Solution to Tutorial 6

2012/2013 Semester I MA4264 Game Theory

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1 Review

Static game of incomplete information

- The normal-form representation of an n-player static Bayesian game:
$$\{A_1, \ldots, A_n; T_1, \ldots, T_n; \mathbf{P}_1, \ldots, \mathbf{P}_n; u_1, \ldots, u_n\}.$$
- A strategy for player i is a function $s_i \colon T_i \to A_i$.
- Player i's expected payoff when her/his type is t_i:
$$\mathbb{E}_{t_{-i}} u_i(s^*_{-i}(t_{-i}), a_i; t_i) = \sum_{t_{-i} \in T_{-i}} \mathbf{P}(t_{-i} \mid t_i) u_i(s^*_{-i}(t_{-i}), a_i; t_i).$$
- Bayesian Nash equilibrium: In the static Bayesian game
$$G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; \mathbf{P}_1, \ldots, \mathbf{P}_n; u_1, \ldots, u_n\},$$
the strategy profile $s^* = (s^*_1, \ldots, s^*_n)$ is a (pure-strategy) Bayesian Nash equilibrium if for each player i and for each of i's type t_i in T_i, $s^*_i(t_i)$ maximizes Player i's expected payoff.
- A static game of incomplete information can be transferred to a game of complete information.

2 Tutorial

Exercise 1. *Consider the following asymmetric-information model of Bertrand duopoly with differentiated products. Demand for firm i is $q_i(p_i, p_j) = a - p_i + b_i \cdot p_j$. Costs are zero for both firms. The sensitivity of firm i's demand to firm j's price is either high or low. That is, b_i is either b_H or b_L, where $b_H > b_L > 0$. For each firm, $b_i = b_H$ with probability θ and $b_i = b_L$ with probability $1 - \theta$, independent of the realization of b_j. Each firm knows its own b_i but not its competitor's. All of this is common knowledge. What are the action spaces, type spaces, beliefs, and utility functions in this game? What are the strategy spaces? Assume that $\theta b_H + (1-\theta) b_L < 2$. Find the pure-strategy Bayesian Nash equilibrium of this game.*

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Solution.

- Firm i's action space: $A_i = \{p\colon p \geq 0\}$.
- Firm i's type space: $T_i = \{H, L\}$.
- Firm i's beliefs: $\theta H + (1-\theta)L$.
- Firm i's strategy space: $S_i = \{(p_{iH}, p_{iL})\colon p_{iH}, p_{iL} \in A_i\}$.
- Firm i's utility function (for type t): $[a - p_{it} + b_t(\theta p_{jH} + (1-\theta)p_{jL})]p_{it}$.
- For type $t = H, L$, Firm i's maximization problem:

$$\max_{p_{it}} [a - p_{it} + b_t(\theta p_{jH} + (1-\theta)p_{jL})]p_{it}.$$

By the first order condition, we have

$$a - 2p_{it} + b_t(\theta p_{jH} + (1-\theta)p_{jL}) = 0.$$

That is, for $i = 1, 2$,

$$\begin{aligned} p_{iH} &= \frac{a}{2} + \frac{b_H(\theta p_{jH} + (1-\theta)p_{jL})}{2}, \\ p_{iL} &= \frac{a}{2} + \frac{b_L(\theta p_{jH} + (1-\theta)p_{jL})}{2}. \end{aligned}$$

Let $b = \theta b_H + (1-\theta)b_L$. Then we have

$$\begin{aligned} p_{iH} &= \frac{a}{2} + \frac{ab_H}{4} + bb_H\frac{\theta p_{iH} + (1-\theta)p_{iL}}{4}, \\ p_{iL} &= \frac{a}{2} + \frac{ab_L}{4} + bb_L\frac{\theta p_{iH} + (1-\theta)p_{iL}}{4}. \end{aligned}$$

Therefore, for $i = 1, 2$,

$$\begin{aligned} p_{iH} &= \frac{1}{1-\frac{1}{4}b^2}\left[\frac{1}{2}a(1+\frac{1}{2}b_H) + \frac{1-\theta}{8}ab(b_H - b_L)\right], \\ p_{iL} &= \frac{1}{1-\frac{1}{4}b^2}\left[\frac{1}{2}a(1+\frac{1}{2}b_L) + \frac{\theta}{8}ab(b_H - b_L)\right]. \end{aligned}$$

□

Exercise 2. *Find all the pure-strategy Bayesian Nash equilibria in the following static Bayesian game:*

(i) Nature determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.

	L	R
T	$1,1$	$0,0$
B	$0,0$	$0,0$

Game 1

	L	R
T	$0,0$	$0,0$
B	$0,0$	$2,2$

Game 2

(ii) Player 1 learns whether nature has drawn Game 1 or Game 2, but Player 2 does not.

