Solution to Tutorial 8

2012/2013 Semester I MA4264 Game Theory

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1 Review

A **perfect Bayesian equilibrium** consists of strategies and beliefs satisfying Requirements 1 through 4.

- Requirement 1: At each information set, the Player with the move must have a belief about which node in the information set has been reached by the play of the game.
- Requirement 2: Given their beliefs, the players' strategies must be sequentially rational.
- Requirement 3: At information sets on the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies.
- Requirement 4: At information sets off the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies where possible.

For the dynamic games, we have

$$\text{PBE} \subseteq \text{SPE} \subseteq \text{NE}$$

which gives us a standard method to find perfect Bayesian equilibrium.

2 Tutorial

Exercise 1. *The following static game of complete information (Matching Pennies) has no pure-strategy Nash equilibrium but has one mixed-strategy Nash equilibrium: each player plays H with probability $1/2$. Provide a pure-strategy Bayesian Nash equilibrium of*

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

Game G

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a corresponding game of incomplete information such that as incomplete information disappears, the players' behavior in the Bayesian Nash equilibrium approaches their behavior in the mixed-strategy Nash equilibrium in the original game of complete information.

Solution. Consider the following game with incomplete information $G(x)$: where

	H	T
H	$1+t_1, -1$	$-1, 1-t_2$
T	$-1, 1$	$1, -1$

Game $G(x)$

- Type spaces: $T_1 = T_2 = [0, x]$, t_1 and t_2 are i.i.d. random variables and uniformly distributed on $[0, x]$.
- Action spaces: $A_1 = A_2 = \{H, T\}$.
- Strategy spaces: $S_1 = S_2 = \{s_i$ is a function from $[0, x]$ to $\{H, T\}\}$.

Note that $G(0) = G$.

In $G(x)$, suppose (s_1^*, s_2^*) is a Bayesian Nash equilibrium, $p = \text{Prob}\{t_1 \colon s_1^*(t_1) = H\}$, and $q = \text{Prob}\{t_2 \colon s_2^*(t_2) = H\}$.

- For Player 1, given his type t_1 and Player 2's strategy s_2^*, his expected payoff is

$$\mathbb{E}[u_1(a_1, s_2^*) \mid t_1] = \begin{cases} (1+t_1)\cdot q - 1\cdot(1-q), & a_1 = H; \\ -1\cdot q + 1\cdot(1-q), & a_1 = T. \end{cases}$$

  Thus H is a best response if and only if $(1+t_1)\cdot q - 1\cdot(1-q) \geq -1\cdot q + 1\cdot(1-q)$, that is, $t_1 \geq \frac{2}{q} - 4$. Hence, we have

$$p = \text{Prob}\{t_1 \colon s_1^*(t_1) = H\} = 1 - \frac{2/q - 4}{x} \tag{1}$$

- For Player 2, given his type t_2 and Player 1's strategy s_1^*, his expected payoff is

$$\mathbb{E}[u_2(a_2, s_1^*) \mid t_2] = \begin{cases} -1\cdot p + 1\cdot(1-p), & a_2 = H; \\ (1-t_2)\cdot p + (-1)\cdot(1-p), & a_2 = T. \end{cases}$$

  Thus H is a best response if and only if $-1\cdot p + 1\cdot(1-p) \geq (1-t_2)\cdot p + (-1)\cdot(1-p)$, that is, $t_2 \geq 4 - \frac{2}{p}$. Hence, we have

$$q = \text{Prob}\{t_2 \colon s_2^*(t_2) = H\} = 1 - \frac{4 - 2/p}{x} \tag{2}$$

Rewriting Equations (1) and (2), we will have

$$p = \frac{2}{4 + (q-1)x}, \quad q = \frac{2}{4 + (1-p)x}.$$

As $x \to 0$, $p, q \to \frac{1}{2}$, that is, the Bayesian Nash equilibrium will converge to the mixed-strategy Nash equilibrium in G. □

Exercise 2. *In the following extensive-form games, derive the normal-form game and find all the pure-strategy Nash, subgame-perfect, and perfect Bayesian equilibria.*

