Solution to Tutorial 3*

2011/2012 Semester I MA4264 Game Theory

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September 7, 2011

1 Review

- By **perfect information**, we mean that at each move in the game, the player with the move knows the full history of the play of the game thus far.
- The **extensive-form** representation of a game specifies
  - (1) the players in the game;
  - (2a) when each player has the move;
  - (2b) what each player can do at each of his or her opportunities to move;
  - (2c) what each player knows at each of his or her opportunities to move;
  - (3) the payoffs received by each player for each combination of moves that could be chosen by the players.
- An **information set** for a player is a collection of decision nodes satisfying:
  - (1) the player needs to move at every node in the information;
  - (2) when the play of the game reached a node in the information set, the player with the move does not know which node in the set has (or has not) been reached.
- A strategy for a player is a complete plan of actions. It specifies a feasible action for the player in every contingency in which the player might be called on to act.

  A player's **strategy** is a function which assigns an action to each information set (not each decision node) belonging to the player.
- A **subgame** in a extensive-form game
  - (1) begins at a decision node n that is a singleton information set (but is not the game's first decision node);

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[*] Corrections are always welcome.

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  (2) includes all the decision and terminal nodes following node n in the game tree (but no nodes that do not follow n);
  (3) does not cut any information set.

- Backwards induction will give us:
  (1) backwards-induction outcome: dynamic game with complete and perfect information;
  (2) subgame-perfect outcome: dynamic game with complete and imperfect information;
  (3) subgame-perfect Nash equilibrium (SPE): dynamic game with complete information (including both perfect and imperfect information).

2 Tutorial

Exercise 1. *Consider a population of voters uniformly distributed along the ideological spectrum from left (*$x = 0$*) to right (*$x = 1$*). Each of the candidates for a single office simultaneously chooses a campaign platform (i.e., a point on the line between* $x = 0$ *and* $x = 1$*). The voters observe the candidates' choices, and then each voter votes for the candidate whose platform is closest to the voter's position on the spectrum. If there are two candidates and they choose platforms* $x_1 = 0.3$ *and* $x_2 = 0.6$*, for example, then all voters to the left of* $x = 0.45$ *vote for candidate 1, all those to the right vote for candidate 2, and candidate 2 wins the election with 55 percent of the vote. Suppose that the candidates care only about being elected—they do not really care about their platforms at all!*

If there are two candidates, what is the pure-strategy Nash equilibrium.

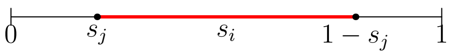
If there are three candidates, exhibit a pure-strategy Nash equilibrium.

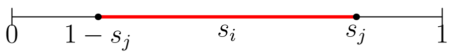
(Assume that any candidates who choose the same platform equally split the votes cast for that platform, and that ties among the leading vote-getters are resolved by coin flips.)

Solution. 1. 2-person game: for each player i, the strategy set is $S_i = [0, 1]$. Player i's payoff function:

$$\pi_i(s_i, s_j) = \begin{cases} 1, & \text{if } s_j < s_i < 1 - s_j, \text{ or } 1 - s_j < s_i < s_j; \\ \frac{1}{2}, & \text{if } s_i = s_j, \text{ or } s_i = 1 - s_j; \\ 0, & \text{otherwise.} \end{cases}$$

Given Player j's strategy $s_j \neq \frac{1}{2}$, from the following figures, we will see that Player i wins only when s_i is in the red regions.





Therefore, we have Player i's best response:

$$R_i^*(s_j) = \begin{cases} (s_j, 1-s_j), & \text{if } s_j < \frac{1}{2}; \\ \{\frac{1}{2}\}, & \text{if } s_j = \frac{1}{2}; \\ (1-s_j, s_j), & \text{if } s_j > \frac{1}{2}. \end{cases}$$

From Figure 1, there is only one Nash equilibrium $(\frac{1}{2}, \frac{1}{2})$.

