Solution to Tutorial 11*

2011/2012 Semester I MA4264 Game Theory

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Exercise 1. *We say that player i is a null player in a game (N, v) if $v(S \cup \{i\}) = v(S)$ for every coalition S not containing i. Thus, a null player cannot help (or harm) any coalition. Suppose that player 1 is a null player. Show that, if $(x_1, x_2, \ldots, x_n)$ is in the core, then $x_1 = 0$.*

Proof. By definition of core, we have

$$\sum_{i=2}^{n} x_i \geq v(\{2, 3, \ldots, n\}) = v(\{1, 2, \ldots, n\}) = \sum_{i=1}^{n} x_i,$$

that is, $x_1 \leq 0$.

On the other hand, we also have

$$x_1 \geq v(\{1\}) = v(\emptyset) = 0^{1},$$

and hence $x_1 = 0$. □

Exercise 2. *Let $\delta_i = v(N) - v(N - \{i\})$, $i = 1, 2, \ldots, n$, for a cooperative game (N, v). Show that the core is empty if $\sum_{i=1}^{n} \delta_i < v(N)$.*

Proof. Assume that the core is nonempty, and $x = (x_1, \ldots, x_n)$ is an element in it. Since

$$\sum_{i \in N} \frac{v(N - \{i\})}{n - 1} \leq \sum_{i \in N} \Big(\frac{1}{n-1} \sum_{j \neq i} x_j \Big) = \sum_{j \in N} x_j \Big(\sum_{i \neq j} \frac{1}{n-1} \Big) = \sum_{j \in N} x_j = v(N),$$

we have $\sum_{i \in N} v(N - \{i\}) \leq (n-1)v(N)$, and hence

$$v(N) \leq n \cdot v(N) - \sum_{i \in N} v(N - \{i\}) = \sum_{i \in N} [v(N) - v(N - \{i\})] = \sum_{i \in N} \delta_i$$

which is a contradiction. □

*Corrections are always welcome.

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[1] Technically, we should assume that coalication could be empty.

Remark 1. *Bondareva-Shapley Theorem: Let (N, v) be a cooperative game, where N is the set of players, and v is the characteristic function. The core of (N, v) is nonempty if and only if for every function $\lambda\colon 2^N - \{\emptyset\} \to [0, 1]$, where for any $i \in N$, $\sum_{S \ni i} \lambda(S) = 1$, the following condition holds:*

$$\sum_{S \in 2^N - \{\emptyset\}} \lambda(S) v(S) \leq v(N).$$

Exercise 3. *Consider the following three-person game:*

$$v(\emptyset) = 0,\ v(\{1\}) = 0.2,\ v(\{2\}) = v(\{3\}) = 0,$$

$$v(\{1, 2\}) = 1.5,\ v(\{1, 3\}) = 1.6,\ v(\{2, 3\}) = 1.8,\ v(\{1, 2, 3\}) = 2.$$

(a) Find the core of this game.

(b) Find the Shapley value of this game.

(c) Find an imputation dominating the imputation $(1, 1/2, 1/2)$.

Solution. (a) Suppose (x_1, x_2, x_3) is in the core, then it should satisfy:

$$x_1 \geq 0.2 \tag{1}$$
$$x_2, x_3 \geq 0 \tag{2}$$
$$x_1 + x_2 \geq 1.5 \tag{3}$$
$$x_1 + x_3 \geq 1.6 \tag{4}$$
$$x_2 + x_3 \geq 1.8 \tag{5}$$
$$x_1 + x_2 + x_3 = 2 \tag{6}$$

By (3), (4) and (5), we have $x_1 + x_2 + x_3 \geq 2.45$, which contradicts to (6). Therefore, the core is empty.

(b) Since $n = 3$, we have $\gamma(0) = \gamma(2) = 1/3$ and $\gamma(1) = 1/6$.

