Solution to Tutorial 10

2012/2013 Semester I MA4264 Game Theory

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Exercise 1. *An entrepreneur has a project that requires* $\$100,000$ *as an investment, and will yield* $\$300,000$ *with probability* $1/2$, $\$0$ *with probability* $1/2$. *There are two types of entrepreneurs: rich ones with a wealth of* $\$1,000,000$, *and poor ones with* $\$0$. *For some reason, the entrepreneur cannot use his wealth as investment towards this project. There is also a bank that can lend money with interest rate* π. *That is, if the entrepreneur borrows* $\$100,000$ *to invest, after the project is completed he will pay back* $\$100,000(1+\pi)$*—if he has that much money. If his wealth is less than this amount at the end of the project, he will pay all he has. The order of events is as follows:*

- *First, the bank decides and announces the interest rate* π.
- *Then, the entrepreneur decides whether to borrow* $\$100,000$ *from the bank and invest in the project.*
- *Then, the project, if invested, is completed and the money is paid back to the bank.*

(i) For the case when the wealth is common knowledge, compute the subgame perfect equilibrium for each type of the entrepreneur.

(ii) Now assume that the bank does not know the wealth of the entrepreneur. The probability that the entrepreneur is rich is $1/4$. *Compute the perfect Bayesian equilibrium.*

Solution. (i) The extensive-form representation is given in Figure 1. A Bank's strategy can be represented by $\pi = (\pi_r, \pi_p)$, where π_r and π_p are the actions when Entrepreneur is rich or poor respectively. A Entrepreneur's strategy can be represented by $s = (s_r(\pi_r), s_p(\pi_p))$.

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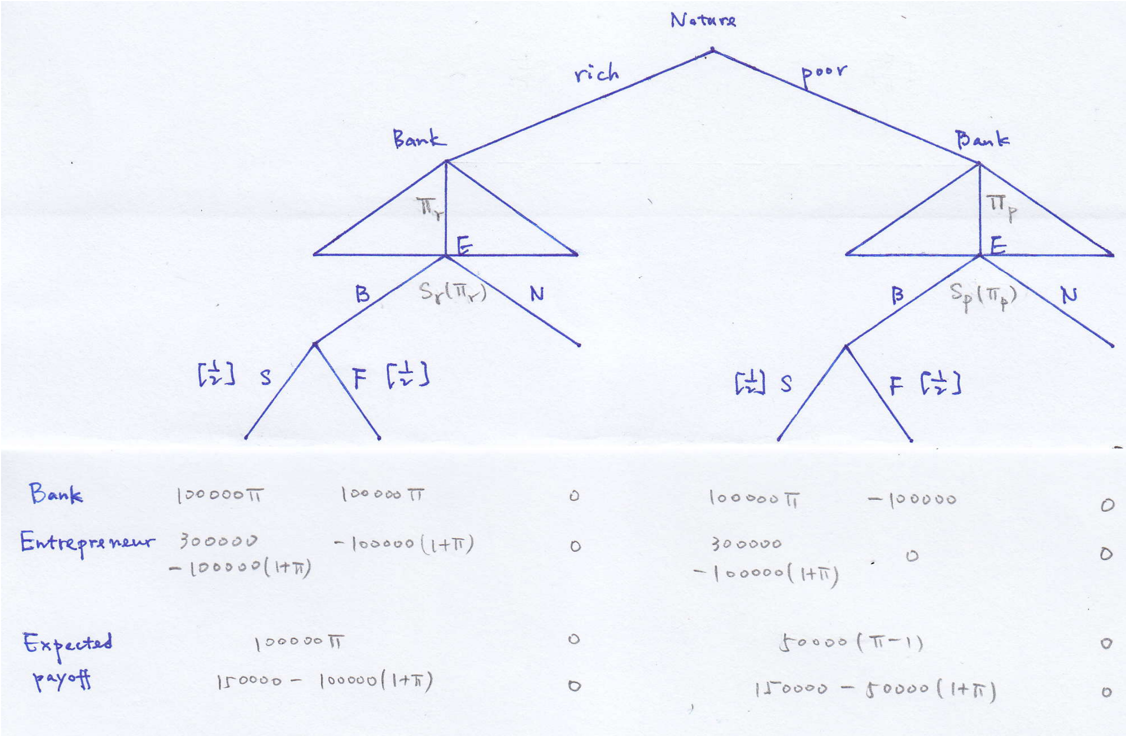

