Solution to Tutorial 5*

2011/2012 Semester I   MA4264   Game Theory

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1 Review

- Sequential bargaining game
  - Three-period bargaining game
  - Infinite-horizon bargaining game, could be reduced to a three-period bargaining game
- Infinitely repeated game: trigger strategy
- Static game of incomplete information
  - The normal-form representation of an n-player static Bayesian game:

    $$\{A_1, \ldots, A_n; T_1, \ldots, T_n; \mathbf{P}_1, \ldots, \mathbf{P}_n; u_1, \ldots, u_n\}.$$

  - A strategy for player i is a function $s_i \colon T_i \to A_i$.
  - The strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a (pure-strategy) Bayesian Nash equilibrium if for each player i and for each of i's type $t_i \in T_i$, $s_i^*(t_i)$ solves

    $$\max_{a_i \in A_i} \mathbb{E}_{t_{-i}} u_i(s_{-i}^*(t_{-i}), a_i; t_i),$$

    where

    $$\mathbb{E}_{t_{-i}} u_i(s_{-i}^*(t_{-i}), a_i; t_i) = \sum_{t_{-i} \in T_{-i}} \mathbf{P}_i(t_{-i}|t_i) \times u_i(s_{-i}^*(t_{-i}), a_i; t_i).$$

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*Corrections are always welcome.

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2 Tutorial

Exercise 1. *Suppose the players in Rubinstein's infinite-horizon bargaining game have different discount factors: δ_1 for Player 1 and δ_2 for Player 2. Adapt the argument in the lecture to show that in the backwards-induction outcome, Player 1 offers the settlement*

$$\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

to Player 2, who accepts.

Proof. Let $(s, 1-s)$ be the (optimal) payoffs players can receive in the backwards-induction outcome. Adapting the argument in the lecture, the game can be reduced to a three-period bargaining game, which is represented by Figure 1.

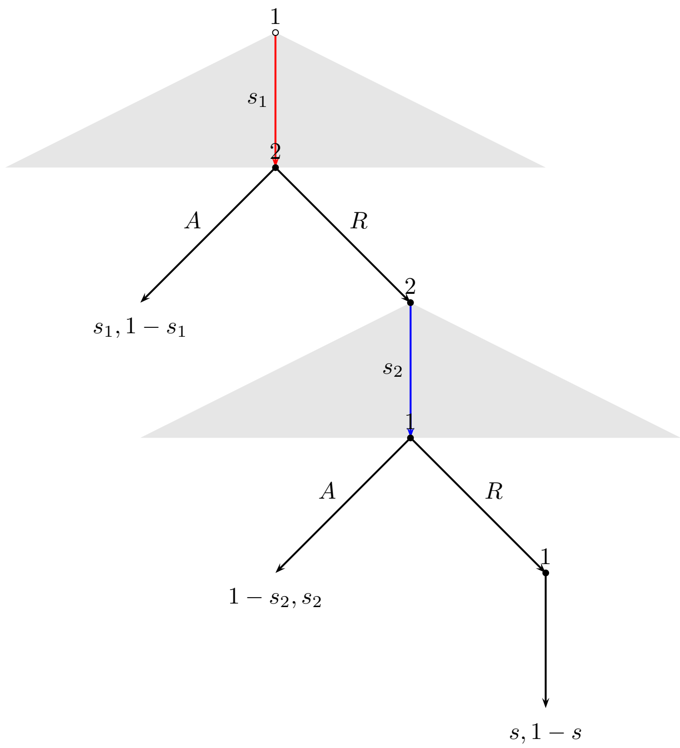

Figure 1: The extensive-form representation

- Consider the period 2. By the Figure 2a, Player 1 accepts if and only if $1 - s_2 \geq \delta_1 s$, i.e., $s_2 \leq 1 - \delta_1 s$.
  - If Player 2 chooses $s_2 > 1 - \delta_1 s$, then Player 1 will reject, and Player 2 will get $1 - s$ (present value is $\delta_2(1-s) \leq 1 - s \leq 1 - \delta_1 s$);
  - If Player 2 chooses $s_2 < 1 - \delta_1 s$, then Player 1 will accept, and Player 2 will get $s_2 < 1 - \delta_1 s$;
  - If Player 2 chooses $s_2 = 1 - \delta_1 s$, then accept and reject are indifferent for Player 1, and Player 2 will get $s_2 = 1 - \delta_1 s$.

