

Solution to Tutorial 8*

2011/2012 Semester I

MA4264

Game Theory

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1 Review

A **perfect Bayesian equilibrium** consists of strategies and beliefs satisfying Requirements 1 through 4.

- Requirement 1: At each information set, the Player with the move must have a belief about which node in the information set has been reached by the play of the game.
- Requirement 2: Given their beliefs, the players' strategies must be sequentially rational.
- Requirement 3: At information sets on the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies.
- Requirement 4: At information sets off the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies where possible.

For the dynamic games, we have

$$\text{PBE} \subseteq \text{SPE} \subseteq \text{NE}$$

which gives us a standard method to find perfect Bayesian equilibrium.

2 Tutorial

Exercise 1. *The following static game of complete information (Matching Pennies) has no pure-strategy Nash equilibrium but has one mixed-strategy Nash equilibrium: each player plays H with probability 1/2. Provide a pure-strategy Bayesian Nash equilibrium of a corresponding game of incomplete information such that as incomplete information disappears, the players' behavior in the Bayesian Nash equilibrium approaches their behavior in the mixed-strategy Nash equilibrium in the original game of complete information.*

*Corrections are always welcome.

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	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Game G

Solution. Consider the following game with incomplete information $G(x)$: where

	H	T
H	$1 + t_1, -1$	$-1, 1 - t_2$
T	$-1, 1$	$1, -1$

Game $G(x)$

- Type spaces: $T_1 = T_2 = [0, x]$, t_1 and t_2 are i.i.d. random variables and uniformly distributed on $[0, x]$.
- Action spaces: $A_1 = A_2 = \{H, T\}$.
- Strategy spaces: $S_1 = S_2 = \{s_i \text{ is a function from } [0, x] \text{ to } \{H, T\}\}$.

Note that $G(0) = G$.

In $G(x)$, suppose (s_1^*, s_2^*) is a Bayesian Nash equilibrium, $p = \text{Prob}\{t_1 : s_1^*(t_1) = H\}$, and $q = \text{Prob}\{t_2 : s_2^*(t_2) = H\}$.

- For Player 1, given his type t_1 and Player 2's strategy s_2^* , his expected payoff is

$$\mathbb{E}[u_1(a_1, s_2^*)|t_1] = \begin{cases} (1 + t_1) \cdot q - 1 \cdot (1 - q), & a_1 = H; \\ -1 \cdot q + 1 \cdot (1 - q), & a_1 = T. \end{cases}$$

Thus H is a best response if and only if $(1 + t_1) \cdot q - 1 \cdot (1 - q) \geq -1 \cdot q + 1 \cdot (1 - q)$, that is, $t_1 \geq \frac{2}{q} - 4$. Hence, we have

$$p = \text{Prob}\{t_1 : s_1^*(t_1) = H\} = 1 - \frac{2/q - 4}{x} \quad (1)$$

- For Player 2, given his type t_2 and Player 1's strategy s_1^* , his expected payoff is

$$\mathbb{E}[u_2(a_2, s_1^*)|t_2] = \begin{cases} -1 \cdot p + 1 \cdot (1 - p), & a_2 = H; \\ (1 - t_2) \cdot p + (-1) \cdot (1 - p), & a_2 = T. \end{cases}$$

Thus H is a best response if and only if $-1 \cdot p + 1 \cdot (1 - p) \geq (1 - t_2) \cdot p + (-1) \cdot (1 - p)$, that is, $t_2 \geq 4 - \frac{2}{p}$. Hence, we have

$$q = \text{Prob}\{t_2 : s_2^*(t_2) = H\} = 1 - \frac{4 - 2/p}{x} \quad (2)$$

Rewriting Equations (1) and (2), we will have

$$p = \frac{2}{4 + (q - 1)x}, \quad q = \frac{2}{4 + (1 - p)x}.$$

As $x \rightarrow 0$, $p, q \rightarrow \frac{1}{2}$, that is, the Bayesian Nash equilibrium will converge to the mixed-strategy Nash equilibrium in G . \square

Exercise 2. In the following extensive-form games, derive the normal-form game and find all the pure-strategy Nash, subgame-perfect, and perfect Bayesian equilibria.