Figure 1

Consider the last period, then Entrepreneur's strategy is

$$s_r^*(\pi_r) = \begin{cases} B, & \text{if } \pi_r \le 0.5 \\ N, & \text{otherwise} \end{cases}, \quad s_p^*(\pi_p) = \begin{cases} B, & \text{if } \pi_p \le 2 \\ N, & \text{otherwise} \end{cases}.$$

Note that, for sake of simplicity, we always assume players will choose "not rejection".

Apply backwards induction, then Bank's strategy is

$$\pi_r = 0.5, \;\; \pi_p = 2.$$

(ii) The extensive-form representation is given in Figure 2. A Bank's strategy is π. A Entrepreneur's strategy can be represented by $s = (s_r(\pi), s_p(\pi))$.

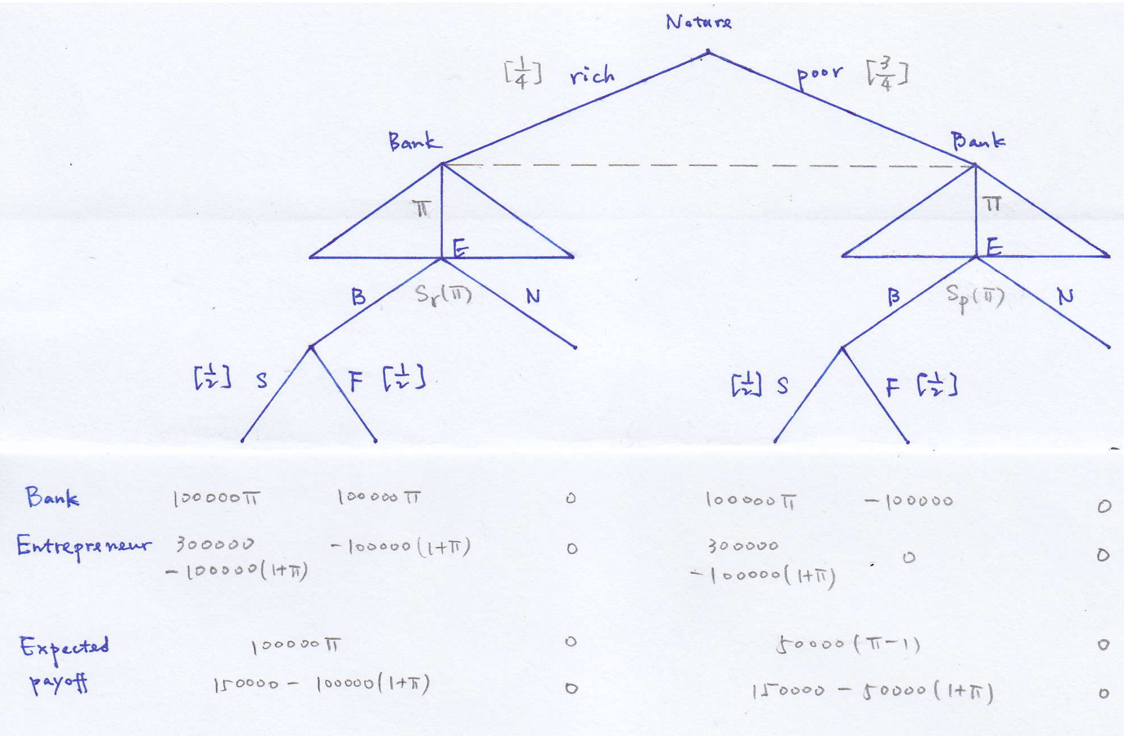

Figure 2

Consider the last period, then Entrepreneur's strategy is

$$s_r^*(\pi) = \begin{cases} B, & \text{if } \pi \leq 0.5 \\ N, & \text{otherwise} \end{cases}, \quad s_p^*(\pi) = \begin{cases} B, & \text{if } \pi \leq 2 \\ N, & \text{otherwise} \end{cases}.$$

