Solution to Tutorial 1

2012/2013 Semester I MA4264 Game Theory

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1 Review

- "**Static**" means one-shot, or simultaneous-move; "**Complete information**" means that the payoff functions are common knowledge.
- Normal-form representation: $G = \{S_1, \dots, S_n; u_1, \dots, u_n\}$, where n is finite.
- s_i' is **strictly dominated** by s_i'', if

$$u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i}), \quad \forall s_{-i} \in S_{-i}.$$

- Rational players do not play strictly dominated strategies, since they are always not optimal no matter what strategies others would choose.
- Iterated elimination of strictly dominated strategies. This process is order-independent.
- Given other players' strategies $s_{-i} \in S_{-i}$, Player i's **best response**, denoted by $R_i(s_{-i})$, is the set of maximizers of $\max_{s_i \in S_i} u_i(s_i, s_{-i})$, i.e.,

$$R_i(s_{-i}) = \left\{ s_i \in S_i \colon u_i(s_i, s_{-i}) = \max_{s_i' \in S_i} u_i(s_i', s_{-i}) \right\} \subset S_i.$$

  We call R_i the **best-response correspondence** for player i.
- Given s_{-i}, the best response $R_i(s_{-i})$ is a set.
- In the n-player normal-form game $G = \{S_1, \dots, S_n; u_1, \dots, u_n\}$, the strategy profile $(s_1^*, \dots, s_n^*)$ is a **pure-strategy Nash equilibrium** if

$$s_i^* \in R_i(s_{-i}^*), \quad \forall i = 1, \dots, n,$$

  equivalently,

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*), \quad \forall i = 1, \dots, n.$$

- $\{$Nash equilibrium(a)$\} \subset \{$Outcomes of IESDS$\}$.

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2 Tutorial 1

Exercise 1. *In the following normal-form games, what strategies survive iterated elimination of strictly dominated strategies? What are the pure-strategy Nash equilibria?*

	L	C	R
T	$2,0$	$1,1$	$4,2$
M	$3,4$	$1,2$	$2,3$
B	$1,3$	$0,2$	$3,0$

	L	R
U	$1,3$	$-2,0$
M	$-2,0$	$1,3$
D	$0,1$	$0,1$

Solution. 1. In the left game, for Player 1, B is strictly dominated by T and will be eliminated. Then the bi-matrix becomes to the reduced bi-matrix G_1.

In the bi-matrix G_1, for Player 2, C is strictly dominated by R and the bi-matrix G_1 becomes to the reduced bi-matrix G_2.

In the bi-matrix G_2, for Players 1 and 2, no strategy is strictly dominated.

Hence the strategies T, M, L and R will survive iterated elimination of strictly dominated strategies.

	L	C	R
T	$2,0$	$1,1$	$4,2$
M	$3,4$	$1,2$	$2,3$

G_1

	L	R
T	$2,0$	$4,2$
M	$3,4$	$2,3$

G_2

	L	R
U	$1,3$	$-2,0$
M	$-2,0$	$1,3$
D	$0,1$	$0,1$

H

In the bi-matrix G_2, we will obtain that the Nash equilibria are (M,L) and (T,R) (red pairs in the bi-matrix).

2. In the right game, it is easy to see that no strategy is strictly dominated. Hence all strategies will survive iterated elimination of strictly dominated strategies.

   From the bi-matrix H, we will obtain that the Nash equilibria are (U,L) and (M,R) (red pairs in the bi-matrix).

□

Exercise 2. *An old lady is looking for help crossing the street. Only one person is needed to help her; more are okay but no better than one. You and I are the two people in the vicinity who can help, each has to choose simultaneously whether to do so. Each of us will get pleasure worth of 3 from her success (no matter who helps her). But each one who goes to help will bear a cost of 1, this being the value of our time taken up in helping. Set this up as a game. Write the payoff table, and find all pure-strategy Nash equilibria.*

Solution.
- There are two players: You (Player 1) and I (Player 2);
- For each player, he/she has 2 strategies: "Help" and "Not Help".

