Solution to Tutorial 5

2012/2013 Semester I MA4264 Game Theory

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1 Review

- Sequential bargaining game
  - Three-period bargaining game
  - Infinite-horizon bargaining game, could be reduced to a three-period bargaining game
- Infinitely repeated game
  - In the stage game, every player has at least two actions: cooperative action and non-cooperative action.
  - Non-cooperative strategy: in every stage game, choosing the non-cooperative action.
  - Cooperative strategy: in every stage game, choosing the cooperative action.
  - Trigger strategy: choosing the cooperative action if previous histories are all cooperative actions, and choosing the non-cooperative action otherwise.

2 Tutorial

Exercise 1. *Consider the following two-period political game with two players, the working class and the elite. In the first period, the elite decides whether to redistribute or not, and then the working class decides whether to carry out a revolution. Redistribution and no revolution gives a utility of 10 to the working class and 15 to the elite. If there is no redistribution and no revolution, the working class gets 0 and the elite gets 25. And if there is a revolution (irrespective of redistribution), the working class gets 15 and the elite gets 0.*

If there is a revolution, then that is the end of the game, and the payoffs are final. If there is no revolution, the game proceeds to the second period, where both parties get additional payoffs. But first, nature determines whether or not the working class has the opportunity to carry out a revolution. The probability that this opportunity exists for the working class is q. Observing whether the working class has the opportunity to carry out a revolution, the elite again decides whether to redistribute. Once again, without redistribution, the elite gets an additional utility of 25 and the working class gets 0. With

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redistribution, the working class gets an additional utility of 10 and the elite 15. Also, again a revolution gives a utility of 15 to the working class and nothing to the elite (irrespective of whether there is redistribution in the second period or not). There is no discounting between the two periods.

(i) For each $q \in [0, 1]$, find all the subgame perfect Nash equilibrium of this game.

(ii) Explain briefly why a high value of q, probability of revolution opportunity in the second period, prevents a revolution in the first period.

Solution. The extensive-form representation is given in Figure 1.