Therefore Player 2's best strategy is to choose $s_2 = 1 - \delta_1 s$, and Player 1 will accept this offer.

- Consider the period 1. By the Figure 2b, Player 2 accepts if and only if $1 - s_1 \geq \delta_2 s_2$, i.e., $s_1 \leq 1 - \delta_2 s_2$.
  - If Player 1 chooses $s_1 > 1 - \delta_2 s_2$, then Player 2 will reject, and Player 1 will get $1 - s_2$ (present value is $\delta_1(1 - s_2) \leq 1 - s_2 \leq 1 - \delta_2 s_2$);
  - If Player 1 chooses $s_1 < 1 - \delta_2 s_2$, then Player 2 will accept, and Player 1 will get $s_1 < 1 - \delta_2 s_2$;
  - If Player 1 chooses $s_1 = 1 - \delta_2 s_2$, then accept and reject are indifferent for Player 2, and Player 1 will get $s_1 = 1 - \delta_2 s_2$.

  Therefore Player 1's best strategy is to choose $s_1 = 1 - \delta_2 s_2 = 1 - \delta_2(1 - \delta_1 s)$, and Player 2 will accept this offer.

- To determine s, using the same trick as in the lecture, we have

$$s = 1 - \delta_2(1 - \delta_1 s).$$

  So Player 1 will get $s = \frac{1-\delta_2}{1-\delta_1\delta_2}$, and Player 2 will get $1 - s = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$.

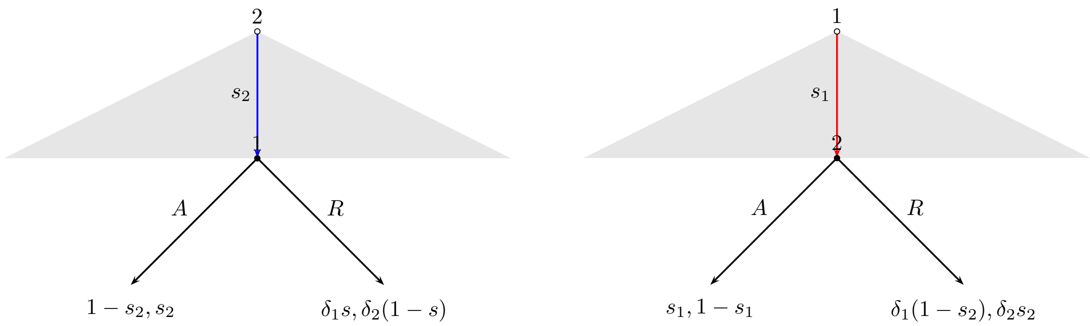

Figure 2

□

Exercise 2. *Let the game given below be the stage game of an infinitely repeated game where* δ_1 *and* δ_2 *are the discount factors for Players 1 and 2.*

	L	R
A	$1, 2$	$5, 0$
B	$1, 8$	$4, 6$

(i) *Determine the ranges of* δ_1 *and* δ_2 *for which the trigger strategies for both players are a Nash equilibrium. The trigger strategy for Player 1 (2) is to play* B *(*R*) if all preceding actions are* (B, R)*; to play* A *(*L*) otherwise.*

(ii) *Show that the Nash equilibrium in part (i) is also a subgame-perfect Nash equilibrium.*

(iii) *Show that playing (A, L) in every stage is a subgame-perfect Nash equilibrium.*

(iv) *In (i), is the trigger strategy a Nash equilibrium if to play B (L) instead of A (L)?*

Solution and Proof. (i) Assume that Player 2 chooses the trigger strategy T_2. We want to find the condition which guarantees the trigger strategy T_1 to be Player 1's best response.