Apply backwards induction, and Bank's expected payoff is

$$\mathbb{E}U_B(\pi) = \begin{cases} \frac{1}{4}\cdot 100000\pi + \frac{3}{4}\cdot 50000(\pi-1) = \frac{50000}{4}(5\pi-3), & \text{if } \pi \leq 0.5 \\ \frac{1}{4}\cdot 0 + \frac{3}{4}\cdot 50000(\pi-1) = \frac{150000}{4}(\pi-1), & \text{if } 0.5 < \pi \leq 2 \\ 0, & \text{if } 2 < \pi \end{cases}.$$

Hence, Bank will choose $\pi^* = 2$, since it will get $-\frac{50000}{8}$ when it chooses $\pi = 0.5$, and will get $\frac{150000}{4}$ when it chooses $\pi = 2$.

Bank's belief is: Prob$(Rich) = \frac{1}{4}$ and Prob$(Poor) = \frac{3}{4}$. Therefore the perfect Bayesian equilibrium is

$$\left(\pi^* = 2, (s_r^*(\pi), s_p^*(\pi)), (\frac{1}{4}, \frac{3}{4})\right).$$

□

Exercise 2. *A firm and a union play the following two-period bargaining game. It is common knowledge that the firm's profit,* π*, is uniformly distributed between zero and one, that the union's reservation wage is* w_r*, and that only the firm knows the true value of* π*. Assume that* $0 < w_r < 1/2$*. Find the perfect Bayesian equilibrium of the following game:*

(a) At the beginning of period one, the union makes a wage offer to the firm, w_1.

(b) The firm either accepts or rejects w_1. If the firm accepts w_1 then production occurs in both periods, so payoffs are $2w_1$ for the union and $2(\pi - w_1)$ for the firm. (There is no discounting.) If the firm rejects w_1 then there is no production in the first period, and payoffs for the first period are zero for both the firm and the union.

(c) At the beginning of the second period (assuming that the firm rejected w_1), the firm makes a wage offer to the union, w_2. (Unlike in the Sobel-Takahashi model, the union does not make this offer.)

(d) The union either accepts or rejects w_2. If the union accepts w_2 then production occurs in the second period, so seocnd-period (and total) payoffs are w_2 for the union and $\pi - w_2$ for the firm. (Recall that first-period payoffs were zero.) If the union rejects w_2 then there is no productioin. The union then earns its alternative wage, w_r, for the second period and the firm shuts down and earns zero.

Solution. Figure 3 is the extensive-form representation of this game.

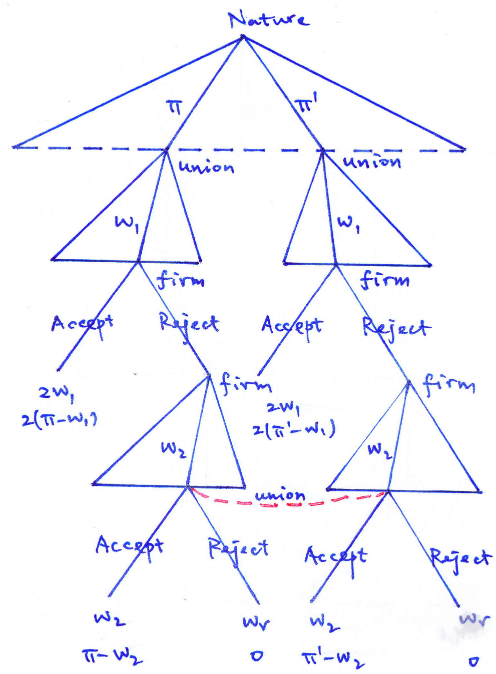

Figure 3

At first, we will apply backwards induction to find the subgame-perfect Nash equilibrium.