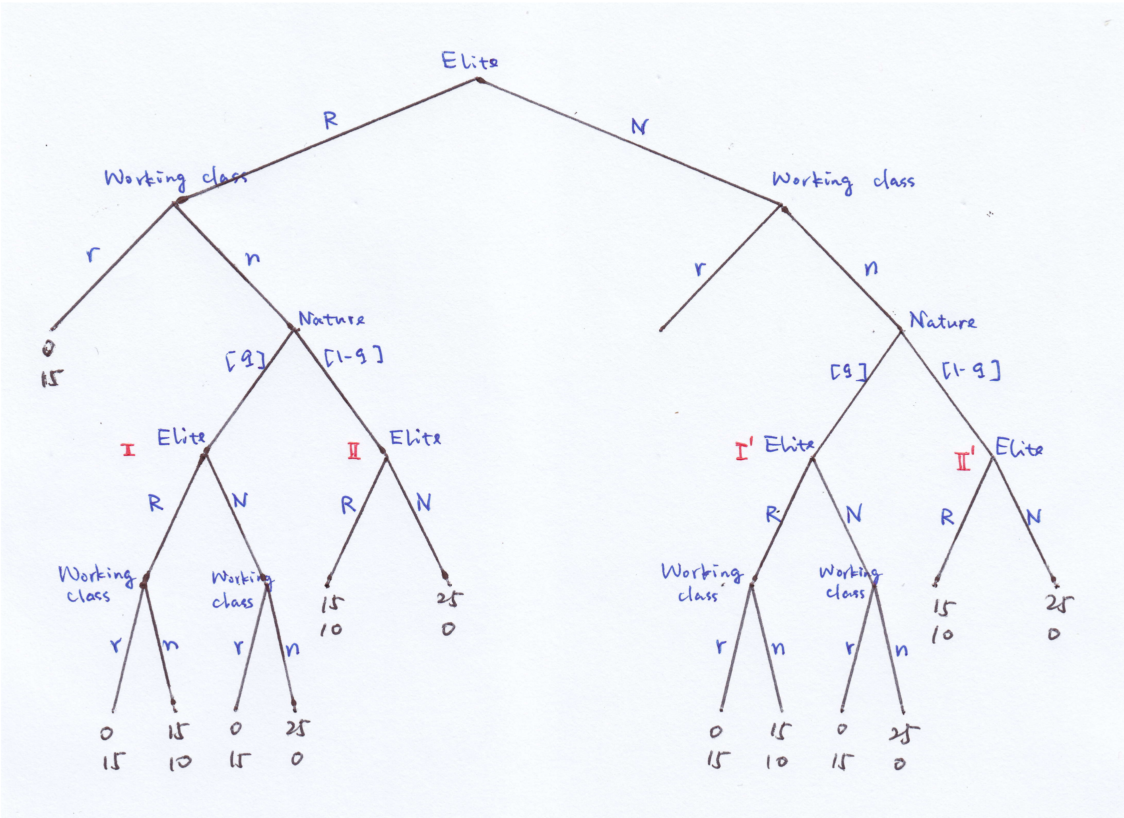

Figure 1

(i) In subgames I and I', the working class will choose "revolution" at every node with her/his move; apply backwards induction, "redistribution" and "no redistribution" are equivalent for the elite.

In subgames II and II', the working class has no opportunity to revolution, so the elite will choose "no redistribution".

Hence, the original game could be reduced to the following game given in Figure 2, where $40 - 25q$ and $15q + 10$ are expected payoffs for the elite and the working class respectively.

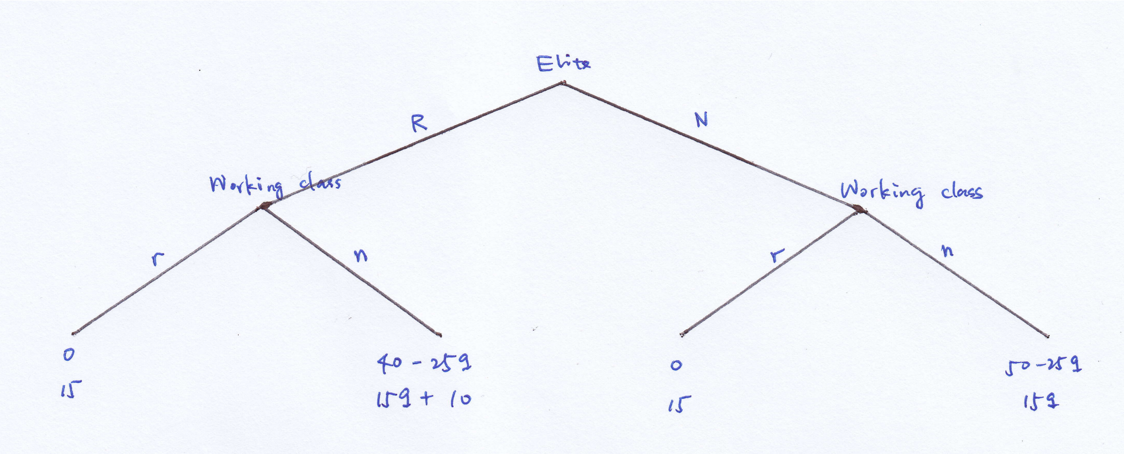

Figure 2

- If $q = 1$, there are 8 subgame-perfect Nash equilibria: $(RRNRN, nrrrrr)$, $(NRNRN, nnrrrr)$, $(RRNNN, nrrrrr)$, $(NRNNN, nnrrrr)$, $(RNNRN, nrrrrr)$, $(NNNRN, nnrrrr)$, $(RNNNN, nrrrrr)$, $(NNNNN, nnrrrr)$.
- If $1 > q > \frac{1}{3}$, there are 4 subgame-perfect Nash equilibria: $(RRNRN, nrrrrr)$, $(RRNNN, nrrrrr)$, $(RNNRN, nrrrrr)$, $(RNNNN, nrrrrr)$.
- If $q = \frac{1}{3}$, there are 12 subgame-perfect Nash equilibria: $(RRNRN, rrrrrr)$, $(RRNNN, rrrrrr)$, $(RNNRN, rrrrrr)$, $(RNNNN, rrrrrr)$, $(NRNRN, rrrrrr)$, $(NRNNN, rrrrrr)$, $(NNNRN, rrrrrr)$, $(NNNNN, rrrrrr)$, $(RRNRN, nrrrrr)$, $(RRNNN, nrrrrr)$, $(RNNRN, nrrrrr)$, $(RNNNN, nrrrrr)$.
- If $\frac{1}{3} > q$, there are 8 subgame-perfect Nash equilibria: $(RRNRN, rrrrrr)$, $(RRNNN, rrrrrr)$, $(RNNRN, rrrrrr)$, $(RNNNN, rrrrrr)$, $(NRNRN, rrrrrr)$, $(NRNNN, rrrrrr)$, $(NNNRN, rrrrrr)$, $(NNNNN, rrrrrr)$.