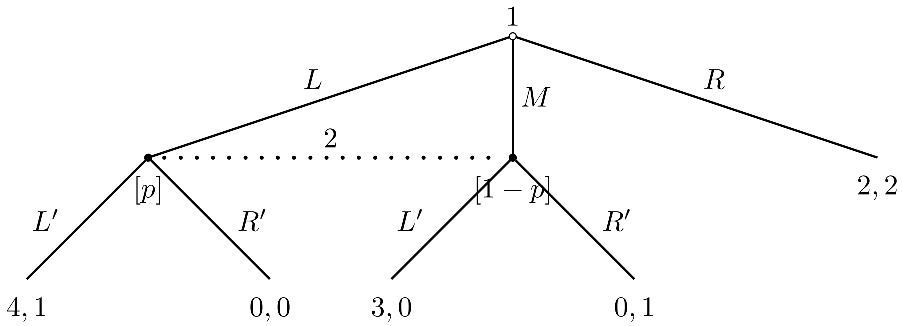


Figure 1: Game (a)

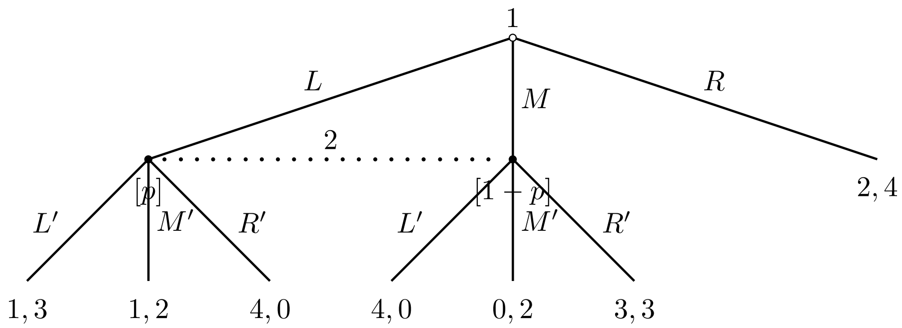


Figure 2: Game (b)

Solution. (a) Game (a). Normal-form representation is as follows:

- $S_1 = \{L, M, R\}$, $S_2 = \{L', R'\}$.
- Payoff table:

		Player 2 L'	Player 2 R'
Player 1	L	4, 1	0, 0
	M	3, 0	0, 1
	R	2, 2	2, 2

(i) There are two pure-strategy Nash equilibria (L, L') and (R, R').

(ii) Since there is no subgame, every Nash equilibrium is subgame-perfect, and hence (L, L') and (R, R') are all the subgame-perfect Nash equilibria.

(iii) To check whether (L, L') is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.

- Requirement 1: For Player 2's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
- Requirement 2: To support L' to be a best response for Player 2, we should take $p \geq \frac{1}{2}$.
- Requirement 3: Since Player 1 chooses L, by Bayes' rule, Player 2's belief should be $(1, 0)$, that is, $p = 1$.

- Requirement 4: No information set is off the path, so Requirement 4 gives no restriction on p.

Hence (L, L') with $p = 1$ is a perfect Bayesian equilibrium.

To check whether (R, R') is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.

- Requirement 1: For Player 2's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
- Requirement 2: To support R' to be a best response for Player 2, we should take $p \leq \frac{1}{2}$.
- Requirement 3: No nontrivial information set is on the path, so Requirement 3 gives no restriction on p.
- Requirement 4: Since Player 1 chooses R, Player 2's information set is off the path, so p could be arbitrary.

Hence (R, R') with $p \leq \frac{1}{2}$ is a perfect Bayesian equilibrium.

(b) Game (b). Normal-form representation is as follows:

- $S_1 = \{L, M, R\}$, $S_2 = \{L', M', R'\}$.
- Payoff table:

		Player 2		
		L'	M'	R'
	L	$1, 3$	$1, 2$	$4, 0$
Player 1	M	$4, 0$	$0, 2$	$3, 3$
	R	$2, 4$	$2, 4$	$2, 4$

(i) (R, M') is the unique pure-strategy Nash equilibrium.

(ii) Since there is no subgame, every Nash equilibrium is subgame-perfect, and hence (R, M') is the unique subgame-perfect Nash equilibrium.

(iii) To check whether (R, M') is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.

- Requirement 1: For Player 2's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
- Requirement 2: To support M' to be a best response for Player 2, we should take $p \in [\frac{1}{3}, \frac{2}{3}]$.
- Requirement 3: No nontrivial information set is on the path, so Requirement 3 gives no restriction on p.
- Requirement 4: Since Player 1 chooses R, Player 2's information set is off the path, so p could be arbitrary.

