Solution to Tutorial 2*

2011/2012 Semester I MA4264 Game Theory

Tutor: Xiang Sun†

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1 Review

- The pure-strategy Nash equilibrium may not exist (e.g. matching pennies); However, the mixed-strategy Nash equilibrium always exists. Nash's Theorem: In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, if S_i is finite for every i, then there exists at least one Nash equilibrium, possibly involving mixed strategies.

- In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, suppose $S_i = \{s_{i1}, \ldots, s_{iK_i}\}$. Then each strategy $s_{ik} \in S_i$ is called a Player i's **pure strategy**. A Player i's **mixed strategy** is a probability distribution $p_i = (p_{i1}, \ldots, p_{iK_i})$, where $p_{i1} + \cdots + p_{iK_i} = 1$ and $0 \leq p_{ik} \leq 1$.

- In the 2-player normal-form game $G = \{S_1, S_2; u_1, u_2\}$, suppose $S_1 = \{s_{11}, \ldots, s_{1J}\}$, and $S_2 = \{s_{21}, \ldots, s_{2K}\}$. If Player 1 believes that Player 2 will play the strategies $(s_{21}, \ldots, s_{2K})$ with the probabilities $p_2 = (p_{21}, \ldots, p_{2K})$, then Player 1's **expected payoff** from playing the mixed strategy $p_1 = (p_{11}, \ldots, p_{1J})$ is

$$U_1(p_1, p_2) = \sum_{j=1}^{J} p_{1j} U_1(s_{1j}, p_2) = \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} p_{2k} u_1(s_{1j}, s_{2k}).$$

  Here we assume that Players 1 and 2 are independent.

  Similarly, if Player 2 believes that Player 1 will play the strategies $(s_{11}, \ldots, s_{1J})$ with the probabilities $p_1 = (p_{11}, \ldots, p_{1J})$, then Player 2's **expected payoff** from playing the mixed strategy $p_2 = (p_{21}, \ldots, p_{2K})$ is

$$U_2(p_1, p_2) = \sum_{k=1}^{K} p_{2k} U_2(p_1, s_{2k}) = \sum_{k=1}^{K} \sum_{j=1}^{J} p_{2k} p_{1j} u_2(s_{1j}, s_{2k}).$$

*Corrections are always welcome.

†Email: xiangsun@nus.edu.sg; Mobile: 9169 7677; Office: S17-06-14.

- In the 2-player normal-form game $G = \{S_1, S_2; u_1, u_2\}$, let $p_1 = (r, 1-r)$ and $p_2 = (q, 1-q)$ be the Players 1 and 2'smixed strategies. Given p_2,

$$r^*(q) \equiv \arg\max_{0 \le r \le 1} U_1(p_1, p_2) \subset [0, 1]$$ [1]

is called Player 1's **best response**. The function $r^*(\cdot)$ is called the best-response correspondence.

- In the 2-player normal-form game $G = \{S_1, S_2; u_1, u_2\}$, the mixed strategy profile (p_1^*, p_2^*) is a **mixed-strategy Nash equilibrium** if each player's mixed strategy is the best response to the other player's mixed strategies:

$$U_1(p_1^*, p_2^*) \ge U_1(p_1, p_2^*), \qquad U_2(p_1^*, p_2^*) \ge U_2(p_1^*, p_2)$$

for all probability distributions p_1 and p_2, on S_1 and S_2, respectively.

2 How to find the Nash equilibrium(a)?

2.1 Pure-strategy Nash equilibrium(a)

- There are 2/3 players, and for each player, the strategy set is **finite**. Then we will represent the game as a bi-matrix or tri-matrix, apply IESDS, underline the best responses for each player, and find the cell in which both/all numbers are underlined. For example, the prisoners' dilemma.

- There are 2 players, for Player 1, the strategy set is **finite**, and for Player 2, the strategy set is **infinite**. Then we will fix Player 1's strategy s_{1j}, find Player 2's best response $R_2^*(s_{1j})$, and then check whether the fixed strategy s_{1j} is a best response for some strategy in $R_2^*(s_{1j})$. For example, Exercise 3 in Tutorial 2.

- There are 2 players, for each player, the strategy set is **infinite**. Then we will find the best response correspondence for each player.
  - If there is a player whose best response correspondence is a **function by cases**, then we will draw the graphs of both best response correspondences, and find the intersection points which give us the NE. For example, Exercise 2 in Tutorial 2.
  - Otherwise, we assume (s_1^*, s_2^*) is a NE, substitute into the Equations derived from the definition of NE and the best response correspondences, and resolve them which will give us the NE. In this subcase, there could be more than 2 players. For example, Exercise 1 in Tutorial 2.

