Solution to Tutorial 4*

2011/2012 Semester I MA4264 Game Theory

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1 Review

- Backwards induction will give us:
  (1) backwards-induction outcome: dynamic game with complete and perfect information;
  (2) subgame-perfect outcome: dynamic game with complete and imperfect information;
  (3) subgame-perfect Nash equilibrium (SPE): dynamic game with complete information (including both perfect and imperfect information).
- Standard methods to find SPE:
  - Backwards induction:
    (1) IESDS;
    (2) Find all information sets (strategies) and subgames;
    (3) Apply backwards induction.
  - SPE $\subset$ NE:
    (1) IESDS;
    (2) Find all information sets (strategies) and subgames;
    (3) Construct the normal-from representation;
    (4) Find all Nash equilibria;
    (5) Check whether each NE is subgame-perfect.

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[*] Corrections are always welcome.

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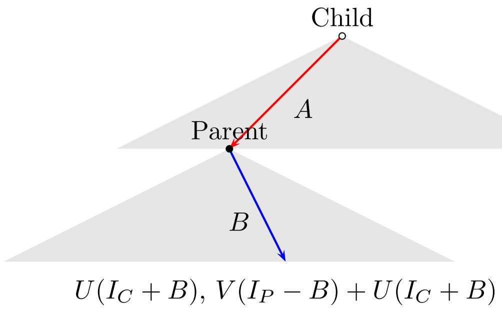

Figure 1: The extensive-form representation

2 Tutorial

Exercise 1. *Suppose a parent and a child play the following game. First, the child takes an action, $A \in \mathbb{R}$, that produces income for the child, $I_C(A) = 5 - (A-3)^2$, and income for the parent, $I_P(A) = 5 - (A-1)^2$. Second, the parent observes the incomes I_C and I_P and then chooses a bequest, B, to leave to the child. The child's payoff is $U(I_C + B)$; the parent's is $V(I_P - B) + U(I_C + B)$, where the utility functions $U(x) = \ln x$ and $V(x) = \ln(4 + x)$.*

(i) Find the backwards-induction outcome of the game.

(ii) Prove the "Rotten Kid" Theorem: in the backwards-induction outcome, the child chooses the action that maximizes the family's aggregate income, $I_C(A) + I_P(A)$, even though only the parent's payoff exhibits altruism.

(iii) Now consider general functions I_C, I_P, U and V. Assume that all functions are differentiable and strictly concave, and U and V are strictly increasing. Assume also that maximizers of the parent's payoff and the child's payoff exist. Show that the Rotten Kid Theorem holds true.

Solution and Proof. (i) Figure 1 is the extensive-form representation of the game. It is a dynamic game with complete and perfect information, and there are two stages. The child and parent's strategy sets are $\mathbb{R}$ and $[0, +\infty)$, respectively.

- In a backwards-induction outcome, after observing I_P and I_C, the parent chooses $B \geq 0$ in the second stage to maximize his utility

$$V(I_P - B) + U(I_C + B) = \ln(4 + I_P - B) + \ln(I_C + B).$$

Given I_C and I_P, $\ln(4+I_P-B)+\ln(I_C+B)$ is a strictly concave function in terms of B since the second derivative is negative. Hence by the first order condition, the unique maximizer is

$$\begin{aligned} B^*(A) &= \frac{4 + I_P(A) - I_C(A)}{2} \\ &= \frac{4 + [5-(A-1)^2] - [5-(A-3)^2]}{2} = 6 - 2A. \end{aligned}$$

- In the first stage, the child chooses A to maximize his utility

  $$\begin{aligned} U(I_C + B) &= \ln(I_C(A) + B^*(A)) \\ &= \ln(5 - (A-3)^2 + 6 - 2A) = \ln(-A^2 + 4A + 2), \end{aligned}$$

  which is also a concave function. By the first order condition, the unique maximizer is $A^* = 2$.

Therefore $B^* = B^*(A^*) = 2$, and hence the backwards-induction outcome is: the child chooses $A^* = 2$ in the first stage, and the parent chooses $B^* = 2$ in the second stage.

(ii) It suffices to show A^* is a maximizer of the function $I_C(A) + I_P(A)$.

$$I_C(A) + I_P(A) = [5 - (A-3)^2] + [5 - (A-1)^2] = -2A^2 + 8A$$

is a strictly concave function, and hence the unique maximizer is $A^* = 2$ by the first order condition.

(iii) We need to prove the child's maximizer A^* will maximize the aggregate income $I_C(A) + I_P(A)$.

