Solution to Tutorial 11

2012/2013 Semester I MA4264 Game Theory

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Exercise 1. *Consider the following three-person game:*

$$v(\emptyset) = 0,\ v(\{1\}) = 0.2,\ v(\{2\}) = v(\{3\}) = 0,$$

$$v(\{1,2\}) = 1.5,\ v(\{1,3\}) = 1.6,\ v(\{2,3\}) = 1.8,\ v(\{1,2,3\}) = 2.$$

(a) Find the core of this game.

(b) Find the Shapley value of this game.

(c) Find an imputation dominating the imputation $(1, 1/2, 1/2)$.

Solution. (a) Suppose (x_1, x_2, x_3) is in the core, then it should satisfy:

$$x_1 \geq 0.2 \quad (1)$$

$$x_2, x_3 \geq 0 \quad (2)$$

$$x_1 + x_2 \geq 1.5 \quad (3)$$

$$x_1 + x_3 \geq 1.6 \quad (4)$$

$$x_2 + x_3 \geq 1.8 \quad (5)$$

$$x_1 + x_2 + x_3 = 2 \quad (6)$$

By (3), (4) and (5), we have $x_1 + x_2 + x_3 \geq 2.45$, which contradicts to (6). Therefore, the core is empty.

(b) Since $n = 3$, we have $\gamma(0) = \gamma(2) = 1/3$ and $\gamma(1) = 1/6$.

For Player 1,

S	$\emptyset$	$\{2\}$	$\{3\}$	$\{2,3\}$
$v(S \cup \{1\}) - v(S)$	0.2	1.5	1.6	0.2

Hence $\phi_1(v) = 0.2\frac{1}{3} + 1.5\frac{1}{6} + 1.6\frac{1}{6} + 0.2\frac{1}{3} = 0.65$.

For Player 2,

S	$\emptyset$	$\{1\}$	$\{3\}$	$\{1,3\}$
$v(S \cup \{2\}) - v(S)$	0	1.3	1.8	0.4

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Hence $\phi_2(v) = 0\frac{1}{3} + 1.3\frac{1}{6} + 1.8\frac{1}{6} + 0.4\frac{1}{3} = 0.65$.

For Player 3, by efficiency, $\phi_3(v) = v(N) - \phi_1(v) - \phi_2(v) = 0.7$.[1]

Therefore, the Shapley value is $(0.65, 0.65, 0.7)$.

(c) It suffices to find an allocation $y = (y_1, y_2, y_3)$, such that $y_1 \geq 0.2$, $y_2, y_3 \geq 0$, $y_1 + y_2 + y_3 = 2$, and y dominates x.

Let $y = (0.2, 0.9, 0.9)$, and $S = \{2, 3\}$. y strictly dominates $x = (1, 0.5, 0.5)$ via S.

□

Exercise 2. *The managing board of a corporation consists of three stock-holders who have respectively 20, 30 and 50 shares of stock and the chairman who has no shares. Any decision can be settled by approval of board members holding a simple majority of the shares and the chairman can decide tie votes. Thus, we define a characteristic function as follows. It has value 1 if a coalition holds* > 50 *shares or holds* ≥ 50 *shares and has the chairman in the coalition. It has value 0 otherwise. Find the Shapley value.*

Solution. We number the players as follows: stock-holder with 20 shares—Player 1, stock-holder with 30 shares—Player 2, stock-holder with 50 shares—Player 3, chairman—Player 4.

The characteristic function is as follows:

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = 0,$$

$$v(\{1,2\}) = v(\{1,4\}) = v(\{2,4\}) = 0,$$

$$v(\{1,3\}) = v(\{2,3\}) = v(\{3,4\}) = 1,$$

$$v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = v(\{1,2,3,4\}) = 1.$$

Since $n = 4$, we have $\gamma(0) = \gamma(3) = \frac{1}{4}$, and $\gamma(1) = \gamma(2) = \frac{1}{12}$.

For Player 1,

S	$\emptyset$	$\{2\}$	$\{3\}$	$\{4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{2,3,4\}$
$v(S \cup \{1\}) - v(S)$	0	0	1	0	0	1	0	0

Hence $\phi_1(v) = 1\frac{1}{12} + 1\frac{1}{12} = \frac{1}{6}$.