[1] $\arg\max_{0 \le r \le 1} U_1(p_1, p_2) = \{r : 0 \le r \le 1, r \text{ is a maximizer of } U_1(p_1, p_2)\}$

2.2 Mixed-strategy Nash equilibrium(a)

There are 2 players, and for each player, the strategy set is finite. Then we will represent the game as a bi-matrix or tri-matrix, apply IESDS to find reduced game, and find the best response correspondence for each player.

- If there is a player whose best response correspondence is a **function by cases**, then we will draw the graphs of both best response correspondences, and find the intersection points which give us the NE. For example, Exercises 5, 6, and 7 in Tutorial 2.
- Otherwise, we assume $((r^*, 1-r^*), (q^*, 1-q^*))$ is a NE, substitute into the Equations derived from the definition of NE and the best response correspondences, and resolve them which will give us the NE.

3 Tutorial

Exercise 1. *Suppose there are n firms in the Cournot oligopoly model. Let q_i denote the quantity produced by firm i, and let $Q = q_1 + \cdots + q_n$ denote the aggregate quantity on the market. Let P denote the market-clearing price and assume that inverse demand is given by $P(Q) = a - Q$ (assuming $Q < a$, else $P = 0$). Assume that the total cost of firm i from producing quantity q_i is $C_i(q_i) = cq_i$. That is, there are no fixed costs and the marginal cost is constant at c, where we assume $c < a$. Following Cournot, suppose that the firms choose their quantities simultaneously. What is the Nash equilibrium? What happens as n approaches infinity?*

Solution. We assume $c > 0$.

- Set of players: $\{1, 2, \ldots, n\}$;
- For each i, Player i's strategy set: $S_i = [0, +\infty)$;
- For each i, Player i's payoff function:

$$\begin{aligned}\pi_i(q_i, q_{-i}) &= q_i(\max\{a - q_i - q_{-i}, 0\} - c) \\ &= \begin{cases} (a - q_i - q_{-i} - c)q_i, & \text{if } q_i + q_{-i} < a; \\ -cq_i, & \text{if } q_i + q_{-i} \geq a, \end{cases}\end{aligned}$$

  where $q_{-i} = \sum_{j \neq i} q_j$.

In the following, given q_{-i}, we try to find Player i's best response:

(i) When $a \leq q_{-i}$, then we have $q_i + q_{-i} \geq a$, and hence

$$\pi_i(q_i, q_{-i}) = -cq_i \begin{cases} < 0, & \text{if } q_i > 0; \\ = 0, & \text{if } q_i = 0. \end{cases}$$

Therefore, in this case, the best response for Player i is $q_i = 0$.

(ii) When $a - c \le q_{-i} < a$, then we have

$$\pi_i(q_i, q_{-i}) = \begin{cases} 0, & \text{if } q_i = 0; \\ (a - q_i - q_{-i} - c)q_i < 0, & \text{if } 0 < q_i \le a - q_{-i}; \\ -cq_{-i} < 0, & \text{if } q_i \ge a - q_{-i}. \end{cases}$$

Therefore, in this case, the best response for Player i is $q_i = 0$.

(iii) When $0 \le q_{-i} < a - c$, then we have

$$\pi_i(q_i, q_{-i}) = \begin{cases} 0, & \text{if } q_i = 0; \\ (a - q_i - q_{-i} - c)q_i > 0, & \text{if } 0 < q_i \le a - q_{-i}; \\ -cq_{-i} < 0, & \text{if } a - q_{-i} < q_i. \end{cases}$$

The function $(a - q_i - q_{-i} - c)q_i$ is concave for q_i, because its 2nd derivative is $-2 < 0$. The local maximum can be determined by the first order condition (the 1st derivative equals zero) $a - q_{-i} - c - 2q_i = 0$, thus the best response for Player i is $\frac{a-c-q_{-i}}{2}$.

Therefore Player i's best response is

$$R_i^*(q_{-i}) = \begin{cases} \{0\}, & \text{if } a - c \le q_{-i}; \\ \{\frac{a-c-q_{-i}}{2}\}, & \text{if } 0 \le q_{-i} < a - c. \end{cases}$$

Remark: We can not draw graphs to find Nash equilibrium(a), since there are more than 2 players.

Claim: There does not exist a NE in which some players choose 0. We will prove this claim by contradiction:

1. Assume there is a NE $(q_1^*, q_2^*, \dots, q_n^*)$, where

$$J \equiv \{i \colon q_i^* = 0\} \ne \emptyset.$$

   Let $J^c = \{1, 2, \dots, n\} - J$, then for any $j \in J^c$, $q_j^* = \frac{a-c-q_{-j}^*}{2}$.