		Player 2	
		Help	Not help
Player 1	Help	$2, 2$	$2, 3$
	Not help	$3, 2$	$0, 0$

K

- Since there are 2 players, and 2 pure strategies for each player, the payoff function can be represented by a bi-matrix K:

From the bi-matrix K, we will find the Nash equilibria are (Help, Not help) and (Not help, Help) (red pairs in the bi-matrix). □

Exercise 3. *There are three computer companies, each of which can choose to make large* (L) *or small* (S) *computers. The choice of company 1 is denoted by* S_1 *or* L_1*, and similarly, the choices of companies 2 and 3 are denoted* S_i *or* L_i *of* $i = 2$ *or 3. The following table shows the profit each company would receive according to the choices which the three companies could make. What is the outcome of IESDS and the Nash equilibria of the game?*

	S_2S_3	S_2L_3	L_2S_3	L_2L_3
S_1	$-10, -15, 20$	$0, -10, 60$	$0, 10, 10$	$20, 5, 15$
L_1	$5, -5, 0$	$-5, 35, 15$	$-5, 0, 15$	$-20, 10, 10$

Solution. 1. (a) From the following table, we can obtain that either S_1 or L_1 can not be strictly dominated.

Player 2's strategy	Player 3's strategy	Player 1's best response
S_2	S_3	L_1
S_2	L_3	S_1
L_2	S_3	S_1
L_2	L_3	S_1

(b) From the following table, we can obtain that either S_2 or L_2 can not be strictly dominated.

Player 1's strategy	Player 3's strategy	Player 2's best response
S_1	S_3	L_2
S_1	L_3	L_2
L_1	S_3	L_2
L_1	L_3	S_2

(c) From the following table, we can obtain that either S_3 or L_3 can not be strictly dominated.

Player 1's strategy	Player 2's strategy	Player 3's best response
S_1	S_2	L_3
S_1	L_2	L_3
L_1	S_2	L_3
L_1	L_2	S_3

To summarize, no strategy will be eliminated in IESDS.

2. From the payoff table G, we will obtain that the Nash equilibrium is (S_1, L_2, L_3).

	S_2S_3	S_2L_3	L_2S_3	L_2L_3
S_1	$-10, -15, 20$	$0, -10, 60$	$0, 10, 10$	$20, 5, 15$
L_1	$5, -5, 0$	$-5, 35, 15$	$-5, 0, 15$	$-20, 10, 10$

G

□

Exercise 4. *Players 1 and 2 are bargaining over how to split one dollar. Both players simultaneously name shares they would like to have, s_1 and s_2, where $0 \leq s_1, s_2 \leq 1$. If $s_1^2 + s_2^2 \leq 1/2$, then the players receive the shares they named; if $s_1^2 + s_2^2 > 1/2$, then both players receive zero. What are the pure-strategy Nash equilibria of this game? Now we change the payoff rule as follows: If $s_1^2 + s_2^2 < 1/2$, then the players receive the shares they named; if $s_1^2 + s_2^2 \geq 1/2$, then both players receive zero. What are the pure-strategy Nash equilibria of this game?*

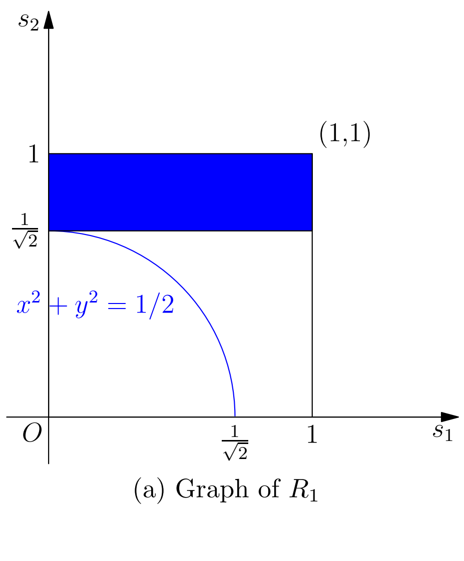
Solution. 1. (1st method for 1st sub-question) Given Player 2's strategy s_2, the best response of Player 1 is:

$$R_1(s_2) = \begin{cases} \left\{\sqrt{\frac{1}{2} - s_2^2}\right\}, & \text{if } s_2 < \frac{1}{\sqrt{2}}; \\ [0, 1], & \text{if } s_2 \geq \frac{1}{\sqrt{2}}. \end{cases}$$