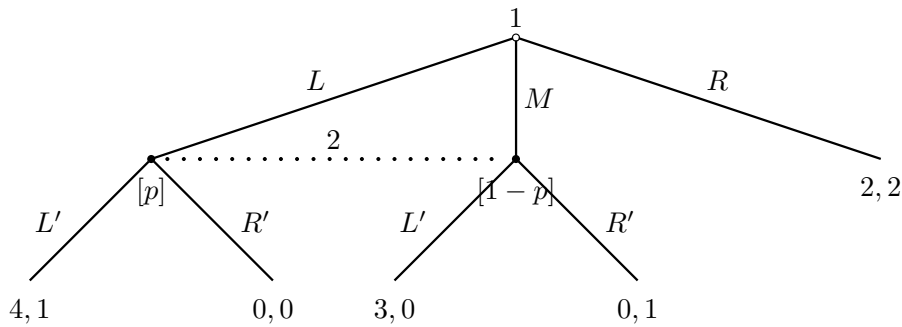


Figure 1: Game (a)

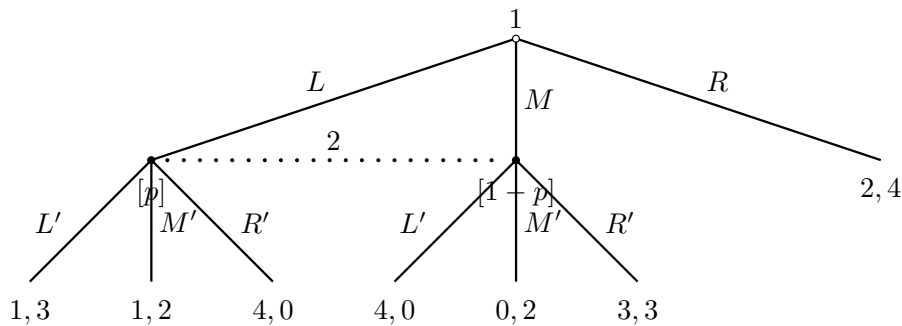


Figure 2: Game (b)

Solution. (a) Game (a). Normal-form representation is as follows:

- $S_1 = \{L, M, R\}$, $S_2 = \{L', R'\}$.
- Payoff table:

		Player 2	
		L'	R'
Player 1	L	$4, 1$	$0, 0$
	M	$3, 0$	$0, 1$
	R	$2, 2$	$2, 2$

- (i) There are two pure-strategy Nash equilibria (L, L') and (R, R') .
- (ii) Since there is no subgame, every Nash equilibrium is subgame-perfect, and hence (L, L') and (R, R') are all the subgame-perfect Nash equilibria.
- (iii) To check whether (L, L') is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.
- Requirement 1: For Player 2's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
 - Requirement 2: To support L' to be a best response for Player 2, we should take $p \geq \frac{1}{2}$.
 - Requirement 3: Since Player 1 chooses L , by Bayes' rule, Player 2's belief should be $(1, 0)$, that is, $p = 1$.
 - Requirement 4: No information set is off the path, so Requirement 4 gives no restriction on p .

Hence (L, L') with $p = 1$ is a perfect Bayesian equilibrium.

To check whether (R, R') is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.

- Requirement 1: For Player 2's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
- Requirement 2: To support R' to be a best response for Player 2, we should take $p \leq \frac{1}{2}$.
- Requirement 3: No nontrivial information set is on the path, so Requirement 3 gives no restriction on p .
- Requirement 4: Since Player 1 chooses R , Player 2's information set is off the path, so p could be arbitrary.