Firstly, we try to find the backwards-induction outcome:

- In the second stage, given A, the best response $B^*(A)$[1] maximizes the parent's payoff

  $$V(I_P(A) - B) + U(I_C(A) + B).$$

  Since V and U are differentiable and strictly concave, $V(I_P(A) - B) + U(I_C(A) + B)$ is also strictly concave in terms of B, and hence $B^*(A)$ should satisfy the first order condition:

  $$-V'(I_P(A) - B^*(A)) + U'(I_C(A) + B^*(A)) = 0 \tag{1}$$

  holds for all A.

- In the first stage, A^* maximizes the child's payoff

  $$U(I_C(A) + B^*(A)).$$

  Since U is strictly increasing, A^* should maximize $I_C(A) + B^*(A)$. Hence by the first order condition, we have

  $$I'_C(A^*) + B^{*\prime}(A^*) = 0.\text{[2]} \tag{2}$$

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[1] $B^*(A)$ may not exist. We need additional assumptions: $V'(-\infty) = U'(-\infty) = \infty$.

[2] We need to show $B^*(A)$ is differentiable: let $f(A, B) = -V'(I_P(A) - B) + U'(I_C(A) - B)$. Then $\frac{\partial f}{\partial B} = U' + V' \neq 0$. By implicit function theorem and uniqueness $B^*(A)$, $B^*(A)$ is continuously differentiable.

Differentiating A in Equation (1), by the chain rule we have

$$-V''(\cdot) \times [I_P'(A) - B^{*\prime}(A)] + U''(\cdot) \times [I_C'(A) + B^{*\prime}(A)] = 0.$$

Taking $A = A^*$, then by Equation (2) we have

$$V''(\cdot) \times [I_P'(A^*) - B^{*\prime}(A^*)] = 0.$$

Since V is strictly concave, we have $V'' < 0$, and hence

$$I_P'(A^*) - B^{*\prime}(A^*) = 0. \tag{3}$$

Combining Equations (2) and (3), we have

$$I_C'(A^*) + I_P'(A^*) = 0.$$

Since $I_C(A) + I_P(A)$ is strictly concave in A, we have A^* is a maximizer.

□

Exercise 2. *Now suppose the parent and child play a different game. Let the incomes $I_C = 80$ and $I_P = 100$ be fixed exogenously. First, the child decides how much of the income I_C to save (S) for the future, consuming the rest ($I_C - S$) today. Second, the parent observes the child's choice of S and chooses a bequest, B. The child's payoff is the sum of current and future utilities: $u_c(S, B) = \ln(I_C - S) + 2\ln(S + B)$. The parent's payoff is $u_p(S, B) = \ln(I_P - B) + u_c(S, B)$.*

(i) Find the backwards-induction outcome of the game.

(ii) Show that there is a "Samaritan's Dilemma": in the backwards-induction outcome, the child saves too little, so as to induce the parent to leave a larger bequest (i.e., both the parent's and child's payoffs could be increased if S were suitably larger and B suitably smaller). (Hint: Let $S = S^* + t\delta$ and $B = B^* - \delta$, where (S^*, B^*) is the backwards-induction outcome and t is any number > 3. Show that both payoffs u_c and u_p increase as δ increases from 0 to a small positive number.)

Solution and Proof. (i) Figure 2 is the extensive-form representation of the game. It is a dynamic game with complete and perfect information, and there are two stages. The child and parent's strategy sets are $[0, 80]$ and $[0, +\infty)$, respectively.

- In the second stage, given the child's action S, the parent chooses $B^*(S)$ to maximize his payoff

$$\begin{aligned} u_p(S, B) &= \ln(I_P - B) + u_c(S, B) \\ &= \ln(100 + B) + \ln(80 - S) + 2\ln(S + B) \end{aligned}$$

which is strictly concave in terms of B since the second derivative is negative. By the first order condition, the unique maximizer is

$$B^*(S) = \frac{200 - S}{3}.$$

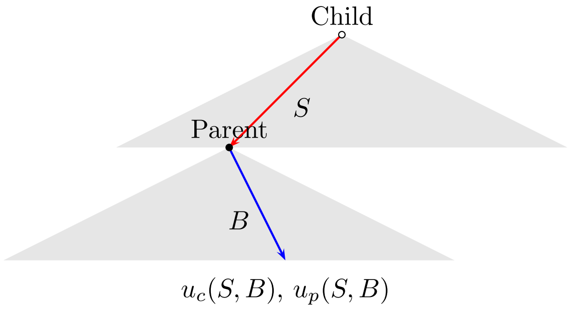

Figure 2: The extensive-form representation

- In the first stage, the child chooses S^* to maximize his payoff

  $$\begin{aligned} u_c(S, B^*(S)) &= \ln(80 - S) + 2\ln(S + B^*(S)) \\ &= \ln(80 - S) + 2\ln\frac{200 + 2S}{3} \end{aligned}$$

  which is strictly concave. Then by the first order condition again, the unique maximizer is $S^* = 20$, and hence $B^* = B^*(S^*) = 60$.