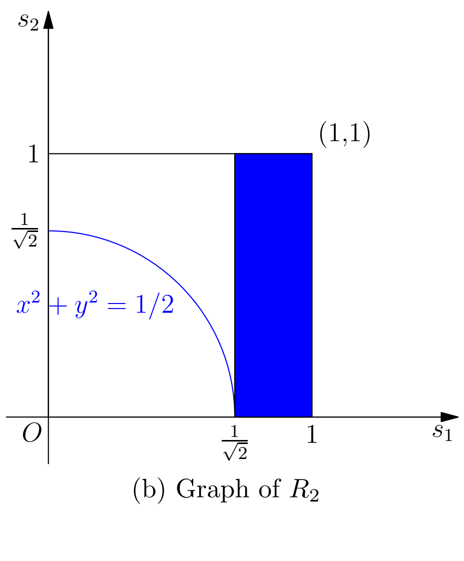
Note that if $s_2 < \frac{1}{\sqrt{2}}$, then Player 1 should choose s_1 as much as possible, so that $s_1^2 + s_2^2 \leq \frac{1}{2}$. Hence, $\left\{\sqrt{\frac{1}{2} - s_2^2}\right\}$ is Player 1's best response to s_2.

If $s_2 \geq \frac{1}{\sqrt{2}}$, no matter what Player 1 chooses, his payoff is always 0. Thus Player 1 can choose any value between 0 and 1.

The graph of R_1 is showed in Figure (a), and by symmetry, we can also get the best response of Player 2, showed in Figure (b).



(a) Graph of R_1



(b) Graph of R_2

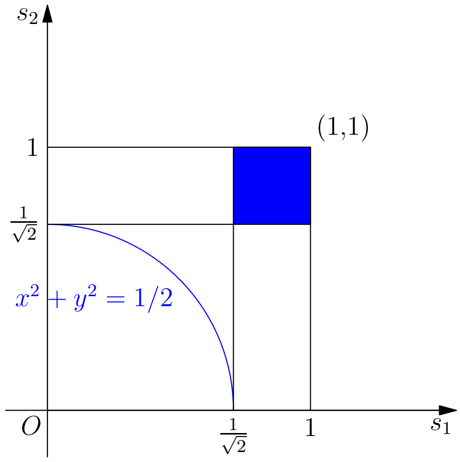


Figure 1: Intersection of R_1 and R_2

Then the intersection of R_1 and R_2 is shown in Figure 1.

So the pure-strategy Nash equilibria are

$$\left\{(s_1, s_2) \mid s_1 \geq 0, s_2 \geq 0, s_1^2 + s_2^2 = \frac{1}{2}\right\} \cup \left(\left[\frac{1}{\sqrt{2}}, 1\right] \times \left[\frac{1}{\sqrt{2}}, 1\right]\right).$$

2. (2nd method for 1st sub-question) Let $s = (s_1, s_2) \in [0,1] \times [0,1]$. We distinguish the following three cases:

   (a) if $s_1^2 + s_2^2 < 1/2$, each player i can do better by choosing $s_i + \epsilon$. Thus, s is not a Nash equilibrium.

   (b) if $s_1^2 + s_2^2 = 1/2$, no player can do better by unilaterally changing his/her strategy (because i's payoff is 0 by choosing $s_i + \epsilon$). Thus, s is a Nash equilibrium.