Hence (R, R') with $p \leq \frac{1}{2}$ is a perfect Bayesian equilibrium.

(b) Game (b). Normal-form representation is as follows:

- $S_1 = \{L, M, R\}$, $S_2 = \{L', M', R'\}$.
- Payoff table:

		Player 2		
		L'	M'	R'
Player 1	L	1, 3	1, 2	4, 0
	M	4, 0	0, 2	3, 3
	R	2, 4	2, 4	2, 4

- (i) (R, M') is the unique pure-strategy Nash equilibrium.
- (ii) Since there is no subgame, every Nash equilibrium is subgame-perfect, and hence (R, M') is the unique subgame-perfect Nash equilibrium.
- (iii) To check whether (R, M') is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.
 - Requirement 1: For Player 2's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
 - Requirement 2: To support M' to be a best response for Player 2, we should take $p \in [\frac{1}{3}, \frac{2}{3}]$.
 - Requirement 3: No nontrivial information set is on the path, so Requirement 3 gives no restriction on p .
 - Requirement 4: Since Player 1 chooses R , Player 2's information set is off the path, so p could be arbitrary.

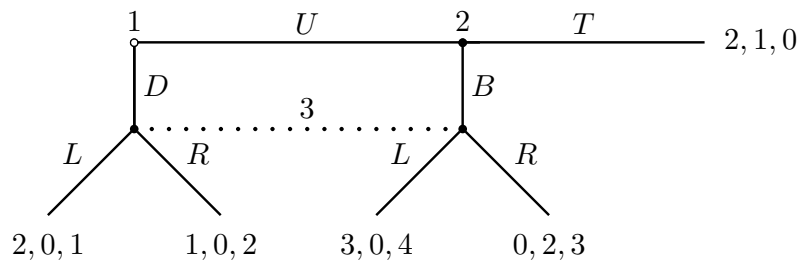
Hence (R, M') with $p \in [\frac{1}{3}, \frac{2}{3}]$ is the unique perfect Bayesian equilibrium.

□

Exercise 3. Consider the following game between three Players:

- Player 1 moves first. He has two actions: U and D . Action U gives the next move to Player 2, action D gives the next move to Player 3.
- If Player 2 is given the move he also has two actions: T and B . Action T ends the game, action B gives the move to Player 3.
- If Player 3 is given the move he also has two actions: L and R . Both actions end the game. Player 3 does not know whether the move was given to him by Player 1 or Player 2.

The extensive form is given as follows: where the payoff vector (x, y, z) means that



Player 1 receives utility x , Player 2 receives utility y and Player 3 receives utility z .

(a) Find all Nash equilibria.

(b) Find all subgame-perfect Nash equilibria.

(c) Find all perfect Bayesian equilibria.

Solution. (a) The normal-form representation is as follows:

- $S_1 = \{D, U\}$, $S_2 = \{B, T\}$, $S_3 = \{L, R\}$.
- Payoff table:

		Player 2 and Player 3			
		<i>BL</i>	<i>BR</i>	<i>TL</i>	<i>TR</i>
Player 1	<i>D</i>	<i>2, 0, 1</i>	<i>1, 0, 2</i>	<i>2, 0, 1</i>	<i>1, 0, 2</i>
	<i>U</i>	<i>3, 0, 4</i>	<i>0, 2, 3</i>	<i>2, 1, 0</i>	<i>2, 1, 0</i>

There are two pure-strategy Nash equilibria (D, B, R) and (U, T, L) .

(b) There is no subgame, so (D, B, R) and (U, T, L) are all the subgame perfect Nash equilibria.

(c) Assume Player 3's belief is p on D and $1 - p$ on B .

To check whether (D, B, R) is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.