2. Since for any $i \in J$, $q_i^* = 0$, we will have $q_{-i}^* \ge a - c$, which implies $\sum_{j \in J^c} q_j^* \ge a - c$.

3. Since for any $i \in J$, $q_i^* = 0$, we will have

$$q_{-j}^* = \sum_{k \in J^c, k \ne j} q_k^*,$$

   for each $j \in J^c$, and hence

$$q_j^* = \frac{a - c - \sum_{k \in J^c, k \ne j} q_k^*}{2}, \quad \forall j \in J^c.$$

Summing this $|J^c|$ equations, we will have

$$\sum_{j\in J^c} q_j^* = \frac{a-c}{2}|J^c| - \frac{1}{2}(|J^c|-1)\sum_{j\in J^c} q_j^*,$$

which implies

$$\sum_{j\in J^c} q_j^* = \frac{|J^c|}{|J^c|+1}(a-c) < a-c.$$

Contradiction.

Assume that $(q_1^*, q_2^*, \ldots, q_n^*)$ is a pure-strategy Nash equilibrium, then based on the claim above, we will have $q_i^* = \frac{a-c-q_{-i}^*}{2}$, for all $i = 1, 2, \ldots, n$. Hence

$$q_i^* = a - c - Q^*, \quad \forall i = 1, 2, \ldots, n,$$

where $Q^* = \sum_{i=1}^n q_i^*$. Summing the n equations above, we obtain

$$Q^* = \frac{n}{n+1}(a-c).$$

Substituting this into each of the above n equations, we obtain

$$q_1^* = q_2^* = \cdots = q_n^* = \frac{a-c}{n+1}.$$

As n approaches infinity, the total output $Q^* = \frac{n}{n+1}(a-c)$ approaches $a-c$ (perfect-competition output) and the price $a - Q^* = \frac{a+nc}{n+1}$ approaches c (the perfect-competition price). □

Exercise 2. *Consider the Cournot duopoly model where inverse demand is $P(Q) = a - Q$ but firms have asymmetric marginal costs: c_1 for firm 1 and c_2 for firm 2. What is the Nash equilibrium if $0 < c_i < a/2$ for each firm? What if $c_1 < c_2 < a$ but $2c_2 > a + c_1$?*

Solution.

- Set of players: $\{1, 2\}$;
- For each i, Player i's strategy set: $S_i = [0, +\infty)$;
- For each i, Player i's payoff function:

$$\pi_i(q_i, q_j) = q_i(\max\{a - q_i - q_j, 0\} - c_i),$$

where $i \neq j$.

By similar method of Exercise 1, we will obtain Player i's best response:

$$R_i^*(q_j) = \begin{cases} \{\frac{a-c_i-q_j}{2}\}, & \text{if } q_j \le a - c_i; \\ \{0\}, & \text{if } q_j > a - c_i. \end{cases}$$

1. If $0 < c_1, c_2 < \frac{a}{2}$, then $\frac{a-c_i}{2} < \frac{a}{2} < a - c_j$, where $i \neq j$. Hence we have the Figure (1a), and from it we will obtain the Nash equilibrium: $(\frac{a-2c_1+c_2}{3}, \frac{a-2c_2+c_1}{3})$.

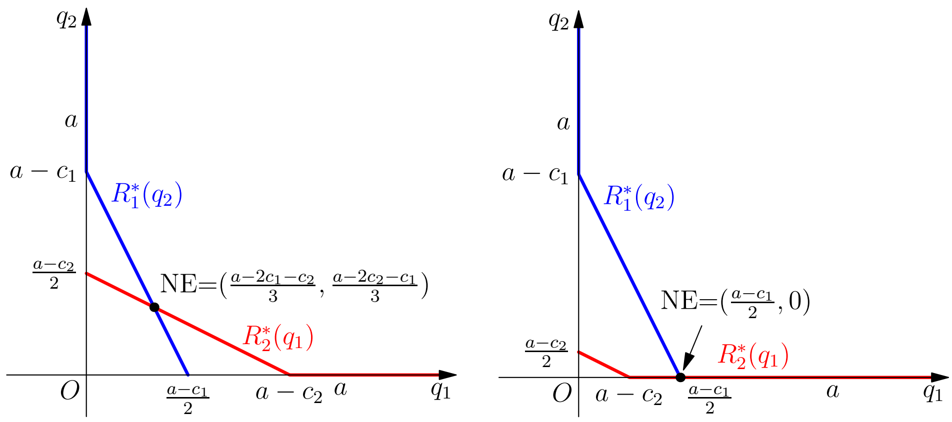

Figure 1: Intersection of best-response correspondences

2. If $0 < c_1 < c_2 < a$ and $2c_2 > a + c_1$, then $a - c_1 > a - c_2 > \frac{a-c_2}{2} > 0$ and $\frac{a-c_1}{2} > a - c_2 > 0$. Hence we have the Figure (1b), and from it we will obtain the Nash equilibrium: $(\frac{a-c_1}{2}, 0)$.