For Player 2,

S	$\emptyset$	$\{1\}$	$\{3\}$	$\{4\}$	$\{1,3\}$	$\{1,4\}$	$\{3,4\}$	$\{1,3,4\}$
$v(S \cup \{2\}) - v(S)$	0	0	1	0	0	1	0	0

Hence $\phi_2(v) = 1\frac{1}{12} + 1\frac{1}{12} = \frac{1}{6}$.

For Player 3,

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{4\}$	$\{1,2\}$	$\{1,4\}$	$\{2,4\}$	$\{1,2,4\}$
$v(S \cup \{3\}) - v(S)$	0	1	1	1	1	1	1	0

Hence $\phi_3(v) = \frac{1}{2}$.

For Player 4,

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S \cup \{4\}) - v(S)$	0	0	0	1	1	0	0	0

[1]It has better to computer the Shapley value for Player 3 by standard way, and to apply efficiency to check whether your calculating is correct.

Hence $\phi_4(v) = \frac{1}{6}$.

Therefore, the Shapley value is $(\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{1}{6})$. □

Exercise 3. *Three doctors have banded together to form a joint practice: the Port Charles Trio. The overhead for the practice is \$40,000 per year. Each doctor brings in annual revenues and incurs annual variable costs as follows: doctor 1—\$155,000 in revenue, \$40,000 in variable cost; doctor 2—\$160,000 in revenue, \$35,000 in variable cost; doctor 3—\$140,000 in revenue, \$38,000 in variable cost.*

The Port Charles Trio wants to use game theory to determine how much each doctor should be paid. Determine the relevant characteristic function and show that the core of the game consists of an infinite number of points. Also determine the Shapley value of the game. Does the Shapley value give a reasonable division of the practice's profits?

Proof and Solution. (i) Characteristic function v:

$$\begin{aligned} v(\emptyset) &= 0 \\ v(\{1\}) &= (155-40)-40 = 75, v(\{2\}) = 85, v(\{3\}) = 62; \\ v(\{1,2\}) &= (155-40)+(160-35)-40 = 200, v(\{1,3\}) = 177, v(\{2,3\}) = 187; \\ v(\{1,2,3\}) &= 302. \end{aligned}$$

(ii) Core. To solve (x_1, x_2, x_3), such that,

$$x_1 \geq 75,\ x_2 \geq 85,\ x_3 \geq 62,$$

$$x_1 + x_2 \geq 200,\ x_1 + x_3 \geq 177,\ x_2 + x_3 \geq 187,$$

$$x_1 + x_2 + x_3 = 302.$$

Solving them, we have

$$C(N, v) = \{(x_1, x_2, x_3) \colon x_1 \in [75, 115],\ x_2 \in [85, 125],\ x_3 \in [62, 102],\ x_1 + x_2 + x_3 = 302\}.$$

Since $(75, 125, 102)$ and $(76, 124, 102)$ are in the core, and for any $\lambda \in [0, 1]$, $\lambda(75, 125, 102) + (1-\lambda)(76, 124, 102)$ is also in the core, that is, there are infinite many elements in the core.

(iii) Shapley value. Since $n = 3$, we have $\gamma(0) = \gamma(2) = 1/3$, and $\gamma(1) = 1/6$.

For Player 1.

S	$\emptyset$	$\{2\}$	$\{3\}$	$\{2,3\}$
$v(S \cup \{1\}) - v(S)$	75	115	115	115

Hence $\phi_1(v) = 75\frac{1}{3} + 115\frac{1}{6} + 115\frac{1}{6} + 115\frac{1}{3} = \frac{305}{3}$.

For Player 2.

S	$\emptyset$	$\{1\}$	$\{3\}$	$\{1,3\}$
$v(S \cup \{2\}) - v(S)$	85	125	125	125

Hence $\phi_2(v) = \frac{335}{3}$.

For Player 3, by efficiency, $\phi_3(v) = v(N) - \phi_1(v) - \phi_2(v) = \frac{266}{3}$.

Therefore, the Shapley value is $\left(\frac{305}{3}, \frac{335}{3}, \frac{266}{3}\right)$.

□

Exercise 4. *The management committee of an association consists of a president, a secretary, a treasurer and two committee members. It requires three votes including one from the president, one from the secretary or treasurer to approve a proposal. Determine the voting power of each member in the committee.*

Solution. We number the players as follows: president—Player 1, secretary—Player 2, Treasurer—Player 3, member 1—Player 4, member 2—Player 5.