- If period 2, the union's best response is

$$a_u^* = \begin{cases} A, & \text{if } w_2 \geq w_r \\ R, & \text{if } w_2 < w_r \end{cases},$$

and firm's best response is

$$w_2^* = \begin{cases} w_r, & \text{if } \pi \geq w_r \\ [0, w_r), & \text{if } \pi < w_r \end{cases}.$$

Then the payoffs are as follows:

$$\pi_f = \max\{\pi - w_r, 0\} = \begin{cases} \pi - w_r, & \text{if } \pi \geq w_r \\ 0, & \text{if } \pi < w_r \end{cases}, \text{ and } \pi_u = w_r.$$

- In period 1, the firm will accept if and only if

$$2(\pi - w_1) \geq \max\{\pi - w_r, 0\}.$$

Thus, the union's payoff by offering w_1 is

$$\pi_u = \begin{cases} 2w_1, & \text{if } 2(\pi - w_1) \geq \max\{\pi - w_r, 0\} \\ w_r, & \text{if } 2(\pi - w_1) < \max\{\pi - w_r, 0\} \end{cases},$$

and union's expected payoff is

$$\begin{aligned} \mathbb{E}\pi_u = {} & 2w_1 \operatorname{Prob}\{2(\pi - w_1) \geq \max\{\pi - w_r, 0\}\} \\ & + w_r \operatorname{Prob}\{2(\pi - w_1) < \max\{\pi - w_r, 0\}\}. \end{aligned}$$

Since $2(\pi - w_1) \geq \max\{\pi - w_r, 0\}$ is equivalent to $\pi \geq 2w_1 - w_r$ and $\pi \geq w_1$,

$$\operatorname{Prob}\{2(\pi - w_1) \geq \max\{\pi - w_r, 0\}\} = \begin{cases} 1 - w_1, & \text{if } w_1 \leq w_r \\ 1 + w_r - 2w_1, & \text{if } w_r < w_1 \leq \frac{1+w_r}{2} \\ 0, & \text{if } w_1 > \frac{1+w_r}{2} \end{cases},$$

and hence the expected payoff is

$$\mathbb{E}\pi_u = \begin{cases} 2w_1(1 - w_1) + w_r w_1, & \text{if } w_1 \leq w_r \\ 2w_1(1 + w_r - 2w_1) + w_r(2w_1 - w_r), & \text{if } w_r < w_1 \leq \frac{1+w_r}{2} \\ w_r, & \text{if } w_1 > \frac{1+w_r}{2} \end{cases}.$$

From Figure 4, the unique maximizer of $\mathbb{E}\pi_u$ is $w_1^* = \frac{w_r + 1/2}{2}$.

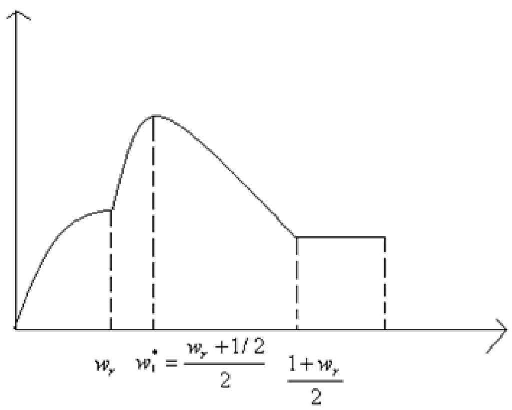

Figure 4

Therefore, the subgame-perfect Nash equilibrium is:

- In period 1,

$$w_1^* = \frac{w_r + 1/2}{2}, \text{ and } a_f^* = \begin{cases} A, & \text{if } 2(\pi - w_1) \geq \max\{\pi - w_r, 0\} \\ R, & \text{otherwise} \end{cases}.$$

- In period 2,

$$w_2^* = \begin{cases} w_r, & \text{if } \pi \geq w_r \\ [0, w_r), & \text{if } \pi < w_r \end{cases} \text{ and } a_u^* = \begin{cases} A, & \text{if } w_2 \geq w_r \\ R, & \text{if } w_2 < w_r \end{cases}.$$

Next we will find the union's belief system, such that the subgame-perfect Nash equilibrium we found above is a perfect Bayesian equilibrium. Assume the union and the firm play the subgame-perfect Nash equilibrium above.

- In period 1, the union has only one information set, and each decision node is reached. Thus, the union's belief about π should be uniformly distributed on $[0, 1]$.
- The firm accepts in period 1 if and only if $2(\pi - w_1^*) \geq \max\{\pi - w_r, 0\}$, that is

$$\pi \geq 2w_1^* - w_r \text{ and } \pi \geq w_1^*.$$

Since $w_r < \frac{1}{2}$, $w_1^* = \frac{2w_r+1}{4} \geq \frac{2w_r+2w_r}{4} = w_r$, and hence the firm accepts in period 1 if and only if $\pi \geq 2w_1^* - w_r = \frac{1}{2}$.

Assume $\pi < \frac{1}{2}$, then the firm will reject in period 1, and game goes into period 2. In period 2, if the union observes that $w_2 = w_r$, then it should know that $\pi \geq w_r$, and hence its belief about π should be a uniform distribution on $[w_r, \frac{1}{2})$. If the union observes that $w_2 < w_r$, then it should know that $\pi < w_r$, and hence its belief about π should be a uniform distribution on $[0, w_r)$. Please note that based on Bayes' law, the belief could be updated further given firm's strategy.

□

Exercise 3. *Suppose the set H consists of the points lying on and within a circle of radius 2, having a center at $(2, 2)$. If the threat point, d, is at $(2, 2)$, what is the Nash bargaining solution? If the threat point, d, is at $(0, 2)$, what is the Nash bargaining solution?*

Solution. $H = \{(u_1, u_2) \colon (u_1 - 2)^2 + (u_2 - 2)^2 \leq 4\}$.

(a) $d = (2, 2)$. Consider the following problem:

$$\begin{aligned} \text{maximize} \quad & (u_1 - 2)(u_2 - 2) && (1) \\ \text{subject to} \quad & (u_1 - 2)^2 + (u_2 - 2)^2 \leq 4 && (2) \\ & u_1 \geq 2, \; u_2 \geq 2 && (3) \end{aligned}$$

Consider (1) and (2), and apply the method of Lagrange multipliers, we will have

$$\begin{aligned} f(u_1, u_2, \lambda) &= (u_1 - 2)(u_2 - 2) - \lambda[(u_1 - 2)^2 + (u_2 - 2)^2 - 4] \\ \frac{\partial f}{\partial u_1} &= 0 \Rightarrow (u_2 - 2) = 2\lambda(u_1 - 2) \\ \frac{\partial f}{\partial u_2} &= 0 \Rightarrow (u_1 - 2) = 2\lambda(u_2 - 2) \\ \frac{\partial f}{\partial \lambda} &= 0 \Rightarrow (u_1 - 2)^2 + (u_2 - 2)^2 = 4 \end{aligned}$$

The solutions are: $(2+\sqrt{2}, 2+\sqrt{2})$ and $(2-\sqrt{2}, 2-\sqrt{2})$. Note that only $(2+\sqrt{2}, 2+\sqrt{2})$ satisfies (3). Therefore, $(2 + \sqrt{2}, 2 + \sqrt{2})$ is the unique Nash bargaining solution.