   (c) if $s_1^2 + s_2^2 > 1/2$, then we further distinguish two subcases:

      i. if $s_i^2 < 1/2$, then j can do better by choosing $s_i + \epsilon$. Thus, s in this subcase is not a Nash equilibrium.

      ii. if $s_1^2 \geq 1/2$ and $s_2^2 \geq 1/2$, then no player can do better by unilaterally changing his/her strategy (because i's payoff is always 0 if $s_j^2 \geq 1/2$). Thus, s in this subcase is a Nash equilibrium.

3. (2nd sub-question) Leave as Question 1 of Assignment 1.

□

Exercise 5. *In the movie, "A Beautiful Mind", John Nash gets the idea for Nash equilibrium in a student hangout where he is sitting with three buddies. Five women walk in, four brunettes and a stunning blonde. Each of the four buddies starts forward to introduce himself to the blonde. Nash stops them, though, saying, "If we all go for the blonde, we will all be rejected and none of the brunettes will talk to us afterwards because they will be offended. So let's go for the brunettes." The next thing we see is the four buddies dancing with the four brunettes and the blonde standing alone, looking unhappy.*

Assume that if more than one buddy goes after a single woman, they will all be rejected by the woman and end up alone. The payoffs are as follows. Ending up with the blonde has a payoff of 4, ending up with a brunette has a payoff of 1, and ending up alone is 0. The four buddies are players in this noncooperative game.

(i) Is the result in the story a Nash equilibrium?

(ii) Find all pure-strategy Nash equilibria for this game.

(iii) Are the Nash equilibria you find better than what Nash suggested?

Solution. (i) It is not a Nash equilibrium. If three guys stick to their strategies of dancing with brunettes, the fourth guy can become better off by going after the blonde.

(ii) There are 4 Nash equilibria: (blonde, brunette, brunette, brunette), (brunette, blonde, brunette, brunette), (brunette, brunette, blonde, brunette), and (brunette, brunette, brunette, blonde), where in the strategy profile (a, b, c, d), Players 1, 2, 3 and 4 choose a, b, c and d, respectively. Here we assume that the four Brunettes are indistinguishable.

Each strategy profile above is a Nash equilibrium, since no one will be better off by changing strategy unilaterally. For example, if a guy who is dancing with one of the brunettes is unhappy and wants to change his strategy. If he goes for the blonde, he will end up being alone since another guy is already dancing with the blonde. So he will not gain by deviating from his current strategy.

Claim: there is no other Nash equilibrium: Any strategy profile is in one of the following 3 types:

(a) One approaches the Blonde;

(b) No one approaches the Blonde;

(c) More than one approaches the Blonde.

It is easy to check that any strategy profile in type (b) or type (c) is not a Nash equilibrium. Thus, all Nash equilibria are of type (a), which are the four listed above.

(iii) In terms of total payoff, Nash equilibrium is better than the outcome in the film.

□

Exercise 6. *Two firms may compete for a given market of total value, V, by investing a certain amount of effort into the project through advertising, securing outlets, etc. Each firm may allocate a certain amount for this purpose. If firm 1 allocates $x \geq 0$ and firm 2 allocates $y \geq 0$, then the proportion of the market that firm 1 corners is $x/(x + y)$. The firms have different difficulties in allocating these resources.*

The cost per unit allocation to firm i is c_i, $i = 1, 2$. Thus the profits to the two firms are

$$\pi_1(x, y) = V \cdot \frac{x}{x+y} - c_1 x,$$
$$\pi_2(x, y) = V \cdot \frac{y}{x+y} - c_2 y.$$

If both x and y are zero, the payoffs to both are $V/2$.

Find the equilibrium allocations, and the equilibrium profits to the two firms, as functions of V, c_1 and c_2.

Solution. It is natural to assume V, c_1 and c_2 are positive.