- Requirement 1: For Player 3's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
- Requirement 2: To support R to be a best response for Player 3, we should take $p \geq \frac{1}{2}$.
- Requirement 3: Since Player 1 chooses D , by Bayes' rule, Player 3's belief should be $(1, 0)$, that is, $p = 1$.
- Requirement 4: No nontrivial information set is off the path, so Requirement 4 gives no restriction on p .

Hence (D, B, R) with $p = 1$ is a perfect Bayesian equilibrium.

To check whether (U, T, L) is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.

- Requirement 1: For Player 3's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
- Requirement 2: To support L to be a best response for Player 3, we should take $p \leq \frac{1}{2}$.
- Requirement 3: No nontrivial information set is on the path, so Requirement 3 gives no restriction on p .

- Requirement 4: Since Player 1 and Player 2 choose U and T , respectively, Player 3's belief could be arbitrary since his information set will not be reached.

Hence (U, T, L) with $p \leq \frac{1}{2}$ is a perfect Bayesian equilibrium.

□

Exercise 4. Find all perfect Bayesian equilibria in the following signaling games.

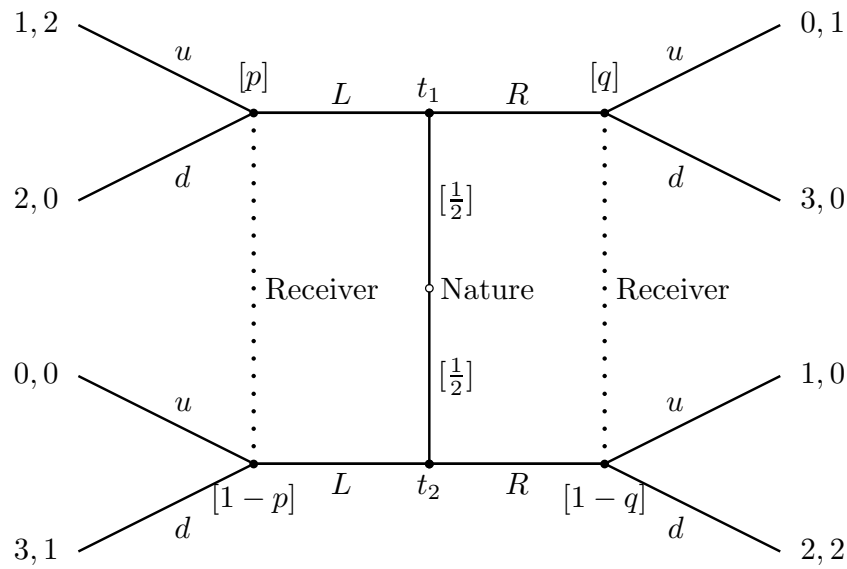


Figure 3: Game 1

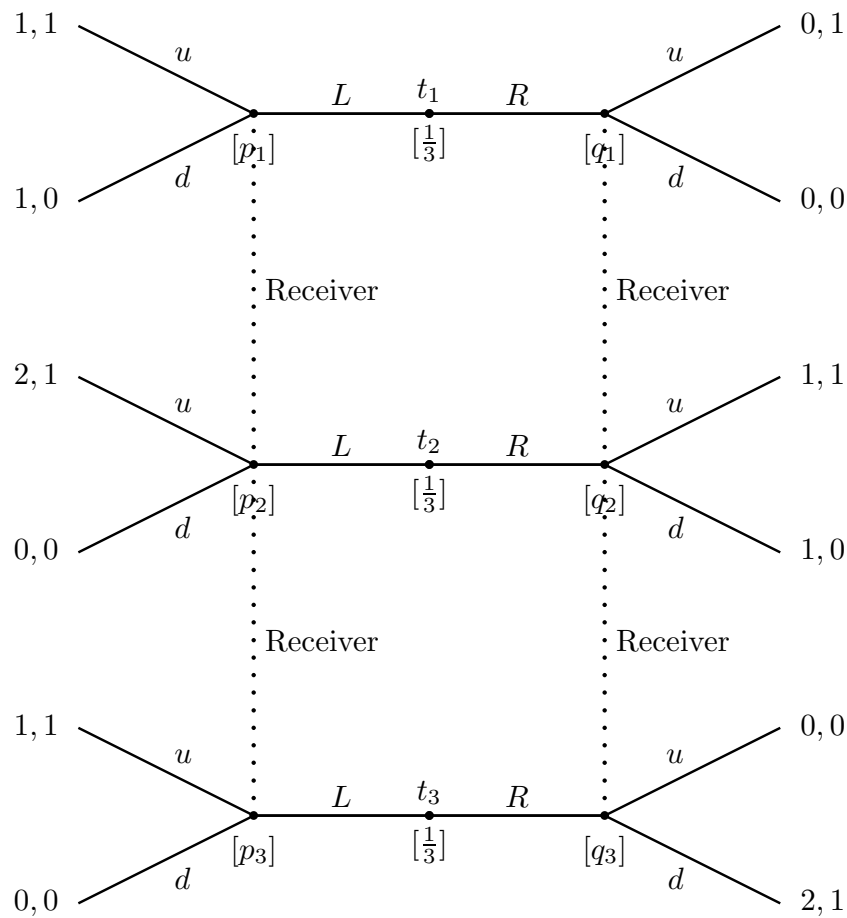


Figure 4: Game 2

Solution. (i) Game 1: The normal-form representation is as follows:

- $T = \{t_1, t_2\}$, $M = \{L, R\}$, $A = \{u, d\}$.
- Payoff table:

		Receiver			
		uu	ud	du	dd
Sender	LL	$1/2, 1$	$1/2, 1$	$5/2, 1/2$	$5/2, 1/2$
	LR	$1, 1$	$3/2, 2$	$3/2, 0$	$2, 1$
	RL	$0, 1/2$	$3/2, 0$	$3/2, 1$	$3, 1/2$