□

Exercise 3. *Consider a market of duopoly. The two firms produce the same product. Let q_i be the quantity of the product produced by firm i, $i = 1, 2$. Let the market price be*

$$P(q_1, q_2) = \begin{cases} 25 - q_1 - q_2, & \text{if } q_1 + q_2 < 25; \\ 0, & \text{if } q_1 + q_2 \geq 25. \end{cases}$$

Let the cost of producing a unit of the product be $c_1 = 6$ for firm 1 and $c_2 = 5$ for firm 2. Due to the restriction of technology, firm 1 can produce either $q_1 = 5$ or $q_1 = 10$. Firm 2 can produce any quantity $q_2 \geq 0$. Firm i's payoff is its profit $q_i(P(q_1, q_2) - c_i)$.

Find the Nash equilibrium of the game.

Solution.

- Set of players: $\{1, 2\}$;
- Player 1 and Player 2's strategy sets are $\{0, 5, 10\}$ and $[0, +\infty)$, respectively;
- Player i's payoff function is

$$\pi_i(q_i, q_j) = q_i(\max\{25 - q_i - q_j, 0\} - c_i),$$

where $i \neq j$.

It is easier to analyze Player 2's best-response first, since Player 1 has only 3 pure strategies.

1. When $q_1 = 0$, Player 2's payoff function is

$$\pi_2(q_2) = q_2(\max\{25 - 0 - q_2, 0\} - 5).$$

When $q_2 > 20$, $\pi_2(q_2) < 0$; when $q_2 \leq 20$, $\pi_2(q_2) \geq 0$. Hence the local maximum should solve the optimization problem

$$\max_{0 \leq q_2 \leq 20} q_2(25 - q_2 - 5).$$

Therefore $R_2^*(0) = \{10\}$.

Now if suffices to check whether 0 is a Player 1's best response to 10: Given $q_2 = 10$, Player 1's payoff function is

$$\pi_1(q_1) = \begin{cases} 0(25 - 0 - 10 - 6) = 0, & \text{if } q_1 = 0; \\ 5(25 - 5 - 10 - 6) = 20, & \text{if } q_1 = 5; \\ 10(25 - 10 - 10 - 6) = -10, & \text{if } q_1 = 10. \end{cases}$$

Therefore Player 1's best response is $R_1^*(10) = \{5\}$, and hence there is no Nash equilibrium in which Player 1's strategy is 0.

2. When $q_1 = 5$, Player 2's payoff function is

$$\pi_2(q_2) = q_2(\max\{25 - 5 - q_2, 0\} - 5).$$

When $q_2 > 15$, $\pi_2(q_2) < 0$; when $q_2 \leq 15$, $\pi_2(q_2) \geq 0$. Hence the local maximum should solve the optimization problem

$$\max_{0 \leq q_2 \leq 15} q_2(20 - q_2 - 5).$$

Therefore $R_2^*(5) = \{\frac{15}{2}\}$.

Now if suffices to check whether 5 is a Player 1's best response to $\frac{15}{2}$: Given $q_2 = \frac{15}{2}$, Player 1's payoff function is

$$\pi_1(q_1) = \begin{cases} 0(25 - 0 - \frac{15}{2} - 6) = 0, & \text{if } q_1 = 0; \\ 5(25 - 5 - \frac{15}{2} - 6) = 32.5, & \text{if } q_1 = 5; \\ 10(25 - 10 - \frac{15}{2} - 6) = 15, & \text{if } q_1 = 10. \end{cases}$$

Therefore the best response for Player 1 is $R_1^*(\frac{15}{2}) = \{5\}$, and hence $(5, \frac{15}{2})$ is a Nash equilibrium.

3. When $q_1 = 10$, Player 2's payoff function is

$$\pi_2(q_2) = q_2(\max\{25 - 10 - q_2, 0\} - 5).$$

When $q_2 > 10$, $\pi_2(q_2) < 0$; when $q_2 \leq 10$, $\pi_2(q_2) \geq 0$. Hence the local maximum should solve the optimization problem

$$\max_{0 \leq q_2 \leq 10} q_2(15 - q_2 - 5).$$

Therefore $R_2^*(10) = \{5\}$.