For Player 1,

S	$\emptyset$	$\{2\}$	$\{3\}$	$\{2, 3\}$
$v(S \cup \{1\}) - v(S)$	0.2	1.5	1.6	0.2

Hence $\phi_1(v) = 0.2\frac{1}{3} + 1.5\frac{1}{6} + 1.6\frac{1}{6} + 0.2\frac{1}{3} = 0.65$.

For Player 2,

S	$\emptyset$	$\{1\}$	$\{3\}$	$\{1, 3\}$
$v(S \cup \{2\}) - v(S)$	0	1.3	1.8	0.4

Hence $\phi_2(v) = 0\frac{1}{3} + 1.3\frac{1}{6} + 1.8\frac{1}{6} + 0.4\frac{1}{3} = 0.65$.

For Player 3, by efficiency, $\phi_3(v) = v(N) - \phi_1(v) - \phi_2(v) = 0.7$.

Therefore, the Shapley value is $(0.65, 0.65, 0.7)$.

(c) It suffices to find an allocation $y = (y_1, y_2, y_3)$, such that $y_1 \geq 0.2$, $y_2, y_3 \geq 0$, $y_1 + y_2 + y_3 = 2$, and y dominates x.

Let $y = (0.2, 0.9, 0.9)$, and $S = \{2, 3\}$. y strictly dominates $x = (1, 0.5, 0.5)$ via S.

□

Exercise 4. *The managing board of a corporation consists of three stock-holders who have respectively 20, 30 and 50 shares of stock and the chairman who has no shares. Any decision can be settled by approval of board members holding a simple majority of the shares and the chairman can decide tie votes. Thus, we define a characteristic function as follows. It has value 1 if a coalition holds* > 50 *shares or holds* ≥ 50 *shares and has the chairman in the coalition. It has value 0 otherwise. Find the Shapley value.*

Solution. We number the players as follows: stock-holder with 20 shares—Player 1, stock-holder with 30 shares—Player 2, stock-holder with 50 shares—Player 3, chairman—Player 4.

The characteristic function is as follows:

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = 0,$$

$$v(\{1,2\}) = v(\{1,4\}) = v(\{2,4\}) = 0,$$
$$v(\{1,3\}) = v(\{2,3\}) = v(\{3,4\}) = 1,$$
$$v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = v(\{1,2,3,4\}) = 1.$$

Since $n = 4$, we have $\gamma(0) = \gamma(3) = \frac{1}{4}$, and $\gamma(1) = \gamma(2) = \frac{1}{12}$.

For Player 1,

S	$\emptyset$	$\{2\}$	$\{3\}$	$\{4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{2,3,4\}$
$v(S \cup \{1\}) - v(S)$	0	0	1	0	0	1	0	0

Hence $\phi_1(v) = 1\frac{1}{12} + 1\frac{1}{12} = \frac{1}{6}$.

For Player 2,

S	$\emptyset$	$\{1\}$	$\{3\}$	$\{4\}$	$\{1,3\}$	$\{1,4\}$	$\{3,4\}$	$\{1,3,4\}$
$v(S \cup \{2\}) - v(S)$	0	0	1	0	0	1	0	0

Hence $\phi_2(v) = 1\frac{1}{12} + 1\frac{1}{12} = \frac{1}{6}$.

For Player 3,

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{4\}$	$\{1,2\}$	$\{1,4\}$	$\{2,4\}$	$\{1,2,4\}$
$v(S \cup \{3\}) - v(S)$	0	1	1	1	1	1	1	0

Hence $\phi_3(v) = \frac{1}{2}$.

For Player 4,

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S \cup \{4\}) - v(S)$	0	0	0	1	1	0	0	0

Hence $\phi_4(v) = \frac{1}{6}$.