1. Given Player 2's strategy $y = 0$, there is no best response for Player 1: The payoff of Player 1 is as follows

$$\pi_1(x, 0) = \begin{cases} V - c_1 x, & \text{if } x > 0; \\ \frac{V}{2}, & \text{if } x = 0. \end{cases}$$

   Player 1 will try to choose $x \neq 0$ as close as possible to 0:

   - We may choose x small enough, such that $\frac{V}{2} < V - c_1 x$, so $x = 0$ can not be a best response;
   - For any $x > 0$, we will have $V - c_1 x < V - c_1 \frac{x}{2}$, so x can not be a best response.

   Hence, the strategy profiles $(x, 0)$ and $(0, y)$ are not Nash equilibria. Therefore, we will assume that $x, y > 0$.

2. Given Player 2's strategy $y > 0$, Player 1's best response $x^*(y)$ should satisfy $\frac{\partial \pi_1}{\partial x}(x) = 0$ and $\frac{\partial^2 \pi_1}{\partial x^2}(x) \leq 0$, which implies

$$\frac{Vy}{(x^*(y) + y)^2} - c_1 = 0.$$

   That is

$$\frac{y}{c_1} = \frac{(x^*(y) + y)^2}{V}. \tag{1}$$

   Similarly, given Player 1's strategy $x > 0$, we will get that Player 2's best response $y^*(x)$ satisfies

$$\frac{x}{c_2} = \frac{(x + y^*(x))^2}{V}. \tag{2}$$

   Let (x^*, y^*) be a Nash equilibrium, that is, x^* and y^* are best responses of each other, and hence (x^*, y^*) should satisfy Equations (1) and (2). From Equations (1) and (2), we will have

$$\frac{y^*}{c_1} = \frac{(x^* + y^*)^2}{V} = \frac{x^*}{c_2}.$$

Substitute this equation into Equations (1) and (2), we will obtain that

$$x^* = \frac{Vc_2}{(c_1+c_2)^2}, \quad y^* = \frac{Vc_1}{(c_1+c_2)^2}.$$

Notice that x^*, y^* are both positive, so they could be the solution of this problem. Hence (x^*, y^*) is the only Nash equilibrium.

Meanwhile, the equilibrium profits to the two firms are

$$\pi_1(x^*, y^*) = \frac{Vc_2^2}{(c_1+c_2)^2}, \quad \pi_2(x^*, y^*) = \frac{Vc_1^2}{(c_1+c_2)^2}.$$

□

Exercise 7. *A two-person game is called a zero-sum game (also called a matrix game) if* $u_1(s_1, s_2) + u_2(s_1, s_2) = 0$ *for all* $s_1 \in S_1$ *and* $s_2 \in S_2$. *Show that* (s_1^*, s_2^*) *is a pure-strategy Nash equilibrium of a two-person zero-sum game if and only if*

$$u_1(s_1, s_2^*) \le u_1(s_1^*, s_2^*) \le u_1(s_1^*, s_2), \quad \forall s_1 \in S_1,\ s_2 \in S_2.$$

Consider a two-person zero-sum game in strategic form with finitely many strategies for each player (not just two), and assume that player I has two particular pure strategies T *and* B *and that player II has two pure strategies* l *and* r *so that both* (T, l) *and* (B, r) *are Nash equilibria of the game. Show that there are at least two further pure-strategy Nash equilibria.*

Prove that, for each player, the payoffs for the given equilibria are equal.

Proof. 1.

"⇒": Assume that (s_1^*, s_2^*) is a Nash equilibrium, by definition, for Player 1, we have

$$u_1(s_1, s_2^*) \le u_1(s_1^*, s_2^*), \quad \forall s_1 \in S_1. \tag{3}$$

Similarly, for Player 2, we have

$$u_2(s_1^*, s_2^*) \ge u_2(s_1^*, s_2), \quad \forall s_2 \in S_2. \tag{4}$$

Notice $u_1(\cdot,\cdot) + u_2(\cdot,\cdot) = 0$, by Equation (4), we have

$$u_1(s_1^*, s_2^*) \le u_1(s_1^*, s_2), \quad \forall s_2 \in S_2. \tag{5}$$

Combining Equations (3) and (5), we have the following characterization of a Nash Equilibrium,

$$u_1(s_1, s_2^*) \le u_1(s_1^*, s_2^*) \le u_1(s_1^*, s_2), \quad \forall s_1 \in S_1, s_2 \in S_2. \tag{6}$$