Now if suffices to check whether 10 is a Player 1's best response to 5: Given $q_2 = 5$, then Player 1' payoff function is

$$\pi_1(q_1) = \begin{cases} 0(25-0-5-6) = 0, & \text{if } q_1 = 0; \\ 5(25-5-5-6) = 45, & \text{if } q_1 = 5; \\ 10(25-10-5-6) = 40, & \text{if } q_1 = 10. \end{cases}$$

Therefore the best response for Player 1 is $R_1^*(5) = \{5\}$, and hence there is no Nash equilibrium when Player 1's strategy is 10.

Therefore there is only one Nash equilibrium: $(5, \frac{15}{2})$. □

Exercise 4. *Prove the following statement for a two-player game. If a strategy $s_{kj} \in S_k (k = 1, 2)$ is played with nonzero probability in a mixed-strategy Nash equilibrium, then s_{kj} cannot be eliminated in the iterated elimination of strictly dominated strategies. (Similar to Proposition 1.1.)*

Proof. Let $S_k = \{s_{k1}, s_{k2}, \ldots, s_{kn_k}\}$, $k = 1, 2$. Assume that (p_1^*, p_2^*) is a mixed-strategy Nash equilibrium, where $p_k^* = (p_{k1}^*, p_{k2}^*, \ldots, p_{kn_k}^*)$ is Player k's mixed strategy and p_{kj}^* is the probability that Player k plays s_{kj}.

Assume that s_{kj} is the **first** of the strategies played with positive probability to be eliminated for being strictly dominated. Then there should exist a strategy s_{kl} that has not yet been eliminated from S_k that strictly dominates s_{kj}. By definition, we have

$$u_k(s_{kj}, s_{-kt}) < u_k(s_{kl}, s_{kt}),$$

for each s_{kt} have not yet been eliminated from the other Player's strategy set.

Since s_{kj} is the first of the strategies played with positive probability to be eliminated for being strictly dominated, we have

$$u_k(s_{kj}, p_{-k}^*) < u_k(s_{kl}, p_{-k}^*).$$

Now we will construct another mixed strategy p_k^{**} for Player k:

$$\begin{cases} p_{kj}^{**} = 0 \\ p_{kl}^{**} = p_{kl}^* + p_{kj}^* \\ p_{ki}^{**} = p_{ki}^*, & i \neq j, l. \end{cases}$$

Since $u_k(s_{kj}, p_{-k}^*) < u_k(s_{kl}, p_{-k}^*)$, we have

$$u_k(p_k^*, p_{-k}^*) < u_k(p_k^{**}, p_{-k}^*),$$

which contracts that (p_k^*, p_{-k}^*) is a mixed-strategy Nash equilibrium.

Hence, s_{kj} will not be eliminated in the iterated elimination of strictly dominated strategies. □

Exercise 5. *Find the mixed-strategy Nash equilibrium of the following normal-form games.*

	L	R
T	2, 1	0, 2
B	1, 2	3, 0

	L	C	R
T	2, 0	1, 1	4, 2
M	3, 4	1, 2	2, 3
B	1, 3	0, 2	3, 0

Solution. 1. In the left game, it is trivial that there is no pure-strategy Nash equilibrium. Let $p_1 = (r, 1-r)$ be a mixed-strategy in which Player 1 plays T with probability r. Let $p_2 = (q, 1-q)$ be a mixed-strategy in which Player 2 plays L with probability q. Then Player 1's expected payoff is:

$$U_1(T, p_2) = 2q,$$
$$U_1(B, p_2) = q + 3(1-q) = 3 - 2q.$$

Hence

$$r^*(q) \equiv \arg\max_{0 \le r \le 1} U_1(p_1, p_2) = \begin{cases} \{1\}, & \text{if } q > \frac{3}{4}; \\ \{0\}, & \text{if } q < \frac{3}{4}; \\ [0,1], & \text{if } q = \frac{3}{4}. \end{cases}$$

Similarly, Player 2's expected payoff is:

$$U_2(p_1, L) = r + 2(1-r) = 2 - r,$$
$$U_2(p_1, R) = 2r.$$

Hence

$$q^*(r) \equiv \arg\max_{0 \le q \le 1} U_2(p_1, p_2) = \begin{cases} \{1\}, & \text{if } r < \frac{2}{3}; \\ \{0\}, & \text{if } r > \frac{2}{3}; \\ [0,1], & \text{if } r = \frac{2}{3}. \end{cases}$$

We draw the graphs of $r^*(q)$ and $q^*(r)$ together in Figure (2):