Therefore, the backwards-induction outcome is: the child chooses 20 in the first stage, and the parent chooses 60 in the second stage.

(ii) Let $S = S^* + t\delta$, $B = B^* - \delta$, and

$$f(\delta) \equiv u_c(S,B) = \ln(60 - t\delta) + 2\ln(80 + (t-1)\delta).$$

In order for f to be increasing for small δ, we only need to verify that $f'(0)$ is positive.

$$f'(\delta) = \frac{t}{\delta - 60} + \frac{2(t-1)}{80 + (t-1)\delta},$$

so

$$f'(0) = -\frac{t}{60} + \frac{t-1}{40} = \frac{t-3}{120}.$$

When $t > 3$, $f'(0) > 0$, and hence there exists $\epsilon > 0$, such that $f'(\delta) > 0$ when $\delta \in [0, \epsilon)$. Therefore, $u_c(S,B) = f$ is increasing in $[0, \epsilon)$.

Note that the parent's payoff is $\ln(40 + \delta) + u_c(S,B)$, so it is also increasing in $[0, \epsilon)$ since each term is increasing in $[0, \epsilon)$.

□

Exercise 3. *Consider two countries denoted by $i = 1, 2$, each of which has one firm producing a homogenous product only for export, to be sold in the world market. The price for the product is $p(Q) = a - Q$, where $Q = q_1 + q_2$ and q_i is the output level of the firm in country i. The pre-innovation cost function of each firm is $C_i(q_i) = cq_i$, $i = 1, 2$. (Assume $0 < \frac{4}{9}a \leq c < a$.) Let x_i denote the amount of research and development (R&D) sponsored by the government in country i. We assume that when government i undertakes R&D at level x_i, the cost function of the firm in country i becomes $C_i(q_i, x_i) = (c - x_i)q_i$, $i = 1, 2$. Also assume that the*

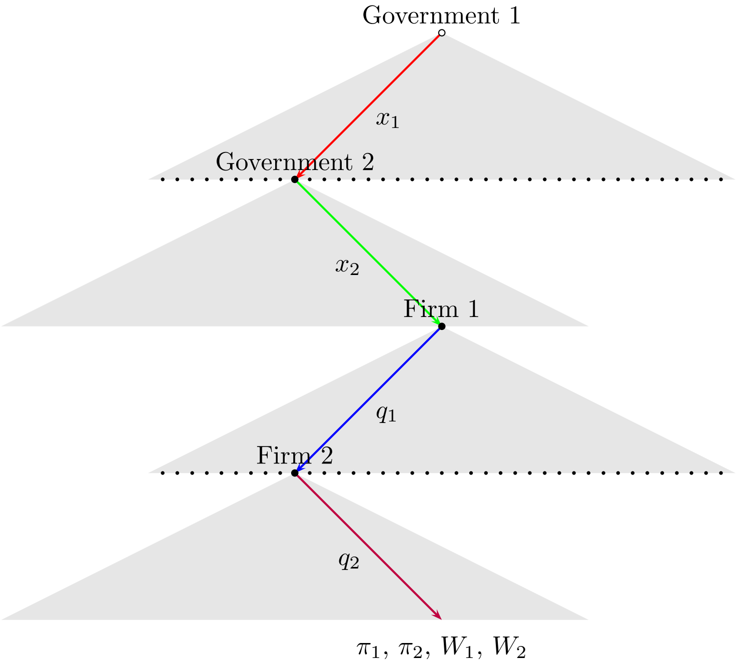

Figure 3: The extensive-form representation

total cost to government i of engaging in R&D at level x_i is $TC_i(x_i) = \frac{x_i^2}{2}$. The game takes place in two stages:

- *Governments choose R&D levels $x_i \geq 0$ simultaneously;*
- *Observing both governments' choice of R&D, firms simultaneously choose output level $q_i \geq 0$.*

The payoff functions of the firms are given by

$$\begin{aligned}\pi_i(q_1, q_2, x_1, x_2) &= q_i(p(Q) - C_i(q_i, x_i)) \\ &= q_i(a - (q_i + q_j) - (c - x_i)), \qquad i = 1, 2,\ j \neq i\end{aligned}$$

and those of the governments by

$$\begin{aligned}W_i(q_1, q_2, x_1, x_2) &= \pi_i(q_1, q_2, x_1, x_2) - TC_i(x_i) \\ &= q_i(a - (q_i + q_j) - (c - x_i)) - \frac{x_i^2}{2}, \qquad i = 1, 2,\ j \neq i\end{aligned}$$

Find the subgame-perfect outcome.