	RR	$1/2, 1/2$	$5/2, 1$	$1/2, 1/2$	$5/2, 1$

For example,

$$\begin{aligned}
 U(RL, du) &= \text{Prob}(t_1)U(R, u|t_1) + \text{Prob}(t_2)U(L, d|t_2) \\
 &= \frac{1}{2}(0, 1) + \frac{1}{2}(3, 1) = (3/2, 1)
 \end{aligned}$$

There is the unique Nash equilibrium (RR, ud) , which is also the unique subgame perfect Nash equilibrium since there is no subgame.

To check whether (RR, ud) is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2S, 2R and 3.

- Requirement 1: Assume the probability distributions on left and right information set are $(p, 1 - p)$ and $(q, 1 - q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since (RR, ud) is a Nash equilibrium)
- Requirement 2R: To support ud (u when Sender chooses L , d when Sender chooses R) to be a best response for Receiver, we should take $p \geq \frac{1}{3}$ and $q \leq \frac{2}{3}$.
- Requirement 3: Since Sender chooses RR , Bayes' rule implies p could be arbitrary, and $q = \frac{1}{2}$.

Hence (RR, ud) with $p \geq \frac{1}{3}$ and $q = \frac{1}{2}$ is a perfect Bayesian equilibrium.

(ii) Game 2: The normal-form representation is as follows:

- $T = \{t_1, t_2, t_3\}$, $M = \{L, R\}$, and $A = \{u, d\}$.
- Payoff table:

		Receiver			
		uu	ud	du	dd
Sender	LLL	$4/3, 1$	$4/3, 1$	$1/3, 0$	$1/3, 0$
	LLR	$1, 2/3$	$5/3, 1$	$1/3, 0$	$1, 1/3$
	LRL	$1, 1$	$1, 2/3$	$2/3, 1/3$	$2/3, 0$
	LRR	$2/3, 2/3$	$4/3, 2/3$	$2/3, 1/3$	$4/3, 1/3$
	RLL	$1, 1$	$1, 2/3$	$0, 1/3$	$0, 0$
	RLR	$2/3, 2/3$	$4/3, 2/3$	$0, 1/3$	$2/3, 1/3$
	RRL	$2/3, 1$	$2/3, 1/3$	$1/3, 2/3$	$1/3, 0$
	RRR	$1/3, 2/3$	$1, 1/3$	$1/3, 2/3$	$1, 1/3$