(ii) Obvious. It suffices to consider the reduced game.

□

Exercise 2. *Suppose the players in Rubinstein's infinite-horizon bargaining game have different discount factors: δ_1 for Player 1 and δ_2 for Player 2. Adapt the argument in the lecture to show that in the backwards-induction outcome, Player 1 offers the settlement*

$$\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

to Player 2, who accepts.

Proof. Let $(s, 1-s)$ be the (optimal) payoffs players can receive in the backwards-induction outcome. Adapting the argument in the lecture, the game can be reduced to a three-period bargaining game, which is represented by Figure 3.

- Consider the period 2. By the Figure 4a, Player 1 accepts if and only if $1 - s_2 \geq \delta_1 s$, i.e., $s_2 \leq 1 - \delta_1 s$.
  - If Player 2 chooses $s_2 > 1 - \delta_1 s$, then Player 1 will reject, and Player 2 will get $1 - s$ (present value is $\delta_2(1-s) \leq 1 - s \leq 1 - \delta_1 s$);

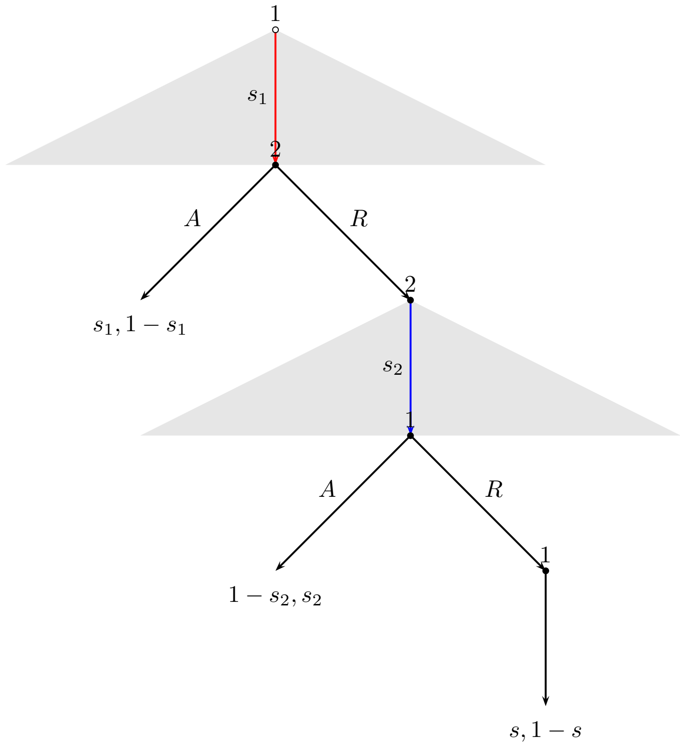

Figure 3: The extensive-form representation

- If Player 2 chooses $s_2 < 1-\delta_1 s$, then Player 1 will accept, and Player 2 will get $s_2 < 1-\delta_1 s$;
- If Player 2 chooses $s_2 = 1-\delta_1 s$, then Player 1 will accept (accept and reject are indifferent for Player 1), and Player 2 will get $s_2 = 1-\delta_1 s$.[1]

Therefore Player 2's best strategy is to choose $s_2 = 1-\delta_1 s$, and Player 1 will accept this offer.