Solution. Figure 3 is the extensive-form representation of the game. It is a dynamic game with complete and imperfect information, and there are two stages. For countries 1 and 2, the strategy set is $[0, c]$. (we need $x_i \leq c$ because firms' marginal cost can not be negative.)

- In the second stage, given x_1 and x_2, two firms play a Cournot duopoly game, where the total demand is a, and marginal cost for firm i is $c_i = c - x_i$. Given

q_j, Firm i's best response is

$$R_i^*(q_j) = \begin{cases} \{\frac{a-c_i-q_j}{2}\}, & \text{if } q_j \leq a - c_i; \\ \{0\}, & \text{if } q_j > a - c_i. \end{cases}$$

Based on Question 2 in Tutorial 2, we have the following 3 cases:

- If $a - c_2 \leq \frac{a-c_1}{2}$ ($x_2 << x_1$), then $q_2^* = 0$, and hence $W_2 \leq 0$.
- If $a - c_1 \leq \frac{a-c_2}{2}$ ($x_1 << x_2$), then $q_1^* = 0$, and hence $W_1 \leq 0$.
- If $a - c_2 > \frac{a-c_1}{2}$ and $a - c_1 > \frac{a-c_2}{2}$, then the unique Nash equilibrium is

$$\begin{aligned}(q_1^*(x_1,x_2), q_2^*(x_1,x_2)) &= \left(\frac{a-2c_1+c_2}{3}, \frac{a-2c_2+c_1}{3}\right) \\ &= \left(\frac{a-c+2x_1-x_2}{3}, \frac{a-c+2x_2-x_1}{3}\right).\end{aligned}$$

  In this case, we will see that W_1 and W_2 may be positive. Therefore, in a subgame-perfect outcome, governments 1 and 2 will not choose x_1 and x_2 so that the first 2 cases occur, and hence we should focus on this case.

- In the first stage, given x_j, government i's best response $R_i^*(x_j)$ is the set

$$\arg\max_{x_i \geq 0} W_i(q_1^*(x_1,x_2), q_2^*(x_1,x_2), x_1, x_2),$$

where

$$W_i(q_1^*(x_1,x_2), q_2^*(x_1,x_2), x_1, x_2) = \left(\frac{a-c+2x_i-x_j}{3}\right)^2 - \frac{x_i^2}{2}$$

is a strictly concave function in terms of x_i since the second derivative is $-\frac{1}{9}$. Then by the first order condition, W_i's unique maximizer is $4(a-c-x_j)$, and hence $R_i^*(x_j) = \{4(a-c-x_j)\}$.

Assume x_1^* and x_2^* are best response to each other, then we have $x_1^* = 4(a - c - x_2^*)$ and $x_2^* = 4(a-c-x_1^*)$, which imply

$$x_1^* = x_2^* = \frac{4}{5}(a-c) \leq c \text{ (because } 4/9a \leq c\text{)},$$

and hence $q_1^* = q_2^* = \frac{3}{5}(a-c)$.

To summarize, the subgame-perfect outcome is: each government chooses $\frac{4}{5}(a-c)$ in the first stage, and each firm chooses $\frac{3}{5}(a-c)$ in the second stage. □

Exercise 4. *Give the extensive-form and normal-form representations and find the Nash equilibria and subgame-perfect equilibria of (i) Game 1 in Tutorial 3 Question 3, and (ii) the bank-runs game.*

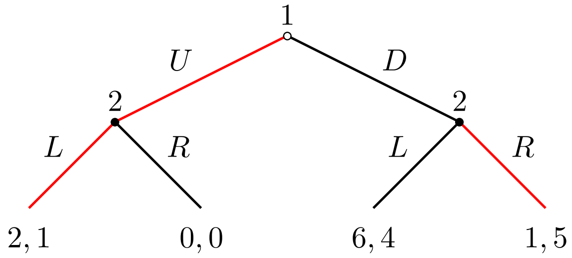

Figure 4: The extensive-form representation and subgame-perfect Nash equilibrium

Player 2

Player 1		LL	LR	RL	RR
	U	$2, 1$	$2, 1$	$0, 0$	$0, 0$
	D	$6, 4$	$1, 5$	$6, 4$	$1, 5$

Figure 5: The normal-form representation and Nash equilibria

Solution. (i) Figures 4 and 5 are the extensive-form and normal-form representations of the game, respectively.