For example,

$$\begin{aligned}
 U(RLR, du) &= \text{Prob}(t_1)U(R, u) + \text{Prob}(t_2)U(L, d) + \text{Prob}(t_3)U(R, u) \\
 &= \frac{1}{3}(0, 1) + \frac{1}{3}(0, 0) + \frac{1}{3}(0, 0) = (0, 1/3)
 \end{aligned}$$

There are two pure-strategy Nash equilibria (LLL, uu) and (LLR, ud) , which are also the subgame perfect Nash equilibria since there is no subgame.

To check whether (LLL, uu) is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2S, 2R and 3.

- Requirement 1: Assume the probability distributions on left and right information set are (p_1, p_2, p_3) and (q_1, q_2, q_3) , respectively, displayed in the figure, where $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$.
- Requirement 2S: Holds automatically. (since (LLL, uu) is a Nash equilibrium)
- Requirement 2S: It is obvious that u is the best response for Receiver when Sender chooses L . To support u to be a best response for Receiver when Sender chooses R , we should take $q_3 \leq \frac{1}{2}$.
- Requirement 3: Since Sender chooses LLL , Bayes' rule implies $p_1 = p_2 = p_3 = \frac{1}{3}$ and q_1, q_2, q_3 could be arbitrary.

Hence, (LLL, uu) with $p_1 = p_2 = p_3 = \frac{1}{3}$ and $q_3 \leq \frac{1}{2}$.

To check whether (LLR, ud) is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2S, 2R and 3.

- Requirement 1: Assume the probability distributions on left and right information set are (p_1, p_2, p_3) and (q_1, q_2, q_3) , respectively, displayed in the figure, where $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$.
- Requirement 2S: Holds automatically. (since (LLR, ud) is a Nash equilibrium)
- Requirement 2S: It is obvious that u is the best response for Receiver when Sender chooses L . To support d to be a best response for Receiver when Sender chooses R , we should take $q_3 \geq \frac{1}{2}$.
- Requirement 3: Since Sender chooses LLR , Bayes' rule implies $p_1 = p_2 = \frac{1}{2}$, $p_3 = 0$ and $q_1 = q_2 = 0$, $q_3 = 1$.

Hence, (LLR, ud) with $p_1 = p_2 = \frac{1}{2}$, $p_3 = 0$ and $q_1 = q_2 = 0$, $q_3 = 1$ is a perfect Bayesian equilibrium. □

Exercise 5. *Two partners must dissolve their partnership. Partner 1 currently owns share s of the partnership, partner 2 owns share $1 - s$. The partners agree to play the following game: partner 1 names a price, p , for the whole partnership, and partner 2 then chooses either to buy 1's share for ps or to sell his or her share to 1 for $p(1 - s)$. Suppose it is common knowledge that the partners' valuations for owning the whole partnership are independently and uniformly distributed on $[0, 1]$, but that each partner's valuation is private information. What is the perfect Bayesian equilibrium?*

Solution. Figure 5 is the extensive-form representation.

- It is easy to see Partner 2's best response is

$$s_2^*(p, v_2) = \begin{cases} \text{buy,} & \text{if } v_2 \geq p; \\ \text{sell,} & \text{if } v_2 < p. \end{cases}$$

Note that we assume Partner 2 will buy if $v_2 = p$. This will not affect our analysis of the game since the probability is zero for $v_2 = p$.