- Consider the period 1. By the Figure 4b, Player 2 accepts if and only if $1-s_1 \geq \delta_2 s_2$, i.e., $s_1 \leq 1-\delta_2 s_2$.
  - If Player 1 chooses $s_1 > 1-\delta_2 s_2$, then Player 2 will reject, and Player 1 will get $1-s_2$ (present value is $\delta_1(1-s_2) \leq 1-s_2 \leq 1-\delta_2 s_2$);
  - If Player 1 chooses $s_1 < 1-\delta_2 s_2$, then Player 2 will accept, and Player 1 will get $s_1 < 1-\delta_2 s_2$;
  - If Player 1 chooses $s_1 = 1-\delta_2 s_2$, then accept and reject are indifferent for Player 2, and Player 1 will get $s_1 = 1-\delta_2 s_2$.

  Therefore Player 1's best strategy is to choose $s_1 = 1-\delta_2 s_2 = 1-\delta_2(1-\delta_1 s)$, and Player 2 will accept this offer.

[1] There is an issue: why Player 1 can not choose "reject" when Player 2 chooses $s_2 = 1-\delta_1 s$. After discussing with Prof. Zhao, we have the following explaining: when finding backwards-induction outcomes, we always assume there is unique optimal action at each decision node (otherwise, there is no backwards-induction outcome in some case, e.g., Remark after Question 2 in "Solution to Assignment 2"). Hence, here we assume Player 1 will choose "accept" rather then "reject" when Player 2 chooses $s_2 = 1-\delta_1 s$ for sake of simplification. Thanks for Mr. Yusheng Luo pointing out this issue.

- To determine s, using the same trick as in the lecture, we have

$$s = 1 - \delta_2(1 - \delta_1 s).$$

So Player 1 will get $s = \frac{1-\delta_2}{1-\delta_1\delta_2}$, and Player 2 will get $1 - s = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$.

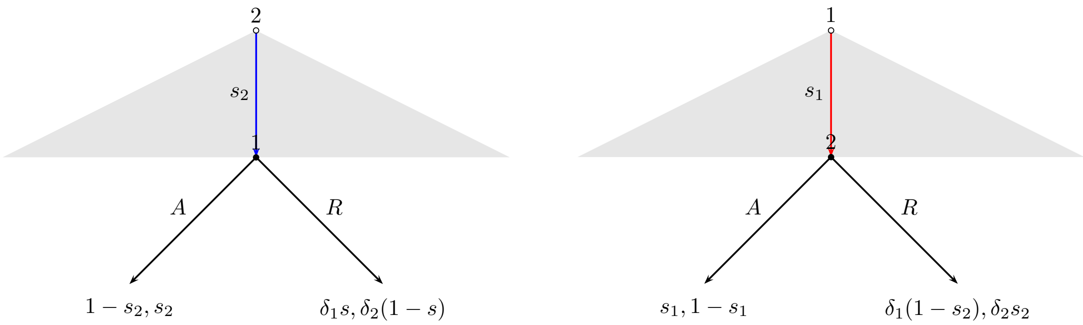

Figure 4

□

Exercise 3. *Let the game given below be the stage game of an infinitely repeated game where δ_1 and δ_2 are the discount factors for Players 1 and 2.*

	L	R
A	$1, 2$	$5, 0$
B	$1, 8$	$4, 6$

(i) Determine the ranges of δ_1 and δ_2 for which the trigger strategies for both players are a Nash equilibrium. The trigger strategy for Player 1 (2) is to play B (R) if all preceding actions are (B, R); to play A (L) otherwise.

(ii) Show that the Nash equilibrium in part (i) is also a subgame-perfect Nash equilibrium.

(iii) Show that playing (A, L) in every stage is a subgame-perfect Nash equilibrium.

(iv) In (i), is the trigger strategy a Nash equilibrium if to play B (L) instead of A (L)?

Solution and Proof. (i) Assume that Player 2 chooses the trigger strategy T_2. We want to find the condition which guarantees the trigger strategy T_1 to be Player 1's best response.