- If Player 1 does not chooses trigger strategy, then in some stage, he will choose A. Without loss of generality, we assume that t-th stage is the first stage when Player 1 chooses A, then he can get 5 at this stage.

  From the $(t+1)$-th stage on, Player 2 will play non-cooperative strategy L to punish Player 1. Thus Player 1 will receive 1 in each of the subsequent stages since A and B are indifferent, and the present value of his payoff from t-th stage onwards is

$$5 + \delta_1 + \delta_1^2 + \cdots = 5 + \frac{\delta_1}{1 - \delta_1}.$$

Stage	t	$t+1$	$t+2$	$t+3$	$\cdots$
Player 1	A	$*$	$*$	$*$	$\cdots$
Player 2	R	L	L	L	$\cdots$
Player 1's payoff	5	1	1	1	$\cdots$

- If Player 1 chooses trigger strategy T_1, then he will receive 4 in each stage, and the present value of his payoff from t-th stage onwards is

$$4 + \delta_1 4 + \delta_1^2 4 + \cdots = \frac{4}{1 - \delta_1}.$$

In order for Player 1 to play trigger strategy T_1, we should have

$$\frac{4}{1 - \delta_1} \geq 5 + \frac{\delta_1}{1 - \delta_1},$$

that is $(1 >)\delta_1 \geq \frac{1}{4}$.

Assume that Player 1 chooses the trigger strategy T_1. We want to find the condition which guarantees the trigger strategy T_2 to be Player 2's best response.

- If Player 2 does not choose the trigger strategy, then in some stage, he will choose L. Without loss of generality, we assume that t-th stage is the first stage when Player 2 chooses L, then he can get 8 at this stage.

  From the $(t+1)$-th stage on, Player 1 will play non-cooperative strategies A to punish Player 2. Thus Player 2 will receive at most 2 in each of the subsequent stages, and the present value of his payoff from t-th stage onwards is at most

$$8 + \delta_2 2 + \delta_2^2 2 + \cdots = 8 + \frac{2\delta_2}{1-\delta_2}.$$

- If Player 2 chooses the trigger strategy T_2, then he will receive 6 in each stage, and the present value of his payoff from t-th stage onwards is

$$6 + \delta_2 6 + \delta_2^2 6 + \cdots = \frac{6}{1-\delta_2}.$$

Stage	t	$t+1$	$t+2$	$t+3$	$\cdots$
Player 1	B	A	A	A	$\cdots$
Player 2	L	$*$	$*$	$*$	$\cdots$
Player 2's payoff	8	≤ 2	≤ 2	≤ 2	$\cdots$

In order for Player 2 to play trigger strategy T_2, we should have

$$\frac{6}{1-\delta_2} \geq 8 + \frac{2\delta_2}{1-\delta_2},$$

that is $(1 >)\delta_2 \geq \frac{1}{3}$.

(ii) In an infinitely repeated game, a subgame is characterized by its previous history. The subgames can be grouped as follows:

(a) Subgames whose previous histories are always finite sequence of (B, R).

(b) Subgames whose previous histories contain other outcomes different from (B, R).

If the trigger strategy is played in the original game, then:

(A) In (a), (T_1, T_2) is played, which is a Nash equilibrium in the subgame;

(B) In (b), (NC_1, NC_2) is played, which is a Nash equilibrium in the subgame.

Therefore, the trigger strategy Nash equilibrium in the original game constitutes a Nash equilibrium in every subgame, i.e., (T_1, T_2) in (a), (NC_1, NC_2) in (b), and hence it is a subgame-perfect Nash equilibrium.

(iii) The strategy profile constitutes a Nash equilibrium in every stage game is a subgame-perfect Nash equilibrium.

(iv) The modified trigger strategies cannot constitute a Nash equilibrium if to play B (L) instead of A (L). The reason is because Player 2 will have an incentive to play L instead R in the first period.