Hence (R, M') with $p \in [\frac{1}{3}, \frac{2}{3}]$ is the unique perfect Bayesian equilibrium.

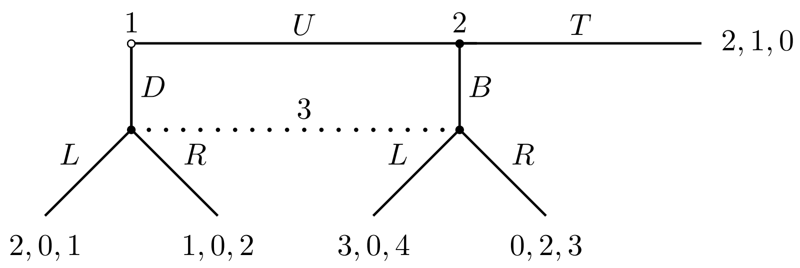
□

Exercise 3. *Consider the following game between three Players:*

- *Player 1 moves first. He has two actions: U and D. Action U gives the next move to Player 2, action D gives the next move to Player 3.*

- *If Player 2 is given the move he also has two actions: T and B. Action T ends the game, action B gives the move to Player 3.*
- *If Player 3 is given the move he also has two actions: L and R. Both actions end the game. Player 3 does not know whether the move was given to him by Player 1 or Player 2.*

The extensive form is given as follows: where the payoff vector (x, y, z) means that Player



1 receives utility x, Player 2 receives utility y and Player 3 receives utility z.

(a) Find all Nash equilibria.

(b) Find all subgame-perfect Nash equilibria.

(c) Find all perfect Bayesian equilibria.

Solution. (a) The normal-form representation is as follows:

- $S_1 = \{D, U\}$, $S_2 = \{B, T\}$, $S_3 = \{L, R\}$.
- Payoff table:

		Player 2 and Player 3			
		BL	BR	TL	TR
Player 1	D	2, 0, 1	1, 0, 2	2, 0, 1	1, 0, 2
	U	3, 0, 4	0, 2, 3	2, 1, 0	2, 1, 0

There are two pure-strategy Nash equilibria (D, B, R) and (U, T, L).

(b) There is no subgame, so (D, B, R) and (U, T, L) are all the subgame perfect Nash equilibria.

(c) Assume Player 3's belief is p on D and $1 - p$ on B.

To check whether (D, B, R) is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.

- Requirement 1: For Player 3's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
- Requirement 2: To support R to be a best response for Player 3, we should take $p \geq \frac{1}{2}$.
- Requirement 3: Since Player 1 chooses D, by Bayes' rule, Player 3's belief should be $(1, 0)$, that is, $p = 1$.

- Requirement 4: No nontrivial information set is off the path, so Requirement 4 gives no restriction on p.

Hence (D, B, R) with $p = 1$ is a perfect Bayesian equilibrium.

To check whether (U, T, L) is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.

- Requirement 1: For Player 3's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
- Requirement 2: To support L to be a best response for Player 3, we should take $p \leq \frac{1}{2}$.
- Requirement 3: No nontrivial information set is on the path, so Requirement 3 gives no restriction on p.
- Requirement 4: Since Player 1 and Player 2 choose U and T, respectively, Player 3's belief could be arbitrary since his information set will not be reached.

Hence (U, T, L) with $p \leq \frac{1}{2}$ is a perfect Bayesian equilibrium.

□

Exercise 4. *Find all perfect Bayesian equilibria in the following signaling games.*