Therefore, the Shapley value is $(\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{1}{6})$. □

Exercise 5. *Three doctors have banded together to form a joint practice: the Port Charles Trio. The overhead for the practice is \$40,000 per year. Each doctor brings in annual revenues and incurs annual variable costs as follows: doctor 1—\$155,000 in revenue, \$40,000 in variable cost; doctor 2—\$160,000 in revenue, \$35,000 in variable cost; doctor 3—\$140,000 in revenue, \$38,000 in variable cost.*

The Port Charles Trio wants to use game theory to determine how much each doctor should be paid. Determine the relevant characteristic function and show that the core of the game consists of an infinite number of points. Also determine the Shapley value of the game. Does the Shapley value give a reasonable division of the practice's profits?

Proof and Solution. (i) Characteristic function v:

$$\begin{aligned} v(\emptyset) &= 0 \\ v(\{1\}) &= (155-40)-40 = 75, v(\{2\}) = 85, v(\{3\}) = 62; \\ v(\{1,2\}) &= (155-40)+(160-35)-40 = 200, v(\{1,3\}) = 177, v(\{2,3\}) = 187; \\ v(\{1,2,3\}) &= 302. \end{aligned}$$

(ii) Core. To solve (x_1, x_2, x_3), such that,

$$x_1 \geq 75,\ x_2 \geq 85,\ x_3 \geq 62,$$

$$x_1 + x_2 \geq 200,\ x_1 + x_3 \geq 177,\ x_2 + x_3 \geq 187,$$
$$x_1 + x_2 + x_3 = 302.$$

Solving them, we have

$$C(N, v) = \{(x_1, x_2, x_3)\colon x_1 \in [75, 115],\ x_2 \in [85, 125],\ x_3 \in [62, 102],\ x_1+x_2+x_3 = 302\}.$$

Since $(75, 125, 102)$ and $(76, 124, 102)$ are in the core, and for any $\lambda \in [0, 1]$, $\lambda(75, 125, 102) + (1-\lambda)(76, 124, 102)$ is also in the core, that is, there are infinite many elements in the core.

(iii) Shapley value. Since $n = 3$, we have $\gamma(0) = \gamma(2) = 1/3$, and $\gamma(1) = 1/6$.

For Player 1.

S	$\emptyset$	$\{2\}$	$\{3\}$	$\{2,3\}$
$v(S \cup \{1\}) - v(S)$	75	115	115	115

Hence $\phi_1(v) = 75\frac{1}{3} + 115\frac{1}{6} + 115\frac{1}{6} + 115\frac{1}{3} = \frac{305}{3}$.

For Player 2.

S	$\emptyset$	$\{1\}$	$\{3\}$	$\{1,3\}$
$v(S \cup \{2\}) - v(S)$	85	125	125	125

Hence $\phi_2(v) = \frac{335}{3}$.

For Player 3, by efficiency, $\phi_3(v) = v(N) - \phi_1(v) - \phi_2(v) = \frac{266}{3}$.

Therefore, the Shapley value is $\left(\frac{305}{3}, \frac{335}{3}, \frac{266}{3}\right)$.

□

Exercise 6. *The management committee of an association consists of a president, a secretary, a treasurer and two committee members. It requires three votes including one from the president, one form the secretary or treasurer to approve a proposal. Determine the voting power of each member in the committee.*

Solution. We number the players as follows: president—Player 1, secretary—Player 2, Treasurer—Player 3, member 1—Player 4, member 2—Player 5.

Let the total voting power be 1. We use the Shapley value to compute each player's voting power.

The characteristic function is

$$v(S) = \begin{cases} 1, & \text{if } |S| \geq 3,\ \{1,2\} \subseteq S \text{ or } \{1,3\} \subseteq S, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that player 2 and player 3 have the same voting power, and player 4 and player 5 have the same voting power. Hence, their Shapley value must be the same.

Since $n = 5$, we have $\gamma(0) = \gamma(4) = \frac{1}{5}$, $\gamma(1) = \gamma(3) = \frac{1}{20}$ and $\gamma(2) = \frac{1}{30}$.