$\square$

Exercise 3. *Suppose there are n firms in a Cournot oligopoly. Inverse demand is given by $P(Q) = a - Q$, where $Q = q_1 + \cdots + q_n$ and q_i is the quantity to be produced by firm i. Each firm has a constant marginal cost of production, c, and no fixed cost. Consider the infinitely repeated game based on this stage game.*

(i) *What is the lowest value of δ such that the firms can use trigger strategies to sustain the monopoly output level in a subgame-perfect Nash equilibrium?*

(ii) *How does the answer vary with n?*

Solution. Calculate Firm i's production and profit in the collusion, Cournot competition, and deviation from punishment cases, respectively:

- Cooperative production and profit: In the collusion, the production is $q_i^c = \frac{a-c}{2n}$, and profit is $\pi_i^c = \frac{(a-c)^2}{4n}$;
- Non-cooperative production and profit: In the Cournot competition, production is $q_i^m = \frac{a-c}{n+1}$, and profit is $\pi_i^m = \frac{(a-c)^2}{(n+1)^2}$;
- Deviation production and profit: For each $j \neq i$, Firm j produces $q_j^c = \frac{a-c}{2n}$, then Firm i can increases its profit by producing $q_i^d = \frac{(n+1)(a-c)}{4n}$, and profit is $\pi_i^d = \frac{(n+1)^2(a-c)^2}{(4n)^2}$.

For each i, consider the following trigger strategy T_i for Firm i:

- In the first stage produce q_i^c.
- In the t-th stage ($t > 1$), produce q_i^c if every Firm j has produced q_j^c in each of the $t-1$ previous stages; otherwise, produce q_i^m.

(i) Fix Firm i, and assume that each other Firm $j \neq i$ chooses the trigger strategy T_j. We want to find the condition which guarantees the trigger strategy T_i to be Firm i's best response.

- If Firm i does not choose the trigger strategy, then in some stage, it will deviate and the profit maximizer is q_i^d. Without loss of generality, we assume that t-th stage is the first stage when Firm i deviates, then it can get at most π_i^d at this stage.

  From the (t+1)-th stage on, every other firm j will produce non-cooperative production q_j^m. Thus Firm i will receive at most π_i^m in each of the subsequent stages, and the present value of its payoff from t-th stage onwards is at most
$$\pi_i^d + \delta\pi_i^m + \delta^2\pi_i^m + \cdots = \pi_i^d + \frac{\delta\pi_i^m}{1-\delta}.$$

Stage	t	$t+1$	$t+2$	$t+3$	$\cdots$
Firm $j \neq i$	q_j^c	q_j^m	q_j^m	q_j^m	$\cdots$
Firm i	q_i^d	$*$	$*$	$*$	$\cdots$
Firm i's payoff	π_i^d	$\leq \pi_i^m$	$\leq \pi_i^m$	$\leq \pi_i^m$	$\cdots$

- If Firm i chooses the trigger strategy T_i, then it will receive π_i^c in each stage, and the present value of its payoff from t-th stage onwards is

$$\pi_i^c + \delta\pi_i^c + \delta^2\pi_i^c + \cdots = \frac{\pi_i^c}{1-\delta}.$$

In order for Firm i to play trigger strategy T_i, we should have

$$\frac{\pi_i^c}{1-\delta} \geq \pi_i^d + \frac{\delta\pi_i^m}{1-\delta},$$

that is $\delta \geq \frac{(n+1)^2}{(n+1)^2+4n}$.

(ii) Since $\lim_{n\to\infty} \frac{(n+1)^2}{(n+1)^2+4n} = 1$, the lowest value of δ approaches 1. That is, as n increases, a larger δ is required to deter the deviation. In other words, there is more incentive to deviate the trigger strategy.

□

Exercise 4. *Find $\delta > 0$ such that the trigger strategy is a subgame-perfect Nash equilibrium for the game which infinitely repeats the stage game of Bertrand model with homogeneous products described in the lecture.*

Solution. Calculate firm i's price and profit in the collusion, Bertrand competition, and deviation from punishment cases, respectively:

- Cooperative price and profit: In the collusion, the price is $p_i^c = \frac{a+c}{2}$, and profit is $\pi_i^c = \frac{(a-c)^2}{8}$;
- Non-cooperative price and profit: In the Bertrand competition, price is $p_i^m = c$, and profit is $\pi_i^m = 0$;
- Deviation price and profit: Firm j's price is $p_j^c = \frac{a+c}{2}$, Firm $i \neq j$ can increases its profit by choosing a price $p_i^d < \frac{a+c}{2}$, but as close as possible to $\frac{a+c}{2}$, and profit is almost equal to monopoly profit $\pi_i^d = \frac{(a-c)^2}{4}$.