(iii) Player 1 chooses either T or B; Player 2 simultaneously chooses either L or R.

(iv) Payoffs are given by the game drawn by nature.

Solution.

- There are two players: Player 1 and Player 2;
- Type spaces: $T_1 = \{1, 2\}$, and $T_2 = \{\{1, 2\}\}$;
- Believes: Player 1's belief on Player 2's type is 1 on $\{T, B\}$, and Player 2's belief on Player 1's types is 1/2 on T and 1/2 on B;
- Action spaces: $A_1 = \{T, B\}$, and $A_2 = \{L, R\}$;
- Strategy spaces: $S_1 = \{TT, TB, BT, BB\}$, and $S_2 = \{L, R\}$.

Now we will find the best-response correspondence for each player and each associated type: let a_1, a_2 be Player 1's actions in Game 1 and Game 2, respectively, b Player 2's action.

- If Game 1 is drawn by Nature, then Player 1's best-response correspondence is

$$a_1^*(b) = \begin{cases} \{T\}, & \text{if } b = L; \\ \{T, B\}, & \text{if } b = R. \end{cases}$$

- If Game 2 is drawn by Nature, then Player 1's best-response correspondence is

$$a_2^*(b) = \begin{cases} \{T, B\}, & \text{if } b = L; \\ \{B\}, & \text{if } b = R. \end{cases}$$

- Since Player 2 does not know which game is being drawn, he will choose b to maximize his expected payoff. The following table is Player 2's expected payoff table:

	L	R
TT	1/2	0
TB	1/2	1
BT	0	0
BB	0	1

Thus we get Player 2's best-response correspondence:

$$b^*(a_1, a_2) = \begin{cases} \{L\}, & \text{if } a_1a_2 = TT; \\ \{R\}, & \text{if } a_1a_2 = TB; \\ \{L, R\}, & \text{if } a_1a_2 = BT; \\ \{R\}, & \text{if } a_1a_2 = BB. \end{cases}$$

Therefore, by definition, we will get all the Bayesian Nash equilibria: (TT, L), (TB, R) and (BB, R). The reason is as follows:

- If Player 2 plays L, then Player 1 must play L in Game 1 (and Player 1 is indifferent between T and B in Game 2). Note that, if Player 1 plays B in Game 2, then Player 2 must play R.

  So, given that Player 2 plays L, the only possible pure-strategy Bayesian Nash equilibrium is (TT, L) in this case.

- If Player 2 plays R, then Player 1 must play B in Game 2 (and Player 1 is indifferent between T and B in Game 1). Note that, R is Player 2's best response for TB and BB.

  So, given that Player 2 plays R, there are two pure-strategy Bayesian Nash equilibria: (TB, R) and (BB, R).

□

Exercise 3. *The worker has an outside opportunity v known by himself. The firm believes that $v = 6$ and $v = 10$ with probabilities $2/3$ and $1/3$ respectively. A wage $w = 8$ is preset by the union. The firm and the worker simultaneously announce whether to accept or reject the wage. The worker will be employed by the firm if and only if both of them accept the wage. If the firm accepts the wage, its payoff is 3 if the worker is employed and -1 otherwise. If the firm rejects the wage, then its payoff is 0 regardless the worker's action. The worker's payoff is w if he is employed and v otherwise. Find the Bayesian Nash equilibria. Depict the extensive-form representation in which Nature draws the outside opportunity for the worker.*

Solution. Let Game 1 and Game 2 be as follows:

Worker \ Firm	A	R
A	$8, 3$	$6, 0$
R	$6, -1$	$6, 0$

Game 1, $v = 6$

Worker \ Firm	A	R
A	$8, 3$	$10, 0$
R	$10, -1$	$10, 0$

Game 2, $v = 10$

- There are two players: firm and worker;
- Type spaces: $T_f = \{\{1, 2\}\}$, and $T_w = \{1, 2\}$;
- Believes: work's belief on firm's type is 1 on $\{1, 2\}$, and firm's belief on work's types is $2/3$ on 1 and $1/3$ on 2;
- Action spaces: $A_w = A_f = \{A, R\}$;
- Strategy spaces: $S_f = \{A, R\}$ and $S_w = \{AA, AR, RA, RR\}$.

Now we will find the best-response correspondence for each player and each associated type: let a_1 and a_2 be worker's actions in Game 1 and Game 2, respectively, b firm's action.