Bi-matrix 5 tells us the all Nash equilibria: (U, LR) and (D, RR).

Since SPE $\subset$ NE, it suffices to check whether each NE is subgame perfect. There are 2 subgames, and L and R are the Nash equilibria in left and right subgames, respectively. Therefore, the unique subgame-perfect equilibrium is (U, LR).

(ii) Figures 6 and 7 are the extensive-form and normal-form representations of the game, respectively.

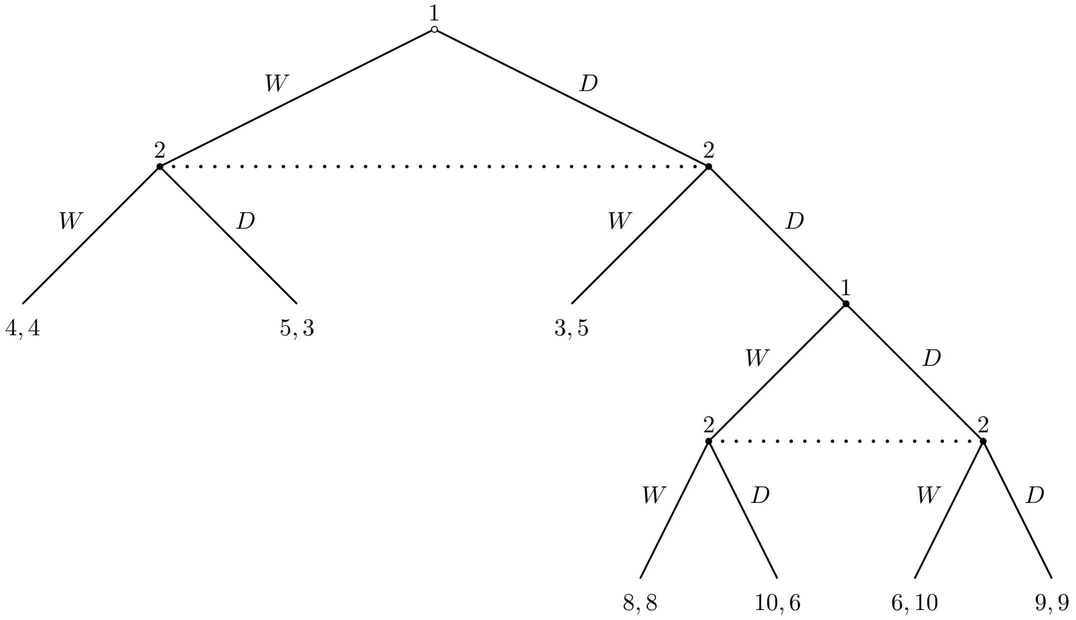

Figure 6: The extensive-form representation

Player 2

Player 1		WW	WD	DW	DD
	WW	4, 4	4, 4	5, 3	5, 3
	WD	4, 4	4, 4	5, 3	5, 3
	DW	3, 5	3, 5	8, 8	10, 6
	DD	3, 5	3, 5	6, 10	9, 9

Figure 7: The normal-form representation and Nash equilibria

Bi-matrix 7 tells us the all Nash equilibria: (WW, WW), (WW, WD), (WD, WW), (WD, WD), and (DW, DW).

There is only one subgame, displayed in Figure 8, and the Nash equilibrium in this subgame is (W, W), Therefore, the all subgame-perfect equilibria are (WW, WW) and (DW, DW).

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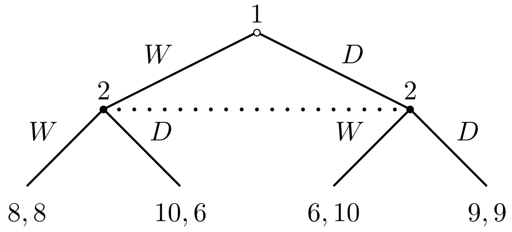

Figure 8

Exercise 5. *For each of the following games.*

(i) find the subgame-perfect outcome;

(ii) give the normal-form representation;

(iii) find all Nash equilibria;

(iv) find all subame-perfect Nash equilibria.