For each i, consider the following trigger strategy T_i for Firm i:

- In the first stage, choose price p_i^c.
- In the t-th stage, choose p_i^c if Firm j chooses price p_j^c in each of the $t-1$ previous stages; otherwise, choose price p_i^m.

For any i, assume that Firm $j \neq i$ chooses the trigger strategy T_j. We want to find the condition which guarantees the trigger strategy T_i to be Firm i's best response.

- If Firm i does not choose the trigger strategy, then in some stage, it will deviate and the profit maxizer is p_i^d. Without loss of generality, we assume that t-th stage is the first stage when Firm i deviates, then it can get at most π_i^d at this stage.

From the $(t+1)$-th stage on, Firm $j \neq i$ will choose non-cooperative price p_j^m. Thus Firm i will receive at most $\pi_m^i = 0$ in each of the subsequent stages, and the present value of its payoff from t-th stage onwards is at most

$$\pi_i^d.$$

Stage	t	$t+1$	$t+2$	$t+3$	$\cdots$
Firm $j \neq i$	p_j^c	p_j^m	p_j^m	p_j^m	$\cdots$
Firm i	p_i^d	$*$	$*$	$*$	$\cdots$
Firm i's payoff	π_i^d	$\leq \pi_i^m$	$\leq \pi_i^m$	$\leq \pi_i^m$	$\cdots$

- If Firm i chooses the trigger strategy T_i, then it will receive π_i^c in each stage, and the present value of its payoff from t-th stage onwards is

$$\pi_i^c + \delta\pi_i^c + \delta^2\pi_i^c + \cdots = \frac{\pi_i^c}{1-\delta}.$$

In order for firm i to play trigger strategy T_i, we should have

$$\frac{\pi_i^c}{1-\delta} \geq \pi_i^d,$$

that is $\delta \geq \frac{1}{2}$. □

Exercise 5. *Consider a Cournot duopoly operating in a market with inverse demand $P(Q) = a - Q$, where $Q = q_1 + q_2$ is the aggregate quantity on the market. Both firms have total costs $c_i(q_i) = cq_i$, but demand is uncertain: it is high ($a = a_H$) with probability θ and low ($a = a_L$) with probability $1-\theta$. Furthermore, information is asymmetric: firm 1 knows whether demand is high or low, but firm 2 does not. All of this is common knowledge. The two firms simultaneously choose quantities. What are the strategy spaces for the two firms? What is the Bayesian Nash equilibrium of this game, (assuming a_H, a_L, θ and c are such that all equilibrium quantities are positive)?*

Solution.

- Firm i's action space is $\{q\colon q \geq 0\}$.
- Firm 1's type space $T_1 = \{H, L\}$; Frim 2 has only one type.
- Strategy space: $S_1 = \{(q_{1H}, q_{1L})\colon q_{1H}, q_{1L} \geq 0\}$, and $S_2 = \{q_2\colon q_2 \geq 0\}$.

Suppose that $(q_{1H}^*, q_{1L}^*, q_2^*)$ is a Bayesian Nash equilibrium, then by definition we will have:

(a) If the demand is high, Firm 1 will choose q_{1H}^* to maximize its payoff

$$q_{1H}[a_H - c - q_2^* - q_{1H}],$$

which is a concave function, and hence

$$q_{1H}^* = \frac{a_H - c - q_2^*}{2}. \tag{1}$$

(b) If the demand is low, Firm 1 will choose q_{1L}^* to maximize its payoff

$$q_{1L}[a_L - c - q_2^* - q_{1L}],$$

which is a concave function, and hence

$$q_{1L}^* = \frac{a_L - c - q_2^*}{2}. \tag{2}$$

(c) Firm 2 does not know the exact type of the demand, so it will choose q_2^* to maximize its expected payoff

$$\theta q_2[a_H - c - q_{1H}^* - q_2] + (1-\theta)q_2[a_L - c - q_2 - q_{1L}^*],$$

and hence

$$q_2^* = \frac{\theta(a_H - q_{1H}^*) + (1+\theta)(a_L - q_{1L}^*)}{2}. \tag{3}$$