- If Game 1 is drawn by Nature, then worker's best-response correspondence is

$$a_1^*(b) = \begin{cases} \{A\}, & \text{if } b = A; \\ \{A, R\}, & \text{if } b = R. \end{cases}$$

- If Game 2 is drawn by Nature, then worker's best-response correspondence is

$$a_2^*(b) = \begin{cases} \{R\}, & \text{if } b = A; \\ \{A, R\}, & \text{if } b = R. \end{cases}$$

		Firm A	Firm R
Worker	AA	3	0
	AR	5/3	0
	RA	1/3	0
	RR	-1	0

- Since firm does not know which game is being drawn, it will choose b to maximize its expected payoff. The following table is firm's expected payoff table:

  Thus we get firm's best-response correspondence is

$$b^*(a_1, a_2) = \begin{cases} \{A\}, & \text{if } a_1a_2 = AA; \\ \{A\}, & \text{if } a_1a_2 = AR; \\ \{A\}, & \text{if } a_1a_2 = RA; \\ \{R\}, & \text{if } a_1a_2 = RR. \end{cases}$$

Therefore, by definition, we will get all the Bayesian Nash equilibria: (AR, A) and (RR, R). The reason is as follows:

- If firm chooses A, then worker should choose A and R in Game 1 and Game 2, respectively. Note that, if worker chooses AR, then firm should choose A.

  So, given that firm chooses A, the only possible pure-strategy Bayesian Nash equilibrium is (AR, A).

- If firm chooses R, then worker can choose any strategy in each game. Note that, only when worker chooses RR, R is firm's best response. So, given that firm chooses R, the only possible pure-strategy Bayesian Nash equilibrium is (RR, R).

□

Exercise 4. *Consider the following static Bayesian game.*

- *Nature selects Game 1 with probability* $1/3$*, Game 2 with probability* $1/3$ *and Game 3 with probability* $1/3$.
- *Player I learns whether Nature has selected Game 1 or not; Player II learns whether Nature has selected Game 2 or not.*
- *Players I and II simultaneously choose their actions: Player I either* T *or* B*, and Player II either* L *or* R.
- *Payoffs are given by the game selected by Nature.*

	L	R
T	$0, 0$	$6, -1$
B	$-1, 6$	$4, 4$

Game 1

	L	R
T	$1, 3$	$0, 0$
B	$0, 0$	$3, 1$

Game 2

	L	R
T	$2, -2$	$-2, 2$
B	$-2, 2$	$2, -2$

Game 3

All of this is common knowledge. Find all the pure-strategy Bayesian Nash equilibria.

Solution. Leave as Question 2 of Assignment 3. □

Exercise 5. *Consider a first-price, sealed-bid auction in which the bidders' valuations are independently and uniformly distributed on* $[0,1]$. *Show that if there are* n *bidders, then the strategy of bidding* $(n-1)/n$ *times one's valuation is a symmetric Bayesian Nash equilibrium of this auction.*

Proof.

- There are n players;
- Type spaces: $T_i = [0,1]$, that is, each $t_i \in T_i$ is a valuation;
- Action spaces: $A_i = [0,1]$, that is, each $a_i \in A_i$ is a bid;
- Strategy spaces: $S_i = \{s_i \colon T_i \to A_i\}$;
- Payoff:

$$u_i(a_i, a_{-i}, t_i) = \begin{cases} t_i - a_i, & \text{if } a_i > a_j, \forall j \neq i; \\ \frac{t_i - a_i}{k}, & \text{if } a_i \text{ is one of the } k \text{ largest bids;} \\ 0, & \text{otherwise.} \end{cases}$$

- Aim: show that $(s_1^*, s_2^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium, where $s_i^*(t_i) = \frac{n-1}{n} t_i$. It suffices to show that for each Player i and each associated type t_i, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \mathbb{E}_{t_{-i}} u_i(s_{-i}^*(t_{-i}), a_i; t_i),$$

where

$$\begin{aligned} \mathbb{E}_{t_{-i}} u_i(s_{-i}^*(t_{-i}), a_i; t_i) &= \sum_{t_{-i} \in T_{-i}} \mathbf{P}_i(t_{-i}|t_i) \times u_i(s_{-i}^*(t_{-i}), a_i; t_i) \\ &= (t_i - a_i) \times \text{Prob}(a_i > s_j^*(t_j),\ \forall j \neq i) \\ &\quad + \sum_{k=2}^{n} \frac{t_i - a_i}{k} \times \text{Prob}(a_i \text{ is one of the } k \text{ largest bids}) \end{aligned}$$

By computation, we have

$$\begin{aligned} &\text{Prob}(a_i \text{ is one of the } k \text{ largest bids}) \\ \leq\ &\text{Prob}(\text{Player } i \text{ shares the winner of the auction with another player, say Player } j) \\ =\ &\text{Prob}(s_j^*(t_j) = a_i) = \text{Prob}(t_j = a_i \frac{n}{n-1}) = 0 \end{aligned}$$

Note that here we use the fact $\text{Prob}(t_j = \ell) = 0$ for any $\ell \in [0,1]$ since t_j is uniformly distributed on $[0,1]$.