(v) In game 3, there is a Nash equilibrium which is not subgame perfect. Explain why it is a Nash equilibrium and why it is not a "good" equilibrium.

Solution. (1) Game 1:

(i) From Figure 9, we have the subgame-perfect outcome: in the first stage Player 1 chooses A, and in the second stage Player 2 chooses L.

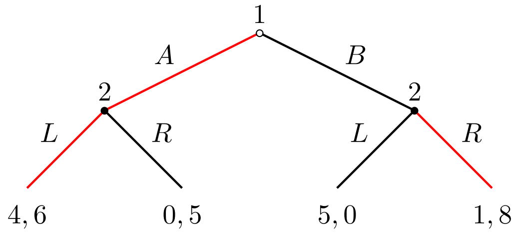

Figure 9: The extensive-form representation and subgame-perfect outcome

(ii,iii) Figure 10 is the normal-form representation, and it tells us the all Nash equilibria: (A, LR) and (B, RR).

		Player 2			
		LL	LR	RL	RR
Player 1	A	$4, 6$	$4, 6$	$0, 5$	$0, 5$
	B	$5, 0$	$1, 8$	$5, 0$	$1, 8$

Figure 10: The normal-form representation and Nash equilibria

(iv) Since it is a dynamic game with complete and perfect information, based on Figure 9, we have the unique subgame-perfect Nash equilibrium: (A, LR).

(2) Game 2:

(i) From Figure 11a, we have the subgame-perfect outcome: since M is strictly dominated by U, we only need to consider the reduced game, displayed in Figure 11b. Hence the subgame-perfect outcome is: Player 1 chooses U in the first stage, and Player 2 chooses L in the second stage.

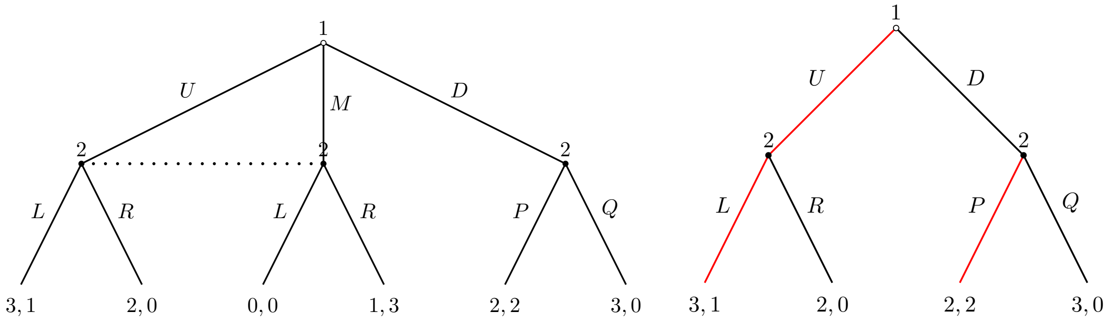

Figure 11: The extensive-form representation and subgame-perfect outcome

(ii,iii) Figure 12 is the normal-form representation, and it tells us the all Nash equilibria: (U, LP), (U, LQ) and (D, RP).

Player 2

Player 1		LP	LQ	RP	RQ
	U	3, 1	3, 1	2, 0	2, 0
	M	0, 0	0, 0	1, 3	1, 3
	D	2, 2	3, 0	2, 2	3, 0

Figure 12: The normal-form representation and Nash equilibria

(iv) There is only one subgame, in which Player 2 will choose P. Therefore the unique subgame-perfect Nash equilibrium is (U, LP).

Remark: there is no subgame-perfect outcome for Game 13.

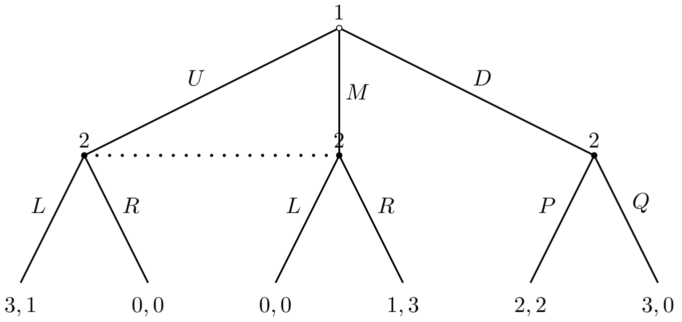

Figure 13: There is no subgame-perfect outcome

(3) Game 3:

(i) From Figure 14, we have the subgame-perfect outcome: in the first stage Player 1 chooses B, and in the second stage Player 2 chooses D.