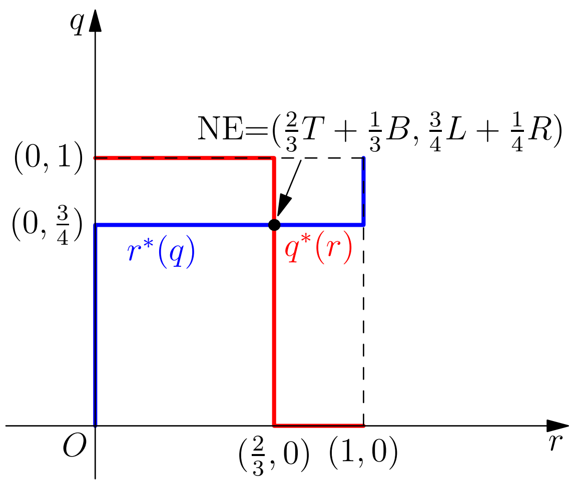

Figure 2: Intersection of best-response correspondences

The graphs of the best response correspondences $r^*(q)$ and $q^*(r)$ intersect at only one point $(r = \frac{2}{3}, q = \frac{3}{4})$. Hence, there is only one mixed-strategy Nash equilibrium $(\frac{2}{3}T + \frac{1}{3}B, \frac{3}{4}L + \frac{1}{4}R)$ (or $((\frac{2}{3}, \frac{1}{3}), (\frac{3}{4}, \frac{1}{4}))$).

2. We have no idea how to find the mixed-strategy Nash equilibrium for the game in which some player has more than 2 strategies. However, after iterated elimination of strictly dominated strategies, we will obtain the reduced game (3):

	L	R
T	$2, 0$	$4, 2$
M	$3, 4$	$2, 3$

Figure 3: Reduced game

It is trivial that there are two pure-strategy Nash equilibria: (M, L) and (T, R).

Let $p_1 = (r, 1-r)$ be a mixed strategy in which Player 1 plays T with probability r. Let $p_2 = (q, 1-q)$ be a mixed strategy in which Player 2 plays L with probability q. Then Player 1's expected payoff is:

$$U_1(T, p_2) = 2q + 4(1-q) = 4 - 2q,$$
$$U_1(B, p_2) = 3q + 2(1-q) = 2 + q.$$

Hence

$$r^*(q) \equiv \arg\max_{0 \le r \le 1} U_1(p_1, p_2) = \begin{cases} \{1\}, & \text{if } q < \frac{2}{3}; \\ \{0\}, & \text{if } q > \frac{2}{3}; \\ [0, 1], & \text{if } q = \frac{2}{3}. \end{cases}$$

Similarly, Player 2's expected payoff is:

$$U_2(p_1, L) = 4(1-r) = 4 - 4r,$$
$$U_2(p_1, R) = 2r + 3(1-r) = 3 - r.$$

Hence

$$q^*(r) \equiv \arg\max_{0 \le q \le 1} U_2(p_1, p_2) = \begin{cases} \{1\}, & \text{if } r < \frac{1}{3}; \\ \{0\}, & \text{if } r > \frac{1}{3}; \\ [0, 1], & \text{if } r = \frac{1}{3}. \end{cases}$$

We draw the graphs of $r^*(q)$ and $q^*(r)$ together in Figure (4):

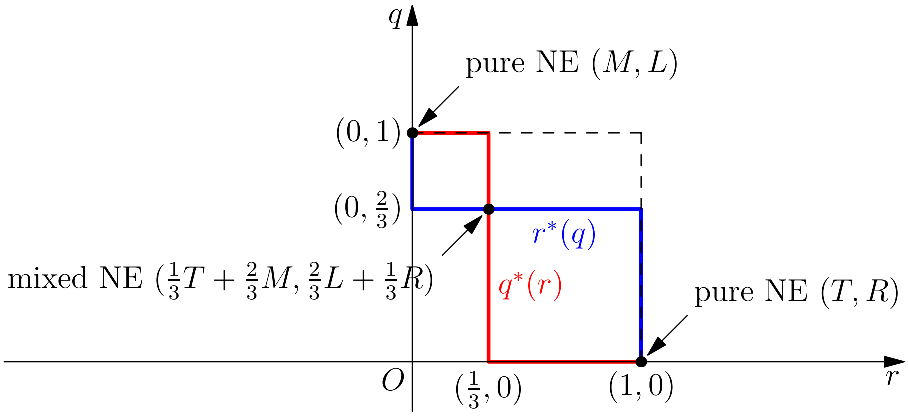

Figure 4: Intersection of best-response correspondences

The graphs of the best response correspondences $r^*(q)$ and $q^*(r)$ intersect at 3 points $(r = \frac{1}{3}, q = \frac{2}{3})$, $(r = 1, q = 0)$ and $(r = 0, q = 1)$. Hence, there are 3 mixed-strategy Nash equilibria

- $(1T, 1R)$ (or $((1, 0, 0), (0, 0, 1))$),
- $(1M, 1L)$ (or $((0, 1, 0), (1, 0, 0))$),
- $(\frac{1}{3}T + \frac{2}{3}M, \frac{2}{3}L + \frac{1}{3}R)$ (or $((\frac{1}{3}, \frac{2}{3}, 0), (\frac{2}{3}, 0, \frac{1}{3}))$).