- If Player 1 does not chooses trigger strategy, then we consider the following two cases:
  - If Player 1 always chooses the cooperative action B in every stage game (it is a strategy for Player 1, but not the trigger strategy), then the payoff is as same as the payoff when she/he chooses trigger strategy.[2]

[2]Thanks for Mr. Yusheng Luo for pointing out this issue.

- If Player 1 chooses the non-cooperative action A in some stage, without loss of generality, we assume that the t-th stage is the first stage when Player 1 chooses A, then she/he can get 5 at this stage.
  From the $(t+1)$-th stage on, Player 2 will play non-cooperative action L to punish Player 1 in each stage. Thus Player 1 will receive 1 in each of the subsequent stages since A and B are indifferent, and the t-th stage's present value of his payoff from the t-th stage onwards is

$$5 + \delta_1 + \delta_1^2 + \cdots = 5 + \frac{\delta_1}{1-\delta_1}.$$

  It is easy to understand when looking at the following table, where $*$ means we do not know exactly the action of Player 1 at that stage.

Stage	1	$\cdots$	$t-1$	t	$t+1$	$t+2$	$t+3$	$\cdots$
Player 1	B	$\cdots$	B	A	$*$	$*$	$*$	$\cdots$
Player 2	R	$\cdots$	R	R	L	L	L	$\cdots$
Player 1's payoff	4	$\cdots$	4	5	1	1	1	$\cdots$

- If Player 1 chooses trigger strategy T_1, then in each stage she/he will choose the cooperative action B, and receive 4. Hence, the t-th stage's present value of his payoff from the t-th stage onwards is

$$4 + \delta_1 4 + \delta_1^2 4 + \cdots = \frac{4}{1-\delta_1}.$$

- In order for Player 1 to play trigger strategy T_1, we should have

$$\frac{4}{1-\delta_1} \geq 5 + \frac{\delta_1}{1-\delta_1},$$

  that is $(1 >)\delta_1 \geq \frac{1}{4}$.

Assume that Player 1 chooses the trigger strategy T_1. We want to find the condition which guarantees the trigger strategy T_2 to be Player 2's best response.

- If Player 2 does not choose the trigger strategy, then we consider the following two cases:
  - If Player 2 always chooses the cooperative action R in every stage game (it is a strategy for Player 2, but not the trigger strategy), then the payoff is as same as the payoff when she/he chooses trigger strategy.
  - If Player chooses L in some stage, without loss of generality, we assume that the t-th stage is the first stage when Player 2 chooses L, then she/he can get 8 at this stage.
    From the $(t+1)$-th stage on, Player 1 will play the non-cooperative action A to punish Player 2 in each stage. Thus Player 2 will receive at most 2 in each of the subsequent stages, and the t-th stage's present value of his payoff from the t-th stage onwards is at most

$$8 + \delta_2 2 + \delta_2^2 2 + \cdots = 8 + \frac{2\delta_2}{1-\delta_2}.$$

    It is easy to understand when looking at the following table, where $*$ means we do not know exactly the action of Player 1 at that stage.

Stage	1	$\cdots$	$t-1$	t	$t+1$	$t+2$	$t+3$	$\cdots$
Player 1	B	$\cdots$	B	B	A	A	A	$\cdots$
Player 2	R	$\cdots$	R	L	$*$	$*$	$*$	$\cdots$
Player 2's payoff	6	$\cdots$	6	8	≤ 2	≤ 2	≤ 2	$\cdots$

- If Player 2 chooses the trigger strategy T_2, then she/he will receive 6 in each stage, and the t-th stage's present value of his payoff from the t-th stage onwards is

$$6 + \delta_2 6 + \delta_2^2 6 + \cdots = \frac{6}{1-\delta_2}.$$

- In order for Player 2 to play trigger strategy T_2, we should have

$$\frac{6}{1-\delta_2} \geq 8 + \frac{2\delta_2}{1-\delta_2},$$

that is $(1 >)\delta_2 \geq \frac{1}{3}$.

(ii) In an infinitely repeated game, a subgame is characterized by its previous history. The subgames can be grouped as follows:

(a) Subgames whose previous histories are always finite sequence of (B, R).

(b) Subgames whose previous histories contain other outcomes different from (B, R).