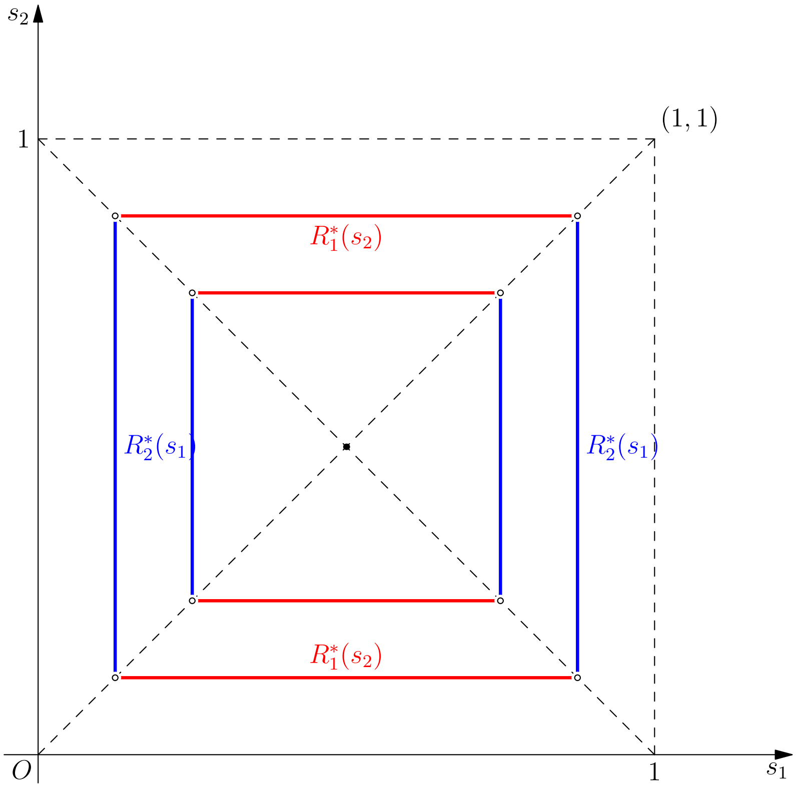


Figure 1: Intersection of the best-response correspondence

2. 3-person game: $(\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$ is a Nash equilibrium. To see this is a Nash equilibrium,

   - Player 3 has no incentive to deviate because he is the winner and obtains the maximal payoff;
   - Players 1 and 2 can not do better given the other two players choose $\frac{1}{3}$ and $\frac{2}{3}$, respectively.

   You may find other equilibria in this case.

□

Exercise 2. *Consider the following game. Player 1 has an infinite set of pure strategies, which is the interval* $[0, 1]$*. Player 2 has only two pure strategies,* L *and* R*. When Player 1 chooses his pure strategy* $x \in [0, 1]$*, payoff to Player 1 are*

$$u_1(x, L) = 2 - 2x, \quad u_1(x, R) = x;$$

and payoff to Player 2 are

$$u_2(x, L) = x, \quad u_2(x, R) = 1 - x.$$

(i) Show that the game has no Nash equilibrium in pure strategies.

(ii) Find all Nash equilibria of the game when Player 2 is allowed to use a mixed strategy. Explain carefully why they are Nash equilibria. Give the payoff to the two players.

Proof. (i) Given Player 2's strategy s_2, Player 1's best response is:

$$R_1^*(s_2) = \begin{cases} \{0\}, & \text{if } s_2 = L; \\ \{1\}, & \text{if } s_2 = R. \end{cases}$$

Given Player 1's strategy s_1, Player 2's best response is:

$$R_2^*(s_1) = \begin{cases} \{L\}, & \text{if } s_1 > \frac{1}{2}; \\ \{L, R\}, & \text{if } s_1 = \frac{1}{2}; \\ \{R\}, & \text{if } s_1 < \frac{1}{2}. \end{cases}$$

From Figure 2, there is no pure-strategy Nash equilibrium.