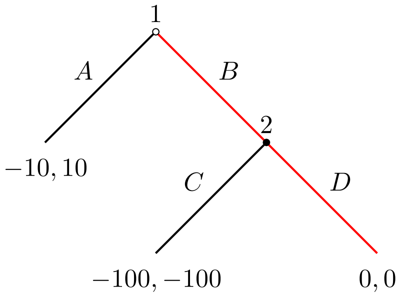

Figure 14: The extensive-form representation and subgame-perfect outcome

(ii,iii) Figure 15 is the normal-form representation, and it tells us the all Nash equilibria: (A, C) and (B, D).

		Player 2	
		C	D
Player 1	A	$-10, 10$	$-10, 10$
	B	$-100, -100$	$0, 0$

Figure 15: The normal-form representation and Nash equilibria

(iv) Since it is a dynamic game with complete and perfect information, based on Figure 14, we have the unique subgame-perfect Nash equilibrium: (B, R).

(v) It is a Nash equilibrium because A is the best response of Player 1 if Player 2 plays C, and C is the best response of Player 2 if Player 1 plays A (actually, Player 2 is indifferent between C and D).

It is not a good equilibrium because it is not subgame-perfect. If the game reaches to the second stage, Player 2 will choose to play D instead of C. This Nash equilibrium is based on a non-credible threat.

(4) Game 4:

(i) From Figure 16, we have the subgame-perfect outcome: in the first stage Player 1 chooses A, in the second stage Player 2 chooses D, and game ends.

(ii,iii) Figure 17 is the normal-form representation, and it tells us the all Nash equilibria: (AG, DE) and (AH, DE).

(iv) Since it is a dynamic game with complete and perfect information, based on Figure 16, we have the unique subgame-perfect Nash equilibrium: (AG, DE).

□

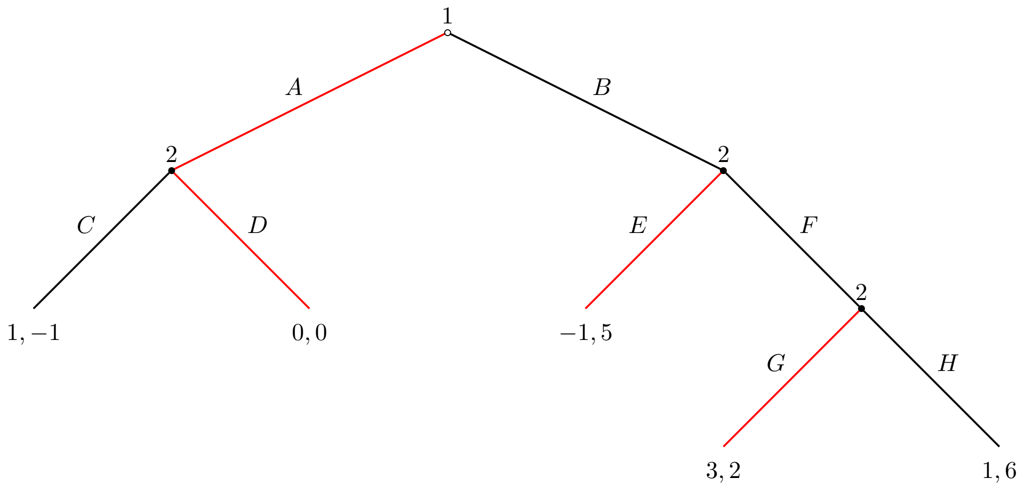

Figure 16: The extensive-form representation and subgame-perfect outcome

Player 2

		CE	CF	DE	DF
Player 1	AG	$1, -1$	$1, -1$	$0, 0$	$0, 0$
	AH	$1, -1$	$1, -1$	$0, 0$	$0, 0$
	BG	$-1, 5$	$3, 2$	$-1, 5$	$3, 2$
	BH	$-1, 5$	$1, 6$	$-1, 5$	$1, 6$

Figure 17: The normal-form representation and Nash equilibria

Exercise 6. *Players 1 and 2 are bargaining over one dollar in two periods: In the first period, Player 1 proposes* s_1 *for himself and* $1 - s_1$ *for player 2. In the second period, player 2 decides whether to accept the offer or to reject the offer. If player 2 accepts the offer, the payoff are* s_1 *for player 1 and* $1 - s_1$ *for player 2. If player 2 rejects the offer, the payoff are zero for both players.*

(i) *Describe all strategies of player 1 and player 2.*

(ii) *Find some (as many as you can) Nash equilibria.*

(iii) *Find a subgame-perfect Nash equilibrium of the game (write down your proof).*

(iv) *Find some Nash equilibria which are not subgame-perfect (write down your proof).*

Solution. Figure 18 is the extensive-form representation of the game.