□

Exercise 6. *Consider the following two-person game.*

		Player 2	
		X	Y
Player 1	A	$9, 9$	$0, 8$
	B	$8, 0$	$7, 7$

(i) Suppose that Player 1 thinks that Player 2 will play her strategy X with probability y and her strategy Y with probability $1 - y$. For what value of y will Player 1 be indifferent between his two strategies?

(ii) If y is less than this value what strategy will Player 1 prefer? If y is greater than that value?

(iii) Graph the best responses of Player 1 to Player 2's mixed strategy.

(iv) Repeat this analysis with the roles of the players reversed.

Solution. (i) Player 1's expected payoff is:

$$U_1(A, p_2) = 9y,$$
$$U_1(B, p_2) = 8y + 7(1 - y) = 7 + y.$$

Hence when $9y = 7 + y$, that is, $y = \frac{7}{8}$, Player 1 will be indifferent between his two strategies.

(ii) If $y > \frac{7}{8}$, then Player 1 prefers A, otherwise Player 1 prefers B.

(iii) Let $p_1 = (x, 1-x)$ be a mixed strategy in which Player 1 plays A with probability x. Then

$$x^*(y) \equiv \arg\max_{0 \le y \le 1} U_1(p_1, p_2) = \begin{cases} \{1\}, & \text{if } y > \frac{7}{8}; \\ \{0\}, & \text{if } y < \frac{7}{8}; \\ [0,1], & \text{if } y = \frac{7}{8}. \end{cases}$$

Then the blue line in the Figure (5) is the graph of the best responses of Player 1 to Player 2's mixed strategy $p_2 = (y, 1-y)$.

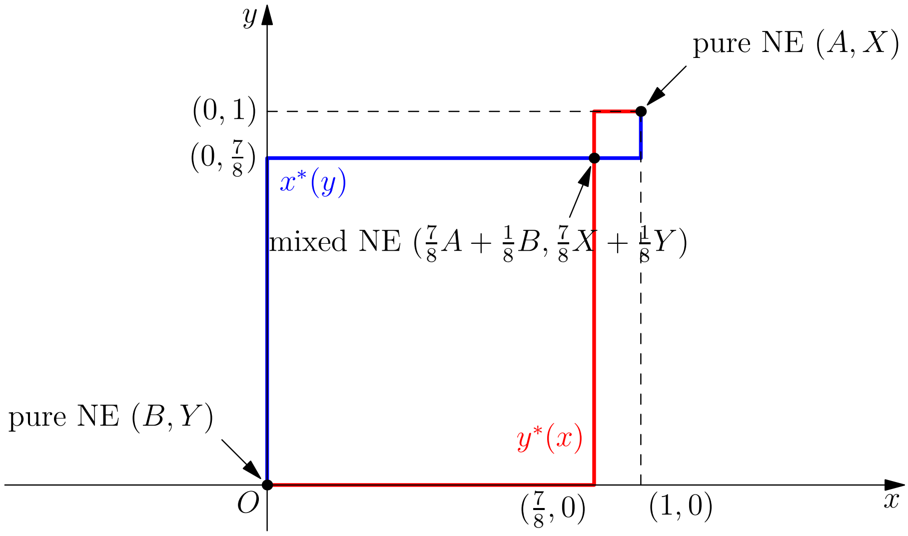

Figure 5: Intersection of best-response correspondences

(iv) By symmetry, we obtain that

$$y^*(x) \equiv \arg\max_{0 \le x \le 1} U_2(p_1, p_2) = \begin{cases} \{1\}, & \text{if } x > \frac{7}{8}; \\ \{0\}, & \text{if } x < \frac{7}{8}; \\ [0,1], & \text{if } x = \frac{7}{8}. \end{cases}$$

Hence, the red line in the Figure (5) is the graph of the best responses of Player 2 to Player 1's mixed strategy $p_1 = (x, 1-x)$.

□

Exercise 7. *Consider the following game:*

(i) Eliminate strictly dominated strategies.

(ii) Find all pure-strategy Nash equilibria and write down the corresponding payoffs.

(iii) Find all mixed-strategy Nash equilibria and write down the corresponding expected payoffs.

		Player 2		
		L	M	R
Player 1	A	$4,3$	$2,5$	$2,0$
	B	$6,2$	$0,3$	$1,4$
	C	$3,1$	$1,0$	$1,2$
	D	$3,0$	$1,1$	$3,3$

Solution. (i) (1) C is strictly dominated by A and will be eliminated;

(2) L is strictly dominated by M and will be eliminated;

(3) B is strictly dominated by D and will be eliminated.