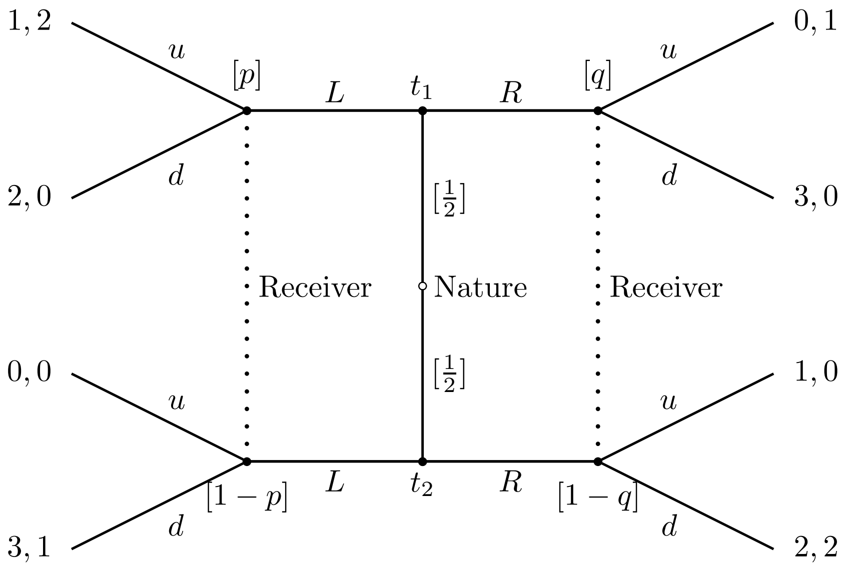


Figure 3: Game 1

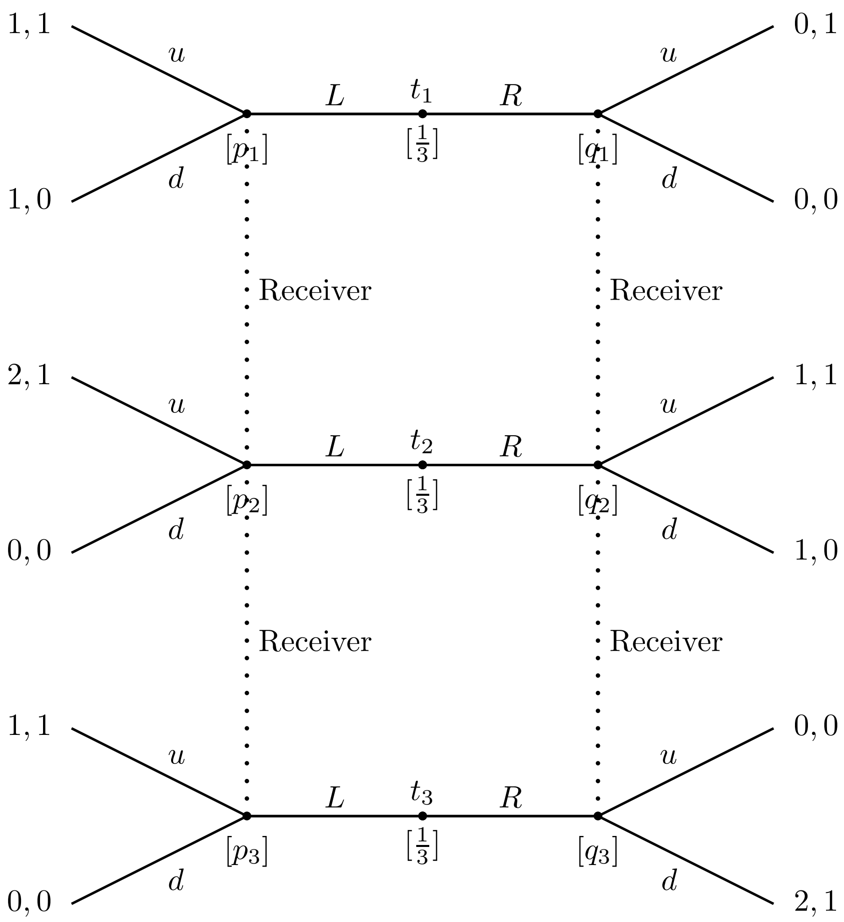


Figure 4: Game 2

Solution. (i) Game 1: The normal-form representation is as follows:

- $T = \{t_1, t_2\}$, $M = \{L, R\}$, $A = \{u, d\}$.
- Payoff table:

Sender \ Receiver	uu	ud	du	dd
LL	$1/2, 1$	$1/2, 1$	$5/2, 1/2$	$5/2, 1/2$
LR	$1, 1$	$3/2, 2$	$3/2, 0$	$2, 1$
RL	$0, 1/2$	$3/2, 0$	$3/2, 1$	$3, 1/2$
RR	$1/2, 1/2$	$5/2, 1$	$1/2, 1/2$	$5/2, 1$

For example,

$$U(RL, du) = \text{Prob}(t_1)U(R, u \mid t_1) + \text{Prob}(t_2)U(L, d \mid t_2)$$
$$= \frac{1}{2}(0, 1) + \frac{1}{2}(3, 1) = (3/2, 1)$$

There is the unique Nash equilibrium (RR, ud), which is also the unique subgame perfect Nash equilibrium since there is no subgame.