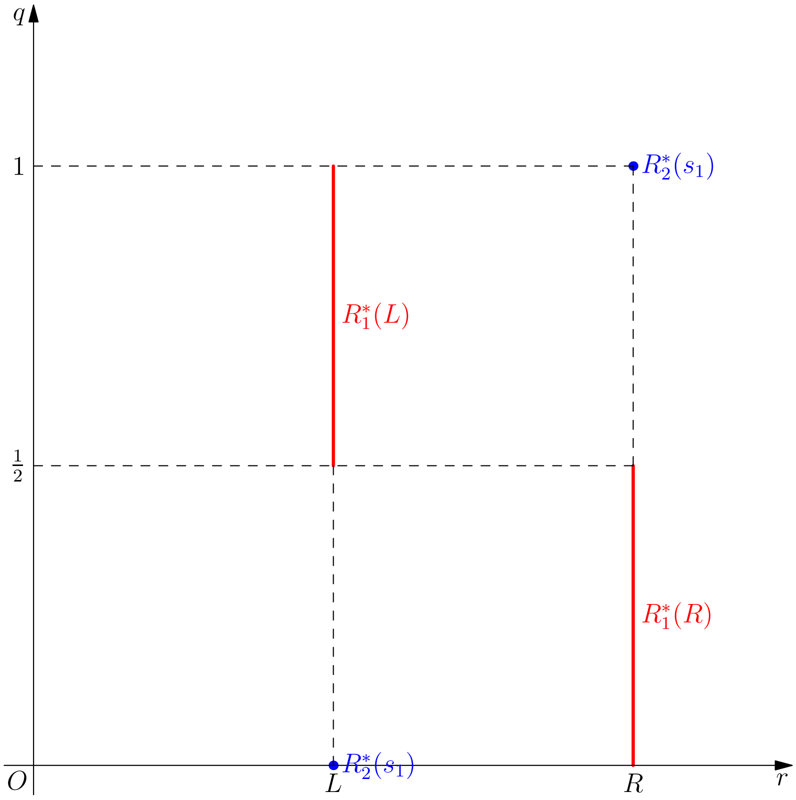


Figure 2: Intersection of best-response correspondences

(ii) Given Player 2's mixed strategy $(p, 1-p)$, Player 1's expected payoff is

$$U_1(s_1, pL + (1-p)R) = p(2-2s_1) + (1-p)s_1 = 2p + s_1(1-3p)$$

and hence Player 1's best response is

$$s_1^*(p) = \begin{cases} \{1\}, & \text{if } p < \frac{1}{3}; \\ [0,1], & \text{if } p = \frac{1}{3}; \\ \{0\}, & \text{if } p > \frac{1}{3}. \end{cases}$$

Given Player 1's strategy s_1, Player 2's expected payoff is

$$U_2(s_1, pL + (1-p)R) = ps_1 + (1-p)(1-s_1) = 1 - s_1 + (2s_1 - 1)p,$$

and hence Player 2's best response is

$$p^*(s_1) = \begin{cases} \{1\}, & \text{if } s_1 > \frac{1}{2}; \\ [0,1], & \text{if } s_1 = \frac{1}{2}; \\ \{0\}, & \text{if } s_1 < \frac{1}{2}. \end{cases}$$

From Figure 3, there is only one mixed-strategy Nash equilibrium $(\frac{1}{2}, \frac{1}{3}L + \frac{2}{3}R)$.