"⇐": Assume

$$u_1(s_1, s_2^*) \le u_1(s_1^*, s_2^*) \le u_1(s_1^*, s_2), \quad \forall s_1 \in S_1, s_2 \in S_2.$$

The 2nd part implies

$$u_2(s_1^*, s_2^*) \ge u_2(s_1^*, s_2), \quad \forall s_2 \in S_2,$$

since $u_1(\cdot,\cdot) + u_2(\cdot,\cdot) = 0$.

Combining the 1st part, we have (s_1^*, s_2^*) is a Nash equilibrium.

2. (Remark for 1st sub-question) Zero-sum games are special in that payoff in each cell of payoff table sums to zero. This allows us to simplify the payoff table by giving only the payoff of Player 1. For example, We can represent a zero-sum game as the following:

	L	R
U	$2, -2$	$3, -3$
D	$-1, 1$	$2, -2$

$\Longrightarrow$

	L	R
U	2	3
D	-1	2

So if Player 1 plays U and Player 2 plays L, Player 1's payoff is 2 and player 2's payoff is -2.

The 1st sub-question gives us a nice property of zero-sum games: Entry (i, j) is a NE with payoff p for Player 1, iff p is the maximum on jth column and the minimum on the ith row. Here is a simple example:

	s_2^*	s_2	s_2'
s_1^*	a	b	c
s_1	d	$*$	$*$
s_1'	e	$*$	$*$

In this game, (s_1^*, s_2^*) is a NE, iff $d, e \leq a \leq b, c$.

3. (1st method for 2nd sub-question) Now we will apply the Remark here: Given the payoff table for a zero-sum game with NE (T, l) and (B, r):

	l	r	$\cdots$
T	a	b	$*$
B	c	d	$*$
$\vdots$	$*$	$*$	$*$

Since (T, l) is a NE, we have $c \leq a \leq b$. Since (B, r) is a NE, we have $b \leq d \leq c$. Therefore $a = b = c = d$.

Since a is the maximum on 1th column and the minimum on the 1th row, we have that b is the minimum on the 1th row, and c is the maximum on 1th column.

Since d is the maximum on 2th column and the minimum on the 2th row, we have that c is the minimum on the 2th row, and b is the maximum on 2th column.

Applying the Remark again, we will have (T, r) and (B, l) are NE.

4. (2nd method for 2nd sub-question) Suppose, $T, B \in S_1$, $l, r \in S_2$, and both (T, l) and (B, r) are Nash equilibria, then by Equation (6), we have,

$$u_1(s_1, l) \leq u_1(T, l) \leq u_1(T, s_2), \quad \forall s_1 \in S_1, s_2 \in S_2; \tag{7}$$

$$u_1(s_1, r) \leq u_1(B, r) \leq u_1(B, s_2), \quad \forall s_1 \in S_1, s_2 \in S_2. \tag{8}$$

Take $s_1 = B, s_2 = r$ in (7) and $s_1 = T, s_2 = l$ (8), combine (7) and (8) together, we have

$$u_1(B,l) \leq u_1(T,l) \leq u_1(T,r) \leq u_1(B,r) \leq u_1(B,l). \tag{9}$$

Using the 2nd part of (7) and the 1st of (8), notice $u_1(T,l) = u_1(B,r) = u_1(T,r)$, we have

$$u_1(s_1,r) \leq u_1(T,r) \leq u_1(T,s_2), \quad \forall s_1 \in S_1, s_2 \in S_2. \tag{10}$$

Therefore, by part 1, (T,r) is also a Nash equilibrium. Similarly, (B,l) is a Nash equilibrium, too.

5. Since $u_1(T,l) = u_1(B,r) = u_1(B,l) = u_1(T,r)$, the all Nash equilibria yield the same payoff for Player 1, and also same for Player 2.

□