(b) $d = (0, 2)$. Consider the following problem:

$$\begin{aligned} \text{maximize} \quad & (u_1 - 0)(u_2 - 2) && (4) \\ \text{subject to} \quad & (u_1 - 2)^2 + (u_2 - 2)^2 \leq 4 && (5) \\ & u_1 \geq 0, \ u_2 \geq 2 && (6) \end{aligned}$$

Consider (4) and (5), and apply the method of Lagrange multipliers, we will have

$$\begin{aligned} f(u_1, u_2, \lambda) &= u_1(u_2 - 2) - \lambda[(u_1 - 2)^2 + (u_2 - 2)^2 - 4] \\ \frac{\partial f}{\partial u_1} &= 0 \Rightarrow (u_2 - 2) = 2\lambda(u_1 - 2) \\ \frac{\partial f}{\partial u_2} &= 0 \Rightarrow u_1 = 2\lambda(u_2 - 2) \\ \frac{\partial f}{\partial \lambda} &= 0 \Rightarrow (u_1 - 2)^2 + (u_2 - 2)^2 = 4 \end{aligned}$$

The solutions are: $(0, 2)$ and $(3, 2 + \sqrt{3})$, where the former is not Pareto optimal. $(3, 2 + \sqrt{3})$ is the unique Nash bargaining solution.

□

Exercise 4. *There are two players who may divide 1 dollar between them. The utility function of player 1 is $u_1(x_1) = x_1^{0.5}$ and of player 2 is $u_2(x_2) = x_2$.*

(a) Calculate and draw the set of possible pairs of utilities that the players can get assuming that they may also divide amounts smaller than 1 dollar, i.e., $x_1 + x_2 \leq 1$.

(b) Assume that if the players do not reach an agreement both get 0 dollars. Calculate the utilities the players will get according to the Nash solution. How much money each player gets?

Solution. (a) Since $u_1(x_1) = x_1^{0.5}$, and $u_2(x_2) = x_2$, we have $x_1 = u_1^2$ and $x_2 = u_2$. Thus, the set of possible pairs of utilities is

$$H = \{(u_1, u_2) \colon x_1 + x_2 \leq 1, x_1, x_2 \geq 0\} = \{(u_1, u_2) \colon u_1^2 + u_2 \leq 1, u_1, u_2 \geq 0\}.$$

(b) Here $(0, 0)$ is the threat point. Consider the following problem:

$$\begin{aligned} \text{maximize} \quad & u_1 u_2 && (7) \\ \text{subject to} \quad & u_1^2 + u_2 \leq 1 && (8) \\ & u_1 \geq 0, \ u_2 \geq 0 && (9) \end{aligned}$$

Consider (7) and (8), and apply the method of Lagrange multipliers, we will have

$$\begin{aligned} f(u_1, u_2, \lambda) &= u_1 u_2 - \lambda[u_1^2 + u_2 - 1] \\ \frac{\partial f}{\partial u_1} &= 0 \Rightarrow u_2 = 2\lambda u_1 \\ \frac{\partial f}{\partial u_2} &= 0 \Rightarrow u_1 = \lambda \\ \frac{\partial f}{\partial \lambda} &= 0 \Rightarrow u_1^2 + u_2 = 1 \end{aligned}$$

The solution is: $(\frac{1}{\sqrt{3}}, \frac{2}{3})$. Note it satisfies (9). Therefore, it is the unique Nash bargaining solution. Besides, the corresponding money is $(x_1^*, x_2^*) = (\frac{1}{3}, \frac{2}{3})$.

□

Exercise 5. *Player 1 and player 2 have been willed equal shares of an estate consisting of \$200,000 cash and 100 acres of farmland. Player 1 has a sentimental attachment to the land and values it at $v_1 = \$3,000$ per acre, whereas player 2 has no such attachment and values it at $v_2 = \$1,000$ per acre. Assume that their payoff functions are linear in money and land at these rates: $u_i = x_i + v_i y_i$ if player i receives x_i dollars of cash and y_i acres of land. The players may reach an agreement on dividing the land and money so as to maximize their payoffs. If they fail to reach agreement they divide the land and money equally.*

(i) Carefully draw the bargaining set and label the disagreement point.