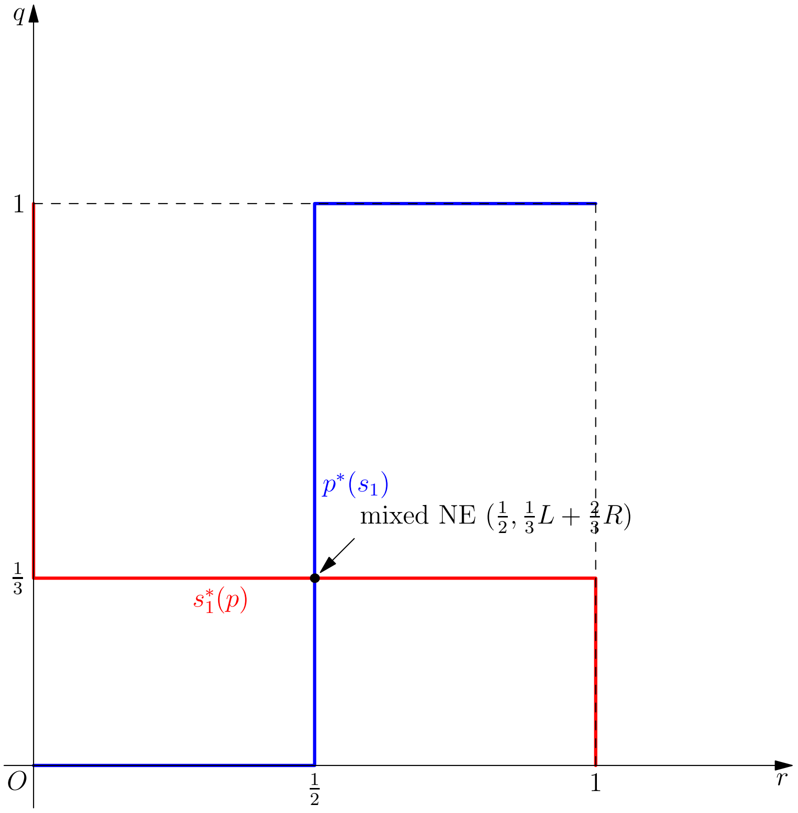


Figure 3: Intersection of best-response correspondences

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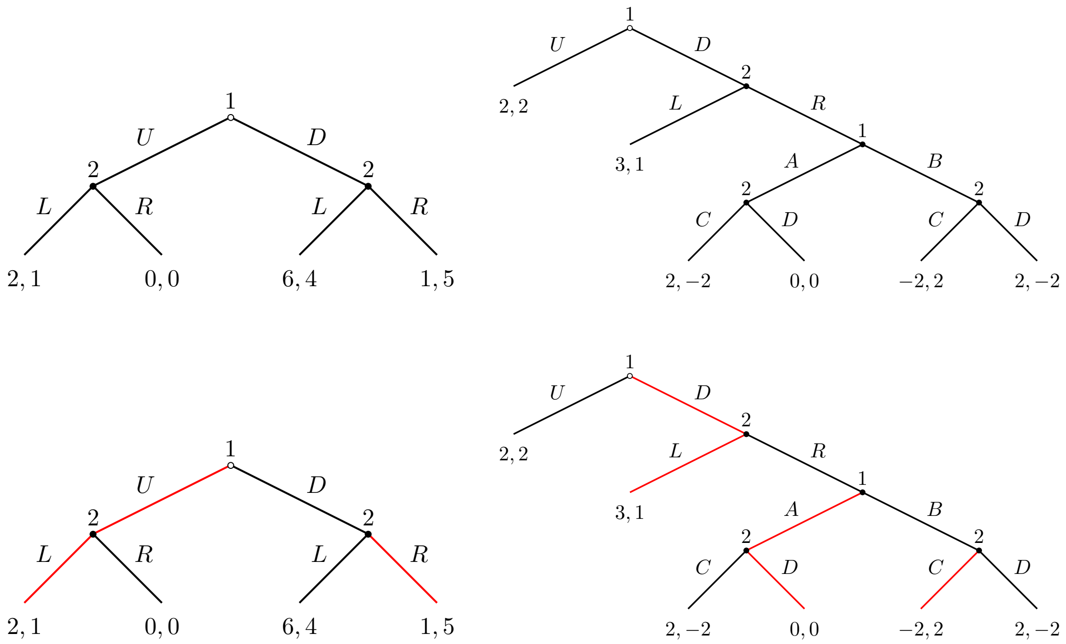

Figure 4: Backwards-induction outcomes

Exercise 3. *Find the backwards-induction outcome of the following games.*

Solution. From Figure 4a, the backwards-induction outcome for Game 1 is: Player 1 chooses U at the first stage, Player 2 chooses L at second stage.

From Figure 4b, the backwards-induction outcome for Game 2 is: Player 1 chooses D at the first stage, Player 2 chooses L at second stage, and the game ends. ☐

Exercise 4. *Denote by G the following game: Game 1:*

		Player 3	
		A	B
Player 2	A	$1,1$	$5,0$
	B	$0,5$	$4,4$

1. *Player 1 chooses p from the set $\{0, 2\}$.*
2. *Players 2 and 3 observe p and then choose actions in G.*
3. *For $i \in \{2, 3\}$, Player i's payoff is the payoff from G plus the amount paid by Player 1, which is p if Player i played B and 0 if he played A. Player 1's payoff is δ minus the total amount he paid to Players 2 and 3, where*

$$\delta = \begin{cases} 5, & \textit{if both players 2 and 3 played } B, \\ 0, & \textit{otherwise.} \end{cases}$$

