\partial u_2} = 0 \Rightarrow u_1 = 2\lambda(u_2 - 2)$$
$$\frac{\partial f}{\partial \lambda} = 0 \Rightarrow (u_1 - 2)^2 + (u_2 - 2)^2 = 4$$

The solutions are: $(0, 2)$ and $(3, 2 + \sqrt{3})$, where the former is not Pareto optimal. $(3, 2 + \sqrt{3})$ is the unique Nash bargaining solution.

□

Exercise 5. *There are two players who may divide 1 dollar between them. The utility function of player 1 is* $u_1(x_1) = x_1^{0.5}$ *and of player 2 is* $u_2(x_2) = x_2$.

(a) Calculate and draw the set of possible pairs of utilities that the players can get assuming that they may also divide amounts smaller than 1 dollar, i.e., $x_1 + x_2 \leq 1$.

(b) Assume that if the players do not reach an agreement both get 0 dollars. Calculate the utilities the players will get according to the Nash solution. How much money each player gets?

Solution. (a) Since $u_1(x_1) = x_1^{0.5}$, and $u_2(x_2) = x_2$, we have $x_1 = u_1^2$ and $x_2 = u_2$. Thus, the set of possible pairs of utilities is

$$H = \{(u_1, u_2) \colon x_1 + x_2 \leq 1, x_1, x_2 \geq 0\} = \{(u_1, u_2) \colon u_1^2 + u_2 \leq 1, u_1, u_2 \geq 0\}.$$

(b) Here $(0, 0)$ is the threat point. Consider the following problem:

$$\max u_1 u_2 \tag{7}$$
$$\text{subject to: } u_1^2 + u_2 \leq 1 \tag{8}$$
$$u_1 \geq 0, \;\; u_2 \geq 0 \tag{9}$$

Consider (7) and (8), and apply the method of Lagrange multipliers, we will have

$$f(u_1, u_2, \lambda) = u_1 u_2 - \lambda[u_1^2 + u_2 - 1]$$
$$\frac{\partial f}{\partial u_1} = 0 \Rightarrow u_2 = 2\lambda u_1$$
$$\frac{\partial f}{\partial u_2} = 0 \Rightarrow u_1 = \lambda$$
$$\frac{\partial f}{\partial \lambda} = 0 \Rightarrow u_1^2 + u_2 = 1$$

The solution is: $(\frac{1}{\sqrt{3}}, \frac{2}{3})$. Note it satisfies (9). Therefore, it is the unique Nash bargaining solution. Besides, the corresponding money is $(x_1^*, x_2^*) = (\frac{1}{3}, \frac{2}{3})$. □

Exercise 6. *Player 1 and player 2 have been willed equal shares of an estate consisting of \$200,000 cash and 100 acres of farmland. Player 1 has a sentimental attachment to the land and values it at* $v_1 = \$3,000$ *per acre, whereas player 2 has no such attachment and values it at* $v_2 = \$1,000$ *per acre. Assume that their payoff functions are linear in money and land at these rates:* $u_i = x_i + v_i y_i$ *if player* i *receives* x_i *dollars of cash and* y_i *acres of land. The players may reach an agreement on dividing the land and money so as to maximize their payoffs. If they fail to reach agreement they divide the land and money equally.*

(i) Carefully draw the bargaining set and label the disagreement point.

(ii) Find the Nash bargaining solution.

Solution. (a) Assume in an agreement, the outcome is (x_1, x_2) and (y_1, y_2), where

$$x_1 + x_2 = 200000,\ y_1 + y_2 = 100,\ x_1, x_2, y_1, y_2 \geq 0,$$

and corresponding payoffs are

$$u_1 = x_1 + 3000y_1,\ u_2 = x_2 + 1000y_2.$$

Hence, we have

$$u_1 + u_2 = 300000 + 2000y_1,\ u_1 + 3u_2 = 500000 + 2000x_1,$$

and hence

$$300000 \leq u_1 + u_2 \leq 500000,\ 500000 \leq u_1 + 3u_2 \leq 900000.$$

Disagreement outcome is $x_1 = x_2 = 100000$, and $y_1 = y_2 = 50$, and hence $u_1 = 250000$ and $u_2 = 150000$, which is a threat point in

$$H = \{(u_1, u_2) \colon 300000 \leq u_1 + u_2 \leq 500000,\ 500000 \leq u_1 + 3u_2 \leq 900000\}.$$

(b) Consider the following problem:

$$\max (u_1 - 250000)(u_2 - 150000) \qquad (10)$$

$$\text{subject to: } u_1 + u_2 \leq 500000 \qquad (11)$$

$$u_1 + 3u_2 \leq 900000 \qquad (12)$$

$$300000 \leq u_1 + u_2 \qquad (13)$$

$$500000 \leq u_1 + 3u_2 \qquad (14)$$

$$u_1 \geq 0,\ u_2 \geq 0 \qquad (15)$$

Consider (10), (11) and (12), and apply the method of Lagrange multipliers, we will have

$$f(u_1, u_2, \lambda) = (u_1 - 250000)(u_2 - 150000) - \lambda_1[u_1 + u_2 - 500000] - \lambda_2[u_1 + 3u_2 - 900000]$$

$$\frac{\partial f}{\partial u_1} = 0 \Rightarrow u_2 - 150000 = \lambda_1 + \lambda_2$$

$$\frac{\partial f}{\partial u_2} = 0 \Rightarrow u_1 - 250000 = \lambda_1 + 3\lambda_2$$

$$\frac{\partial f}{\partial \lambda_1} = 0 \Rightarrow u_1 + u_2 = 500000$$

$$\frac{\partial f}{\partial \lambda_2} = 0 \Rightarrow u_1 + 3u_2 = 900000$$

The solution is: $(300000, 200000)$. Note it satisfies (13), (14) and (15). Therefore, it is the unique Nash bargaining solution.

□