Moreover, we have

$$\begin{aligned} &\text{Prob}(a_i > s_j^*(t_j),\ \forall j \neq i) \\ =\ &\text{Prob}\left(a_i > \frac{n-1}{n} t_j,\ \forall j \neq i\right) && \text{definition of } s_j^*(t_j) \\ =\ &\Pi_{j \neq i} \text{Prob}\left(a_i > \frac{n-1}{n} t_j\right) && \text{independence} \\ =\ &\Pi_{j \neq i} \text{Prob}\left(t_j < \frac{n}{n-1} a_i\right) \end{aligned}$$

When $a_i \geq \frac{n-1}{n}$, $\Pi_{j\neq i}\,\text{Prob}\left(t_j < \frac{n}{n-1}a_i\right) = 1$, so Player i's expected payoff is $t_i - a_i$, and hence the maximizer is $\frac{n-1}{n}$.[1]

When $a_i \leq \frac{n-1}{n}$,

$$\begin{aligned} &\Pi_{j\neq i}\,\text{Prob}\left(t_j < \frac{n}{n-1}a_i\right) \\ =\ &\Pi_{j\neq i}\left(\frac{n}{n-1}a_i\right) = \left(\frac{n}{n-1}a_i\right)^{n-1} \qquad \text{uniform distribution} \end{aligned}$$

Therefore the expected payoff of Player i is

$$\left(\frac{n}{n-1}\right)^{n-1} a_i^{n-1}(t_i - a_i),$$

and the unique maximizer is $\frac{n-1}{n}t_i = s_i^*(t_i)$.

Therefore, the global maximizer is $\frac{n-1}{n}t_i = s_i^*(t_i)$, and every Player i's strategy $s_i^*(t_i) = \frac{n-1}{n}t_i$ constitutes a (symmetric) Bayesian Nash equilibrium. □

Exercise 6. *There are 2 players who were at the scene where a crime was committed. But neither player knows whether she has been the only witness to the crime, or whether there was another witness as well. Let π be the probability with which each player believes the other player is a witness. Each player, if she is a witness, can call the police or not. The payoff to Player i is 2/3 if she calls the police, 1 if someone else calls the police, and 0 if nobody calls.*

(i) Write down each player's types and strategies.

(ii) For each value of $\pi \in [0,1]$, find the Bayesian Nash equilibria.

Solution. (i) Since each player knows that he is in the crime scene, each one has only one type: Player 1's type is "Player 1 is a witness", and Player 2's type is "Player 2's type is a witness". There is no possibility that they are not in the crime scene.[2]

However, they don't know whether the other person is also in the crime scene or not. Hence, what they are uncertain about is the other player's type.

Each Player i has one types: t_i = "on the scene". For $\pi \in [0,1]$, each Player i has two strategies C (call) and N (not call).

(ii) Each Player i thinks that he is playing the following games:

- Game 1: if Player i thinks that Player j is also on the spot (probability π). Then Player i's payoff table is as follows:

		Player j	
		C	N
Player i	C	2/3	2/3
	N	1	0

Game 1: Player j is on the scene

- Game 2: if Player i thinks that Player j is not on the spot (probability $1-\pi$). Then Player i think that he will get 2/3 if he chooses C, and 0 otherwise, no matter what Player j chooses.

[1] Thanks for Mr. Yusheng Luo for pointing out this issue.

[2] Another acceptable solution is: Player i's type space is {Player i is a witness, Player i is not a witness}. While there is no available action when the type is "Player i is not a witness".

Player j

		C	N
Player i	C	$2/3$	$2/3$
	N	0	0

Game 2: Player j is not on the scene

Therefore, Player i's expected payoff is in the payoff table G_1, and the game in fact can be represented by the payoff table G_2.

Player j

		C	N
Player i	C	$2/3$	$2/3$
	N	π	0

G_1

Player j

		C	N
Player i	C	$2/3, 2/3$	$2/3, \pi$
	N	$\pi, 2/3$	$0, 0$

G_2

Thus the Bayesian Nash equilibria are as follows:

- If $2/3 > \pi \geq 0$, then there is only one Bayesian Nash equilibrium (C, C);
- If $\pi = 2/3$, then there are three Bayesian Nash equilibria (C, C), (C, N) and (N, C);
- If $1 \geq \pi > 2/3$, then there are two Bayesian Nash equilibria (C, N) and (N, C).

$\square$

End of Solution to Tutorial 6