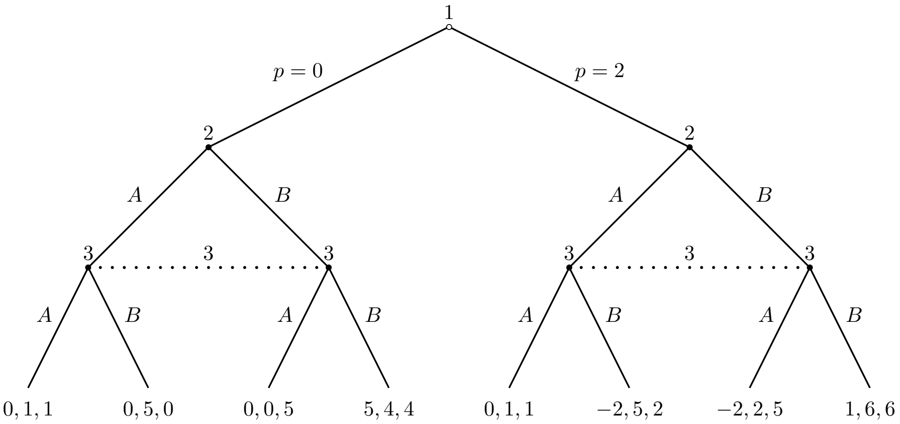

Figure 5

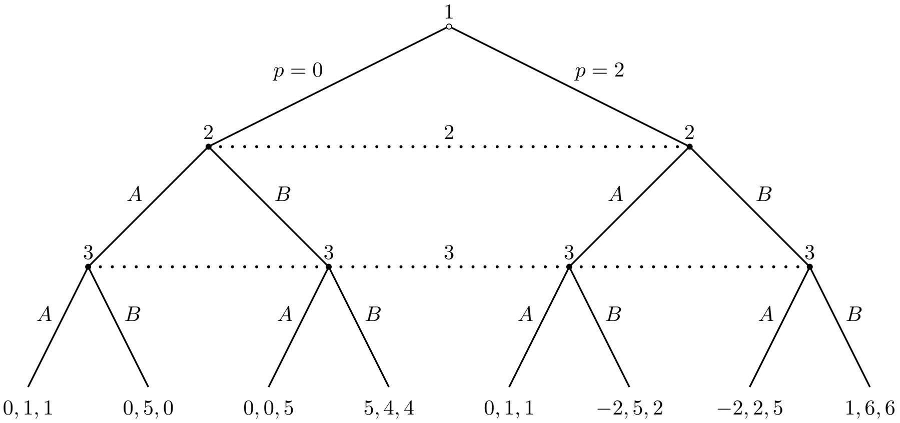

Figure 6

Game 2: Same as Game 1, except that in stage 2 players 2 and 3 do not observe player 1's choice p. (Hint: write the game in a tri-matrix.)

Find the subgame-perfect outcome for each game.

Solution. (i) Figure 5 is the extensive-form representation of Game 1, and there are 2 subgames.

- If Player 1 chooses $p = 0$, Players 2 and 3 will play Game G_1 in the second stage. In G_1, B is strictly dominated by A for both players, so there is only one Nash equilibrium (A, A). Accordingly, Player 1 will get $\delta = 0 - 0 - 0 = 0$.

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Player 3}} \\ & & A & B \\ \hline \text{Player 2} & A & 1,1 & 5,0 \\ \hline & B & 0,5 & 4,4 \\ \hline \end{array}$$

G_1

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Player 3}} \\ & & A & B \\ \hline \text{Player 2} & A & 1,1 & 5,2 \\ \hline & B & 2,5 & 6,6 \\ \hline \end{array}$$

G_2

- If Player 2 chooses $p = 2$, Players 2 and 3 will play Game G_2 in the second stage. In G_2, A is strictly dominated by B for both players, so there is only one Nash equilibrium (B, B). Accordingly, Player 1 will get $\delta = 5 - 2 - 2 = 1$.

Therefore, the subgame-perfect outcome is: Player 1 chooses $p = 2$ in the first stage, and Players 2 and 3 choose B in the second stage.