(i) It is easy to see that Player 1's strategy space is $S_1 = [0, 1]$. Since a strategy is a complete plan of actions in every contingency when a player is called upon to make, a strategy for Player 2 can be represented as a function

$$f\colon [0, 1] \to \{A, R\}.$$

For example,

$$f(s_1) = \begin{cases} A, & \text{if } 0 \le s_1 \le \frac{1}{2}; \\ R, & \text{otherwise} \end{cases}$$

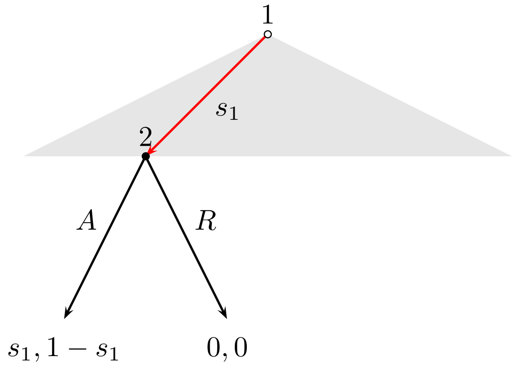

Figure 18: The extensive-form representation of the game

is a strategy of Player 2 in which Player 2 will accept if Player 1 offers any $s_1 \leq \frac{1}{2}$ and otherwise she will reject.

Thus, the space of all strategies of Player 2 is the set of all functions from $[0,1]$ to $\{A, R\}$. We denote it by S_2.[3]

(ii)
- Player 1's best-response correspondence: Given a strategy f of Player 2, note that for any $s_1 \in f^{-1}(A)$, Player 2 will accept the offer. Hence, given f, Player 2 will choose the maximum in $f^{-1}(A)$ if it exists. Thus, Player 1's best-response correspondence is

$$B_1^*(f) = \begin{cases} [0,1], & \text{if } f^{-1}(A) = \emptyset; \\ \{s^*\}, & \text{if } f^{-1}(A) \text{ has a maximum } s^*; \\ \emptyset, & \text{if } f^{-1}(A) \text{ has no maximum.} \end{cases}$$

- Player 2's best-response correspondence: note that Player 2's strategy is a function

$$B_2^*(s_1) = \begin{cases} \{f \in S_2 \colon f(s_1) = A\}, & \text{if } 0 \leq s_1 < 1; \\ S_2, & \text{if } s_1 = 1. \end{cases}$$

  That means for any $s_1 < 1$, Player 1 will accept. If $s_1 = 1$, Player 1 is indifferent between the two actions (accept or reject).

- We can use various combinations of the conditions in the expression of B_1^* and B_2^* to construct all the Nash equilibria:
  - When $f^{*-1}(A) \neq \emptyset$, (s_1^*, f^*) is a Nash equilibrium if and only if $s_1^* = \sup f^{*-1}(A) = \max f^{*-1}(A)$;
  - When $f^{*-1}(A) = \emptyset$, (s_1^*, f^*) is a Nash equilibrium if and only if $s_1^* = 1$.

(iii) For each given s_1, we need to consider a corresponding subgame, displayed in Figure 19. We know if f^* is subgame-perfect, $f^*(s_1) = A$ for any $s_1 < 1$.

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[3] There are other ways to represent the strategies of Player 2, but this seems the most natural way.

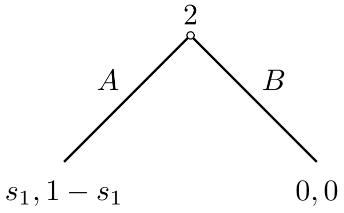

Figure 19

Hence, if (s_1^*, f^*) is subgame-perfect, f^* should be either f_1^* or f_2^*:

$$f_1^*(s_1) = \begin{cases} A, & \text{if } s_1 < 1; \\ R, & \text{if } s_1 = 1. \end{cases} \text{ or } f_2^*(s_1) \equiv A \text{ for all } s_1.$$

It is easy to check that only $(s_1^* = 1, f_2^*)$ is the unique subgame-perfect Nash equilibrium.

(iv) $(s_1^* = 1, f^* \equiv R)$ is a Nash equilibrium but not a subgame-perfect Nash equilibrium.

□