Let the total voting power be 1. We use the Shapley value to compute each player's voting power.

The characteristic function is

$$v(S) = \begin{cases} 1, & \text{if } |S| \geq 3,\ \{1,2\} \subseteq S \text{ or } \{1,3\} \subseteq S, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that player 2 and player 3 have the same voting power, and player 4 and player 5 have the same voting power. Hence, their Shapley value must be the same.

Since $n = 5$, we have $\gamma(0) = \gamma(4) = \frac{1}{5}$, $\gamma(1) = \gamma(3) = \frac{1}{20}$ and $\gamma(2) = \frac{1}{30}$.

For player 1, $v(S \cup \{1\}) - v(S) = 1$ if and only if S is one of the followings:

$$\{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}, \{2,3,4,5\}.$$

Hence the Shapley value of player 1 is $\phi_1(v) = 5\frac{1}{30} + 4\frac{1}{20} + 1\frac{1}{5} = \frac{17}{30}$.

For player 2, $v(S \cup \{2\}) - v(S) = 1$ if and only if S is one of the followings:

$$\{1,3\}, \{1,4\}, \{1,5\}, \{1,4,5\}.$$

Hence the Shapley value of player 2 is $\phi_2(v) = 3\frac{1}{30} + \frac{1}{20} = \frac{3}{20}$, which is $\phi_3(v)$.

For player 4, $v(S \cup \{4\}) - v(S) = 1$ if and only if S is one of the followings:

$$\{1,2\}, \{1,3\}.$$

Hence the Shapley value of player 4 is $\phi_4(v) = 2\frac{1}{30} = \frac{1}{15}$, which is $\phi_5(v)$.

Therefore, the voting power is $(\frac{17}{30}, \frac{3}{20}, \frac{3}{20}, \frac{1}{15}, \frac{1}{15})$. □

Exercise 5. *Consider an n-person game in which the only winning coalitions are those coalitions containing player 1 and at least one other player. If a winning coalition receives a reward of \$1, find the core and the Shapley value of the game.*

Solution. When $n = 2$, the solution is quite easy. (Exercise)

In the following, we assume $n \geq 3$. The characteristic function is

$$v(S) = \begin{cases} 1, & \text{if } |S| \geq 2,\ \text{Player 1 belongs to } S, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Core. Suppose $(x_1, x_2, \ldots, x_n)$ is in the core. Then we have

$$\sum_{i=1}^{n} x_i = 1,\ x_i \geq 0,$$

$$x_1 + \sum_{i \in S} x_i \geq 1, \text{ for all } S \subset \{2, 3, \ldots, n\} \text{ and } S \text{ is nonempty.}$$

It is easy to see that the only solution is $x_1 = 1$, $x_2 = \cdots = x_n = 0$: take $S = \{3, 4, \ldots, n\}$, we have $x_1 + x_3 + \cdots + x_n \geq 1$, and hence $x_2 = 0$. Similarly $x_3 = x_4 = \cdots = x_n = 0$. Therefore the core is $\{(1, 0, 0, \ldots, 0)\}$.

(ii) Shapley value.

For Player $i \neq 1$, $v(S \cup \{i\}) - v(S) = 1$ if and only if $S = \{1\}$. Otherwise it is zero. Since $\gamma(1) = \frac{1}{n-1}$, Player i's Shapley value is

$$\phi_i(v) = \frac{1}{n(n-1)}.$$

For Player 1, we have

$$\phi_1(v) = 1 - \sum_{i=2}^{n} \phi_i(v) = \frac{n-1}{n}.$$

Therefore, the Shapley value is

$$\left(\frac{n-1}{n}, \frac{1}{n(n-1)}, \ldots, \frac{1}{n(n-1)} \right).$$

□

Exercise 6. *Find the Shapley values of the game with $N = \{1,2\}$ and the characteristic function v. Now consider the bargaining game where $H = I(N,v)$ and $d = (v(\{1\}), v(\{2\}))$. Find the bargaining solution of the game (H,d).*

Solution. Since $n = 2$, we have $\gamma(0) = \gamma(1) = \frac{1}{2}$. Denote $v = v(N)$, $v_1 = v(\{1\})$ and $v_2 = v(\{2\})$.