(ii) Figure 6 is the extensive-form representation of Game 2, and there is no subgame. From Game H, there is only Nash equilibrium $(p = 0, A, A)$, and the unique subgame-perfect outcome is: Player 1 chooses $p = 0$, and Players 2 and 3 choose A.

		Players 2 and 3			
		AA	AB	BA	BB
Player 1	$p = 0$	$0, 1, 1$	$0, 5, 0$	$0, 0, 5$	$5, 4, 4$
	$p = 2$	$0, 1, 1$	$-2, 5, 2$	$-2, 2, 5$	$1, 6, 6$

H

□

Exercise 5. *Now suppose games of Prisoners' Dilemma, Battle of Sexes and Matching Pennies are played sequentially: Player 1 chooses action first, player 2 observes player 1's action before choosing an action. Find the backward-induction outcome for each game.*

Solution. (i) The backward-induction outcome for Prisoners' Dilemma is: Player 1 chooses "Confess" in the first stage, and Player 2 chooses "Confess" in the second stage.

	Holdout	Confess
Holdout	$-1, -1$	$-20, 0$
Confess	$0, -20$	$-5, -5$

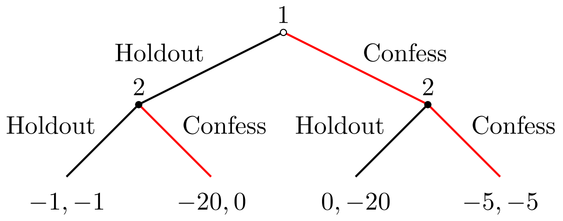

Figure 7: Prisoners' Dilemma

(ii) The backward-induction outcome for Battle of Sexes is: Player 1 chooses "Fight" in the first stage, and Player 2 chooses "Fight" in the second stage.

(iii) The backward-induction outcomes for Matching Pennies are: Player 1 chooses "Heads" in the first stage, Player 2 chooses "Head" in the second stage; and Player 1 chooses "Tails" in the first stage, Player 2 chooses "Tails" in the second stage.

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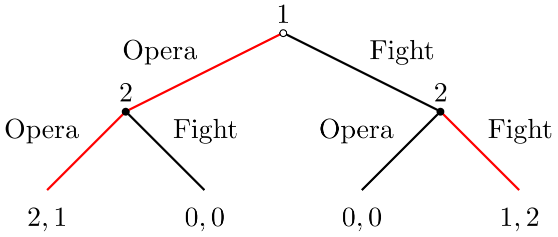
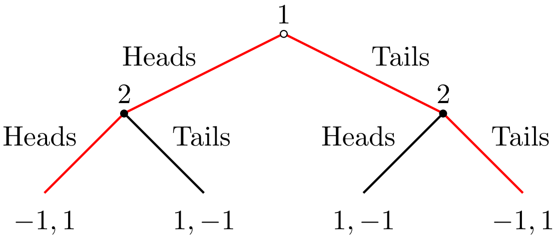


	Opera	Fight
Opera	$2, 1$	$0, 0$
Fight	$0, 0$	$1, 2$

Figure 8: Battle of Sexes



	Heads	Tails
Heads	$-1, 1$	$1, -1$
Tails	$1, -1$	$-1, 1$

Figure 9: Matching Pennies

Exercise 6. *Three oligopolists operate in a market with inverse demand given by* $P(Q) = a - Q$*, where* $Q = q_1 + q_2 + q_3$ *and* q_i *is the quantity produced by firm* i*. Each firm has a constant marginal cost of production,* c*, and no fixed cost. The firms choose their quantities as follows: (1) firm 1 chooses* $q_1 \geq 0$*; (2) firms 2 and 3 observe* q_1 *and then simultaneously choose* q_2 *and* q_3*, respectively. What is the subgame-perfect outcome?*

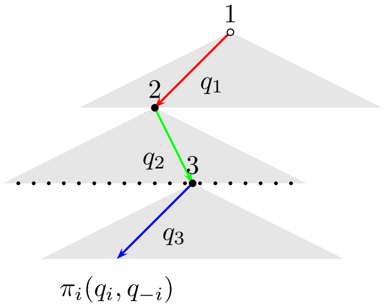


Figure 10

Solution. Figure 10 is the extensive-form representation of the game. Given q_1, suppose $q_1 \leq a - c$ (otherwise Player 1's payoff is nonpositive), which implies $a - q_1 \geq c$.