If the trigger strategy is played in the original game, then:

(A) In (a), (T_1, T_2) is played, which is a Nash equilibrium in the subgame;

(B) In (b), (NC_1, NC_2) is played, which is a Nash equilibrium in the subgame.

Therefore, the trigger strategy Nash equilibrium in the original game constitutes a Nash equilibrium in every subgame, i.e., (T_1, T_2) in (a), (NC_1, NC_2) in (b), and hence it is a subgame-perfect Nash equilibrium.

(iii) The strategy profile constitutes a Nash equilibrium in every stage game is a subgame-perfect Nash equilibrium.

(iv) The modified trigger strategies cannot constitute a Nash equilibrium if to play B (L) instead of A (L). The reason is because Player 2 will have an incentive to play L instead R in the first period.

$\square$

Exercise 4. *Suppose there are n firms in a Cournot oligopoly. Inverse demand is given by $P(Q) = a - Q$, where $Q = q_1 + \cdots + q_n$ and q_i is the quantity to be produced by firm i. Each firm has a constant marginal cost of production, c, and no fixed cost. Consider the infinitely repeated game based on this stage game.*

(i) What is the lowest value of δ such that the firms can use trigger strategies to sustain the monopoly output level in a subgame-perfect Nash equilibrium?

(ii) How does the answer vary with n?

Solution. Calculate Firm i's production and profit in the collusion, Cournot competition, and deviation from punishment cases, respectively:

- Cooperative production and profit: In the collusion, the production is $q_i^c = \frac{a-c}{2n}$, and profit is $\pi_i^c = \frac{(a-c)^2}{4n}$;

- Non-cooperative production and profit: In the Cournot competition, production is $q_i^m = \frac{a-c}{n+1}$, and profit is $\pi_i^m = \frac{(a-c)^2}{(n+1)^2}$;
- Deviation production and profit: For each $j \neq i$, Firm j produces $q_j^c = \frac{a-c}{2n}$, then Firm i can increases its profit by producing $q_i^d = \frac{(n+1)(a-c)}{4n}$, and profit is $\pi_i^d = \frac{(n+1)^2(a-c)^2}{(4n)^2}$.

For each i, consider the following trigger strategy T_i for Firm i:

- In the first stage produce q_i^c.
- In the t-th stage ($t > 1$), produce q_i^c if every Firm j has produced q_j^c in each of the $t-1$ previous stages; otherwise, produce q_i^m.

(i) Fix Firm i, and assume that each other Firm $j \neq i$ chooses the trigger strategy T_j. We want to find the condition which guarantees the trigger strategy T_i to be Firm i's best response.

- If Firm i does not choose the trigger strategy, then we consider the following two cases:
  - If Firm i always chooses the cooperative production q_i^c in every stage game (it is a strategy for Firm i, but not the trigger strategy), then the payoff is as same as the payoff when it chooses trigger strategy.
  - If Firm i deviates in some stage and the profit maximizer is q_i^d. Without loss of generality, we assume that the t-th stage is the first stage when Firm i deviates, then it can get at most π_i^d at this stage.
    From the $(t+1)$-th stage on, every other Firm j will produce non-cooperative production q_j^m. Thus Firm i will receive at most π_i^m in each of the subsequent stages, and the t-th stage's present value of its payoff from the t-th stage onwards is at most

    $$\pi_i^d + \delta\pi_i^m + \delta^2\pi_i^m + \cdots = \pi_i^d + \frac{\delta\pi_i^m}{1-\delta}.$$

    It is easy to understand when looking at the following table, where $*$ means we do not know exactly the action of Firm i at that stage.