For player 1, $v(S \cup \{1\}) - v(S) = 1$ if and only if S is one of the followings:

$$\{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}, \{2,3,4,5\}.$$

Hence the Shapley value of player 1 is $\phi_1(v) = 5\frac{1}{30} + 4\frac{1}{20} + 1\frac{1}{5} = \frac{17}{30}$.

For player 2, $v(S \cup \{2\}) - v(S) = 1$ if and only if S is one of the followings:

$$\{1,3\}, \{1,4\}, \{1,5\}, \{1,4,5\}.$$

Hence the Shapley value of player 2 is $\phi_2(v) = 3\frac{1}{30} + \frac{1}{20} = \frac{3}{20}$, which is $\phi_3(v)$.

For player 4, $v(S \cup \{4\}) - v(S) = 1$ if and only if S is one of the followings:

$$\{1,2\}, \{1,3\}.$$

Hence the Shapley value of player 4 is $\phi_4(v) = 2\frac{1}{30} = \frac{1}{15}$, which is $\phi_5(v)$.

Therefore, the voting power is $(\frac{17}{30}, \frac{3}{20}, \frac{3}{20}, \frac{1}{15}, \frac{1}{15})$. □

Exercise 7. *Consider an n-person game in which the only winning coalitions are those coalitions containing player 1 and at least one other player. If a winning coalition receives a reward of \$1, find the core and the Shapley value of the game.*

Solution. When $n = 2$, the solution is quite easy. (Exercise)

In the following, we assume $n \geq 3$.

The characteristic function is

$$v(S) = \begin{cases} 1, & \text{if } |S| \geq 2, \text{ Player 1 belongs to } S, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Core. Suppose $(x_1, x_2, \ldots, x_n)$ is in the core. Then we have

$$\sum_{i=1}^{n} x_i = 1,$$

$$x_1 + \sum_{i \in S} x_i \geq 1, \text{ for all } S \subset \{2, 3, \ldots, n\} \text{ and } S \text{ is nonempty},$$

$$x_i \geq 0.$$

It is easy to see that the only solution is $x_1 = 1$, $x_2 = \cdots = x_n = 0$. That is, the core is $\{(1, 0, 0, \ldots, 0)\}$.

(ii) Shapley value.

For Player $i \neq 1$, $v(S \cup \{i\}) - v(S) = 1$ if and only if $S = \{1\}$. Otherwise it is zero. Since $\gamma(1) = \frac{1}{n-1}$, Player i's Shapley value is

$$\phi_i(v) = \frac{1}{n(n-1)}.$$

For Player 1, we have

$$\phi_1(v) = 1 - \sum_{i=2}^{n} \phi_i(v) = \frac{n-1}{n}.$$

Therefore, the Shapley value is

$$\left(\frac{n-1}{n}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right).$$

□

Exercise 8. *Find the Shapley values of the game with $N = \{1, 2\}$ and the characteristic function v. Now consider the bargaining game where $H = I(N, v)$ and $d = (v(\{1\}), v(\{2\}))$. Find the bargaining solution of the game (H, d).*

Solution. Since $n = 2$, we have $\gamma(0) = \gamma(1) = \frac{1}{2}$. Denote $v = v(N)$, $v_1 = v(\{1\})$ and $v_2 = v(\{2\})$.

(i) Shapley value. For Player i,

S	$\emptyset$	$\{j\}$
$v(S \cup \{i\}) - v(S)$	v_i	$v - v_j$

Hence the Shapley value for Player i is $\frac{v_i + v - v_j}{2}$.

(ii) To get the Nash bargaining solution, we solve the following problem

$$\max_{x_1 + x_2 = v} (x_1 - v_1)(x_2 - v_2).$$

The solution is $x_i^* = \frac{v_i + v - v_j}{2}$.

Hence, both Nash bargaining solution and the Shapley value given the same result.

□