The second stage is exactly a Cournot model of Duopoly, with total demand $a' = a - q_1$, and marginal cost $c_2 = c_3 = c \leq a - q_1$. Therefore the unique Nash equilibrium is $(q_2^*(q_1), q_3^*(q_1)) = (\frac{a-q_1-c}{3}, \frac{a-q_1-c}{3})$[1].

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[1] For details, please see page 25 in lecture notes.

For Player 1, consider the following optimization problem

$$\max_{q_1 \leq a-c} q_1(a - c - q_1 - q_2^*(q_1) - q_3^*(q_1)) = \max_{q_1 \leq a-c} \frac{1}{3} q_1(a - q_1 - c),$$

which has a unique maximizer $q_1^* = \frac{a-c}{2}$. Hence $q_2^* = q_3^* = \frac{a-c}{6}$.

Hence, the subgame-perfect outcome is: Player 1 chooses $\frac{a-c}{2}$ in the first stage, and Players 2 and 3 choose $\frac{a-c}{6}$. ☐

Exercise 7. *Consider strategic investment in a duopoly model. Firm 1 and firm 2 currently both have a constant average cost of* 2 *per unit. Firm 1 can install a new technology with an average cost of* 0 *per unit; installing the technology costs 8. Firm 2 will observe whether or not firm 1 invests in the new technology. Once firm 1's investment decision is observed, the two firms will simultaneously choose output levels* q_1 *and* q_2 *as in Cournot model. Here let the price be* $P(Q) = 14 - Q$ *if* $Q < 14$ *and* 0 *otherwise. What is the subgame-perfect outcome of the game? (Hint: You can use the result of Question 2 in Tutorial 2.)*

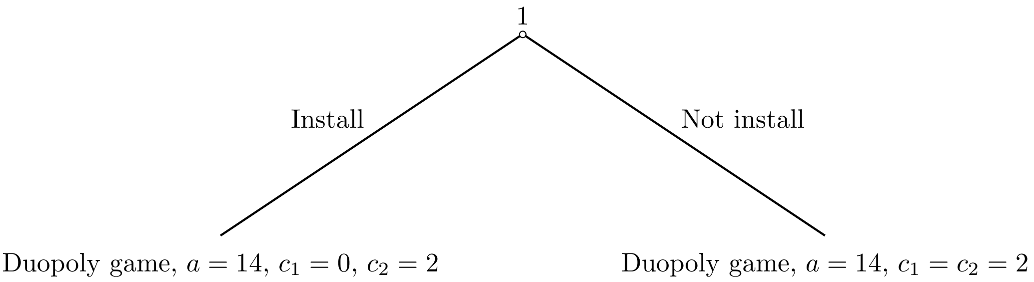

Figure 11

Solution. Figure 11 is the extensive-form representation of the game. There are 2 stages:

- In the first stage, Player 1 choose "Install" or "Not install";
- In the second stage, Players 1 and 2 play the Cournot Duopoly Game.

1. If Player 1 chooses "Install" in the first stage, then $a = 14$, $c_1 = 0$, $c_2 = 2$. Since $0 \leq c_i < \frac{a}{2}$, by Question 2 in Tutorial 2, the unique Nash equilibrium is $(q_1^*, q_2^*) = (\frac{a-2c_1+c_2}{3}, \frac{a-2c_2+c_1}{3}) = (\frac{16}{3}, \frac{10}{3})$, and Player 1's payoff is $\frac{16}{3}(14 - \frac{16}{3} - \frac{10}{3}) - 8 = 20\frac{4}{9}$.
2. If Player 1 chooses "Not install" in the first stage, then $a = 14$, $c_1 = c_2 = 2 < a$, and the unique Nash equilibrium is $(q_1^*, q_2^*) = (\frac{a-c}{3}, \frac{a-c}{3}) = (4, 4)$. Player 1's payoff is $4(14 - 8 - 2) = 16$.

Since $16 < 20\frac{4}{9}$, the subgame-perfect outcome is: Player 1 chooses "Install" in the first stage, and Players 1 and 2 choose $\frac{16}{3}$ and $\frac{10}{3}$, respectively in the second stage. ☐