Hence we will obtain the reduced game G_1.

		Player 2	
		M	R
Player 1	A	$2,5$	$2,0$
	D	$1,1$	$3,3$

G_1

		Player 2	
		M	R
Player 1	A	$2,5$	$2,0$
	D	$1,1$	$3,3$

G_2

(ii) From the bi-matrix G_2, we obtain the pure-strategy Nash equilibria: (A, M) and (D, R) (red pairs) with payoffs $(2,5)$ and $(3,3)$, respectively.

(iii) Let $p_1 = (r, 1-r)$ be a mixed strategy in which Player 1 plays A with probability r. Let $p_2 = (q, 1-q)$ be a mixed strategy in which Player 2 plays M with probability q. Then Player 1's expected payoff is:

$$U_1(A, p_2) = 2q + 2(1-q) = 2,$$
$$U_1(D, p_2) = q + 3(1-q) = 3 - 2q.$$

Hence

$$r^*(q) \equiv \arg\max_{0 \le r \le 1} U_1(p_1, p_2) = \begin{cases} \{1\}, & \text{if } q > \frac{1}{2}; \\ \{0\}, & \text{if } q < \frac{1}{2}; \\ [0,1], & \text{if } q = \frac{1}{2}. \end{cases}$$

Similarly, Player 2's expected payoff is:

$$U_2(p_1, M) = 5r + (1-r) = 1 + 4r,$$
$$U_2(p_1, R) = 3(1-r).$$

Hence

$$q^*(r) \equiv \arg\max_{0 \le q \le 1} U_2(p_1, p_2) = \begin{cases} \{1\}, & \text{if } r > \frac{2}{7}; \\ \{0\}, & \text{if } r < \frac{2}{7}; \\ [0,1], & \text{if } r = \frac{2}{7}. \end{cases}$$

We draw the graphs of $r^*(q)$ and $q^*(r)$ together:

The graphs of the best response correspondences $r^*(q)$ and $q^*(r)$ intersect at 3 points $(r = \frac{2}{7}, q = \frac{1}{2})$, $(0,0)$ and $(1,1)$. Hence, there are 3 mixed-strategy Nash equilibria:

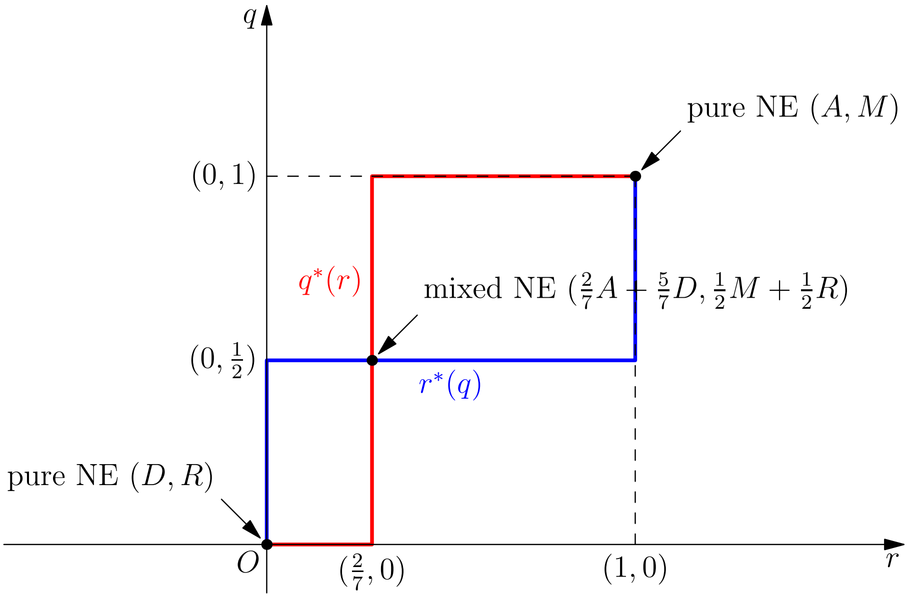

Figure 6: Intersection of best-response correspondences

- $(1A, 1M)$ (or $((1, 0, 0, 0), (0, 1, 0))$) with expected payoff $(2, 5)$,
- $(1D, 1R)$ (or $((0, 0, 0, 1), (0, 0, 1))$) with expected payoff $(3, 3)$,
- $(\frac{2}{7}A + \frac{5}{7}D, \frac{1}{2}M + \frac{1}{2}R)$ (or $((\frac{2}{7}, 0, 0, \frac{5}{7}), (0, \frac{1}{2}, \frac{1}{2}))$), with expected payoff $(2, \frac{15}{7})$.

□