(i) Shapley value. For Player i,

S	$\emptyset$	$\{j\}$
$v(S \cup \{i\}) - v(S)$	v_i	$v - v_j$

Hence the Shapley value for Player i is $\frac{v_i + v - v_j}{2}$.

(ii) To get the Nash bargaining solution, we solve the following problem

$$\max_{x_1 + x_2 = v, x_1 \geq v_1, x_2 \geq v_2} (x_1 - v_1)(x_2 - v_2).$$

The solution is $x_i^* = \frac{v_i + v - v_j}{2}$. Note that we need to check whether $x_i^* \geq v_i$.

Hence, both Nash bargaining solution and the Shapley value given the same result. □

Exercise 7. *Consider the following cost allocation problem. Building an airfield will benefit n players. Player j requires an airfield that costs c_j to build, so to accommodate all the players, the field will be built at a cost of $\max_{1 \leq j \leq n} c_j$. How should this cost be split among the players? Suppose all the costs are distinct and let $0 < c_1 < c_2 < \cdots < c_n$. Take the characteristic function of the game to be $v(S) = -\max_{j \in S} c_j$ for $S \subset \{1, 2, \ldots, n\}$.*

(i) Let $R_k = \{k, k+1, \ldots, n\}$ for $k = 1, 2, \ldots, n$, and define the characteristic function v_k through the equation

$$v_k(S) = \begin{cases} -(c_k - c_{k-1}), & \text{if } S \cap R_k \neq \emptyset \\ 0, & \text{if } S \cap R_k = \emptyset \end{cases}$$

For convenience, let $c_0 = 0$. Show that $v = \sum_{k=1}^{n} v_k$.

(ii) *Find the Shapley value of the game v in the form of $\phi_i(v) = \sum_{k=1}^{i} \alpha_{ik}(c_k - c_{k-1})$, $i = 1, 2, \ldots, n$, where the coefficients α_{ik} are independent of $c_1, c_2, \ldots, c_n$.*

Solution. (i) For every coalition S, we have

$$\begin{aligned}\sum_{k=1}^{n} v_k(S) &= \sum_{k=1}^{\max(S)} v_k(S) + \sum_{k=\max(S)+1}^{n} v_k(S)\\ &= \sum_{k=1}^{\max(S)} v_k(S) = -\sum_{k=1}^{\max(S)} (c_k - c_{k-1})\\ &= -c_{\max(S)} = v(S)\end{aligned}$$

(ii) Since

$$v_k(S) = \begin{cases} -(c_k - c_{k-1}), & \text{if } \max(S) \geq k \\ 0, & \text{if } \max(S) < k \end{cases}$$

we have

$$v_k(S \cup \{i\}) - v_k(S) = \begin{cases} -(c_k - c_{k-1}), & \text{if } \max(S) < k \leq i \\ 0, & \text{otherwise} \end{cases}$$

and hence $\phi_1(v_k) = \cdots = \phi_{k-1}(v_k) = 0$, and

$$\begin{aligned}&\phi_k(v_k) = \cdots = \phi_n(v_k)\\ &= \frac{1}{n}\sum_{s=0}^{n-1} \frac{1}{\binom{n-1}{s}} \sum_{|S|=s} [v_k(S \cup \{i\}) - v_k(S)]\\ &= \frac{1}{n}\sum_{s=0}^{n-1} \frac{1}{\binom{n-1}{s}} \left\{ \sum_{|S|=s, \max(S)<k} [v_k(S \cup \{i\}) - v_k(S)] + \sum_{|S|=s, \max(S)\geq k} [v_k(S \cup \{i\}) - v_k(S)] \right\}\\ &= \frac{1}{n}\sum_{s=0}^{n-1} \frac{1}{\binom{n-1}{s}} \binom{k-1}{s} [-(c_k - c_{k-1})]\end{aligned}$$

Therefore, we have

$$\begin{aligned}\phi_i(v) &= \sum_{k=1}^{n} \phi_i(v_k) = \sum_{k=1}^{i} \phi_i(v_k)\\ &= -\frac{1}{n}\sum_{k=1}^{i}\sum_{s=0}^{k-1} \frac{\binom{k-1}{s}}{\binom{n-1}{s}} (c_k - c_{k-1})\end{aligned}$$

□

End of Solution to Tutorial 11