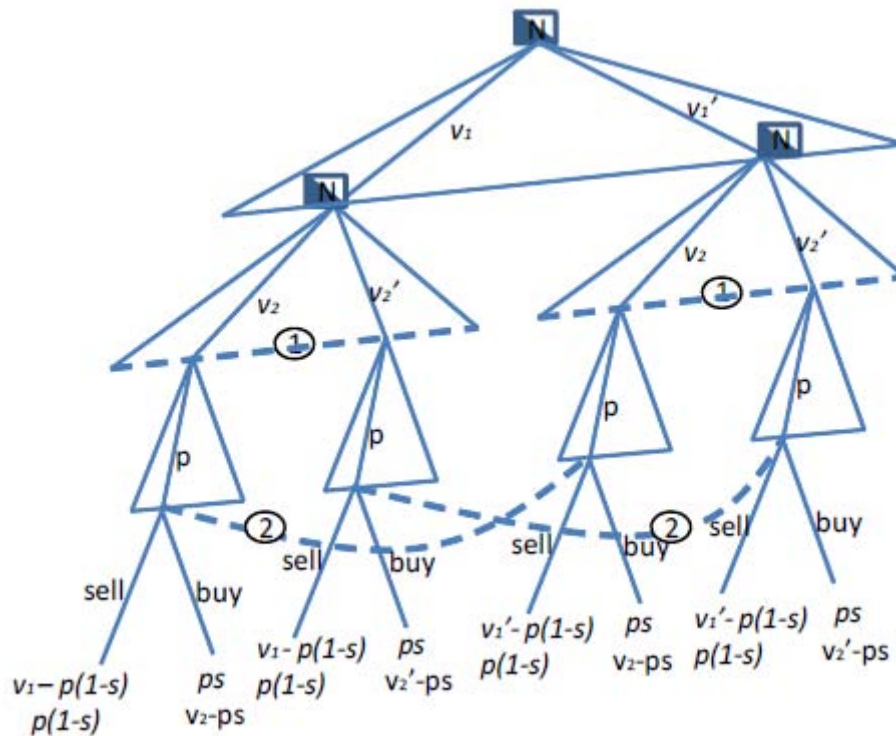


Figure 5

- Given Partner 2's strategy s_2^* , Partner 1's payoff is

$$\pi_1 = \begin{cases} ps, & \text{if } v_2 \geq p \\ v_1 - (1-s)p, & \text{if } v_2 < p \end{cases}$$

and expected payoff is

$$\begin{aligned} \mathbb{E}[\pi_1] &= ps \times \text{Prob}\{v_2: v_2 \geq p\} + (v_1 - (1-s)p) \times \text{Prob}\{v_2: v_2 < p\} \\ &= ps(1-p) + (v_1 - (1-s)p)p = (v_1 + s - p)p \end{aligned}$$

By the first order condition, we have $p^*(v_1) = \frac{v_1+s}{2}$.

- Each information set of Partner 1 is reached, so the belief on it should be determined by Bayes' rule, and hence, Partner 1's belief on each information set is a uniform distribution on $[0, 1]$.
- Since $v_1 \in [0, 1]$, $p^* \in [\frac{s}{2}, \frac{1+s}{2}]$, and hence Partner 2's beliefs should be as follows:
 - If $p \in [\frac{s}{2}, \frac{1+s}{2}]$, then the information set p is on the path, so Partner 2's belief about Partner 1's valuation is a uniform distribution on $[0, 1]$;
 - Otherwise, the information set p is off the path, so Partner 2's belief could be arbitrary.

Therefore, the perfect Bayesian equilibrium is:

$$s_1^*(v_1) = p^* = \frac{v_1 + s}{2}, \quad s_2^*(v_2 | p) = \begin{cases} \text{buy,} & \text{if } v_2 \geq p \\ \text{sell,} & \text{if } v_2 < p \end{cases},$$

Partner 1's belief about the Partner 2's valuation is a uniform distribution on $[0,1]$, and Partner 2's belief is given above. \square