Combining Equations (1), (2) and (3), we get

$$\begin{aligned} q_{1H}^* &= \frac{a_H - c}{2} - \frac{\theta a_H + (1-\theta)a_L - c}{6}, \\ q_{1L}^* &= \frac{a_L - c}{2} - \frac{\theta a_H + (1-\theta)a_L - c}{6}, \\ q_2^* &= \frac{\theta a_H + (1-\theta)a_L - c}{3}. \end{aligned}$$

□

Exercise 6. *Consider the following asymmetric-information model of Bertrand duopoly with differentiated products. Demand for firm i is $q_i(p_i, p_j) = a - p_i + b_i \cdot p_j$. Costs are zero for both firms. The sensitivity of firm i's demand to firm j's price is either high or low. That is, b_i is either b_H or b_L, where $b_H > b_L > 0$. For each firm, $b_i = b_H$ with probability θ and $b_i = b_L$ with probability $1-\theta$, independent of the realization of b_j. Each firm knows its own b_i but not its competitor's. All of this is common knowledge. What are the action spaces, type spaces, beliefs, and utility functions in this game? What are the strategy spaces? Assume that $\theta b_H + (1-\theta)b_L < 2$. Find the pure-strategy Bayesian Nash equilibrium of this game.*

Solution.

- Firm i's action space: $A_i = \{p\colon p \geq 0\}$.
- Firm i's type space: $T_i = \{H, L\}$.
- Firm i's beliefs: $\theta H + (1-\theta)L$.
- Firm i's strategy space: $S_i = \{(p_{iH}, p_{iL})\colon p_{iH}, p_{iL} \in A_i\}$.
- Firm i's utility function (for type t): $[a - p_{it} + b_t(\theta p_{jH} + (1-\theta)p_{jL})]p_{it}$.

- For type $t = H, L$, Firm i's maximization problem:

$$\max_{p_{it}} \pi_{it} = [a - p_{it} + b_t(\theta p_{jH} + (1-\theta)p_{jL})]p_{it}.$$

By the first order condition, we have

$$a - 2p_{it} + b_t(\theta p_{jH} + (1-\theta)p_{jL}) = 0.$$

That is, for $i = 1, 2$,

$$\begin{aligned} p_{iH} &= \frac{a}{2} + \frac{b_H(\theta p_{jH} + (1-\theta)p_{jL})}{2}, \\ p_{iL} &= \frac{a}{2} + \frac{b_L(\theta p_{jH} + (1-\theta)p_{jL})}{2}. \end{aligned}$$

Let $b = \theta b_H + (1-\theta)b_L$. Then we have

$$\begin{aligned} p_{iH} &= \frac{a}{2} + \frac{ab_H}{4} + bb_H\frac{\theta p_{iH} + (1-\theta)p_{iL}}{4}, \\ p_{iL} &= \frac{a}{2} + \frac{ab_L}{4} + bb_L\frac{\theta p_{iH} + (1-\theta)p_{iL}}{4}. \end{aligned}$$

Therefore, for $i = 1, 2$,

$$\begin{aligned} p_{iH} &= \frac{1}{1 - \frac{1}{4}b^2}\left[\frac{1}{2}a(1 + \frac{1}{2}b_H) + \frac{1-\theta}{8}ab(b_H - b_L)\right], \\ p_{iL} &= \frac{1}{1 - \frac{1}{4}b^2}\left[\frac{1}{2}a(1 + \frac{1}{2}b_L) + \frac{\theta}{8}ab(b_H - b_L)\right]. \end{aligned}$$

□