(ii) Find the Nash bargaining solution.

Solution. (a) Assume in an agreement, the outcome is (x_1, x_2) and (y_1, y_2), where

$$x_1 + x_2 = 200000, \ y_1 + y_2 = 100, \ x_1, x_2, y_1, y_2 \geq 0,$$

and corresponding payoffs are

$$u_1 = x_1 + 3000y_1, \ u_2 = x_2 + 1000y_2.$$

Hence, we have

$$u_1 + u_2 = 300000 + 2000y_1, \ u_1 + 3u_2 = 500000 + 2000x_1,$$

and hence

$$300000 \leq u_1 + u_2 \leq 500000, \ 500000 \leq u_1 + 3u_2 \leq 900000.$$

Disagreement outcome is $x_1 = x_2 = 100000$, and $y_1 = y_2 = 50$, and hence $u_1 = 250000$ and $u_2 = 150000$, which is a threat point in

$$H = \{(u_1, u_2) \colon 300000 \leq u_1 + u_2 \leq 500000, \ 500000 \leq u_1 + 3u_2 \leq 900000\}.$$

(b) Consider the following problem:

$$\begin{aligned}
\text{maximize} \quad & (u_1 - 250000)(u_2 - 150000) && (10)\\
\text{subject to} \quad & u_1 + u_2 \leq 500000 && (11)\\
& u_1 + 3u_2 \leq 900000 && (12)\\
& 300000 \leq u_1 + u_2 && (13)\\
& 500000 \leq u_1 + 3u_2 && (14)\\
& u_1 \geq 0, \ u_2 \geq 0 && (15)
\end{aligned}$$

Consider (10), (11) and (12), and apply the method of Lagrange multipliers, we will have

$$f(u_1, u_2, \lambda) = (u_1 - 250000)(u_2 - 150000) - \lambda_1[u_1 + u_2 - 500000] - \lambda_2[u_1 + 3u_2 - 900000]$$

$$\frac{\partial f}{\partial u_1} = 0 \Rightarrow u_2 - 150000 = \lambda_1 + \lambda_2$$

$$\frac{\partial f}{\partial u_2} = 0 \Rightarrow u_1 - 250000 = \lambda_1 + 3\lambda_2$$

$$\frac{\partial f}{\partial \lambda_1} = 0 \Rightarrow u_1 + u_2 = 500000$$

$$\frac{\partial f}{\partial \lambda_2} = 0 \Rightarrow u_1 + 3u_2 = 900000$$

The solution is: $(300000, 200000)$. Note it satisfies (13), (14) and (15). Therefore, it is the unique Nash bargaining solution.

□

Exercise 6. *We say that player i is a null player in a game (N, v) if $v(S \cup \{i\}) = v(S)$ for every coalition S not containing i. Thus, a null player cannot help (or harm) any coalition. Suppose that player 1 is a null player. Show that, if $(x_1, x_2, \ldots, x_n)$ is in the core, then $x_1 = 0$.*

Proof. By definition of core, we have

$$\sum_{i=2}^{n} x_i \geq v(\{2, 3, \ldots, n\}) = v(\{1, 2, \ldots, n\}) = \sum_{i=1}^{n} x_i,$$

that is, $x_1 \leq 0$.

On the other hand, we also have

$$x_1 \geq v(\{1\}) = v(\emptyset) = 0^1,$$

and hence $x_1 = 0$. □

Exercise 7. *Let $\delta_i = v(N) - v(N - \{i\})$, $i = 1, 2, \ldots, n$, for a cooperative game (N, v). Show that the core is empty if $\sum_{i=1}^{n} \delta_i < v(N)$.*

Proof. Leave as Question 3 of Assignment 4. □

End of Solution to Tutorial 10

[1]Technically, we should assume that coalication could be empty.