Stage	1	$\cdots$	$t-1$	t	$t+1$	$t+2$	$t+3$	$\cdots$
Firm $j \neq i$	q_j^c	$\cdots$	q_j^c	q_j^c	q_j^m	q_j^m	q_j^m	$\cdots$
Firm i	q_i^c	$\cdots$	q_j^c	q_i^d	$*$	$*$	$*$	$\cdots$
Firm i's payoff	π_i^c	$\cdots$	π_i^c	π_i^d	$\leq \pi_i^m$	$\leq \pi_i^m$	$\leq \pi_i^m$	$\cdots$

- If Firm i chooses the trigger strategy T_i, then it will receive π_i^c in each stage, and the t-th stage's present value of its payoff from the t-th stage onwards is

$$\pi_i^c + \delta\pi_i^c + \delta^2\pi_i^c + \cdots = \frac{\pi_i^c}{1-\delta}.$$

- In order for Firm i to play trigger strategy T_i, we should have

$$\frac{\pi_i^c}{1-\delta} \geq \pi_i^d + \frac{\delta\pi_i^m}{1-\delta},$$

that is $\delta \geq \frac{(n+1)^2}{(n+1)^2+4n}$.

(ii) Since $\lim_{n\to\infty} \frac{(n+1)^2}{(n+1)^2+4n} = 1$, the lowest value of δ approaches 1. That is, as n increases, a larger δ is required to deter the deviation. In other words, there is more incentive to deviate the trigger strategy.

□

Exercise 5. *Find $\delta > 0$ such that the trigger strategy is a subgame-perfect Nash equilibrium for the game which infinitely repeats the stage game of Bertrand model with homogeneous products described in the lecture.*

Solution. Leave as Question 1 of Assignment 3. □

Exercise 6. *Consider a Cournot duopoly operating in a market with inverse demand $P(Q) = a - Q$, where $Q = q_1 + q_2$ is the aggregate quantity on the market. Both firms have total costs $c_i(q_i) = cq_i$, but demand is uncertain: it is high ($a = a_H$) with probability θ and low ($a = a_L$) with probability $1 - \theta$. Furthermore, information is asymmetric: firm 1 knows whether demand is high or low, but firm 2 does not. All of this is common knowledge. The two firms simultaneously choose quantities. What are the strategy spaces for the two firms? What is the Bayesian Nash equilibrium of this game, (assuming a_H, a_L, θ and c are such that all equilibrium quantities are positive)?*

Solution.

- Firm i's action space is $\{q\colon q \geq 0\}$.
- Firm 1's type space $T_1 = \{H, L\}$; Frim 2 has only one type.
- Strategy space: $S_1 = \{(q_{1H}, q_{1L})\colon q_{1H}, q_{1L} \geq 0\}$, and $S_2 = \{q_2\colon q_2 \geq 0\}$.

Suppose that $((q_{1H}^*, q_{1L}^*), q_2^*)$ is a Bayesian Nash equilibrium, then by definition we will have:

- If the demand is high, Firm 1 will choose q_{1H}^* to maximize its payoff
$$q_{1H}[a_H - c - q_2^* - q_{1H}],$$
which is a concave function, and hence
$$q_{1H}^* = \frac{a_H - c - q_2^*}{2}. \tag{1}$$
- If the demand is low, Firm 1 will choose q_{1L}^* to maximize its payoff
$$q_{1L}[a_L - c - q_2^* - q_{1L}],$$
which is a concave function, and hence
$$q_{1L}^* = \frac{a_L - c - q_2^*}{2}. \tag{2}$$
- Firm 2 does not know the exact type of the demand, so it will choose q_2^* to maximize its expected payoff
$$\theta q_2[a_H - c - q_{1H}^* - q_2] + (1-\theta)q_2[a_L - c - q_{1L}^* - q_2],$$
and hence
$$q_2^* = \frac{\theta(a_H - q_{1H}^*) + (1-\theta)(a_L - q_{1L}^*) - c}{2}.[3] \tag{3}$$

[3] I correct a typo here. Thanks for a student, and I am sorry for not remebering her name.

Combining Equations (1), (2) and (3), we get

$$\begin{aligned} q_{1H}^* &= \frac{a_H - c}{2} - \frac{\theta a_H + (1-\theta)a_L - c}{6}, \\ q_{1L}^* &= \frac{a_L - c}{2} - \frac{\theta a_H + (1-\theta)a_L - c}{6}, \\ q_2^* &= \frac{\theta a_H + (1-\theta)a_L - c}{3}. \end{aligned}$$

□

End of Solution to Tutorial 5