To check whether (RR, ud) is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2S, 2R and 3.

- Requirement 1: Assume the probability distributions on left and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since (RR, ud) is a Nash equilibrium)
- Requirement 2R: To support ud (u when Sender chooses L, d when Sender chooses R) to be a best response for Receiver, we should take $p \geq \frac{1}{3}$ and $q \leq \frac{2}{3}$.
- Requirement 3: Since Sender chooses RR, Bayes' rule implies p could be arbitrary, and $q = \frac{1}{2}$.

Hence (RR, ud) with $p \geq \frac{1}{3}$ and $q = \frac{1}{2}$ is a perfect Bayesian equilibrium.

(ii) Game 2: The normal-form representation is as follows:

- $T = \{t_1, t_2, t_3\}$, $M = \{L, R\}$, and $A = \{u, d\}$.
- Payoff table:

		Receiver			
		uu	ud	du	dd
Sender	LLL	$4/3, 1$	$4/3, 1$	$1/3, 0$	$1/3, 0$
	LLR	$1, 2/3$	$5/3, 1$	$1/3, 0$	$1, 1/3$
	LRL	$1, 1$	$1, 2/3$	$2/3, 1/3$	$2/3, 0$
	LRR	$2/3, 2/3$	$4/3, 2/3$	$2/3, 1/3$	$4/3, 1/3$
	RLL	$1, 1$	$1, 2/3$	$0, 1/3$	$0, 0$
	RLR	$2/3, 2/3$	$4/3, 2/3$	$0, 1/3$	$2/3, 1/3$
	RRL	$2/3, 1$	$2/3, 1/3$	$1/3, 2/3$	$1/3, 0$
	RRR	$1/3, 2/3$	$1, 1/3$	$1/3, 2/3$	$1, 1/3$

For example,

$$\begin{aligned} U(RLR, du) &= \text{Prob}(t_1)U(R, u \mid t_1) + \text{Prob}(t_2)U(L, d \mid t_2) + \text{Prob}(t_3)U(R, u \mid t_3) \\ &= \frac{1}{3}(0,1) + \frac{1}{3}(0,0) + \frac{1}{3}(0,0) = (0, 1/3) \end{aligned}$$

There are two pure-strategy Nash equilibria (LLL, uu) and (LLR, ud), which are also the subgame perfect Nash equilibria since there is no subgame.

To check whether (LLL, uu) is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2S, 2R and 3.

- Requirement 1: Assume the probability distributions on left and right information set are (p_1, p_2, p_3) and (q_1, q_2, q_3), respectively, displayed in the figure, where $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$.
- Requirement 2S: Holds automatically. (since (LLL, uu) is a Nash equilibrium)
- Requirement 2S: It is obvious that u is the best response for Receiver when Sender chooses L. To support u to be a best response for Receiver when Sender chooses R, we should take $q_3 \leq \frac{1}{2}$.
- Requirement 3: Since Sender chooses LLL, Bayes' rule implies $p_1 = p_2 = p_3 = \frac{1}{3}$ and q_1, q_2, q_3 could be arbitrary.

Hence, (LLL, uu) with $p_1 = p_2 = p_3 = \frac{1}{3}$ and $q_3 \le \frac{1}{2}$.

To check whether (LLR, ud) is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2S, 2R and 3.

- Requirement 1: Assume the probability distributions on left and right information set are (p_1, p_2, p_3) and (q_1, q_2, q_3), respectively, displayed in the figure, where $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$.
- Requirement 2S: Holds automatically. (since (LLR, ud) is a Nash equilibrium)
- Requirement 2S: It is obvious that u is the best response for Receiver when Sender chooses L. To support d to be a best response for Receiver when Sender chooses R, we should take $q_3 \ge \frac{1}{2}$.
- Requirement 3: Since Sender chooses LLR, Bayes' rule implies $p_1 = p_2 = \frac{1}{2}$, $p_3 = 0$ and $q_1 = q_2 = 0$, $q_3 = 1$.

Hence, (LLL, uu) with $p_1 = p_2 = \frac{1}{2}$, $p_3 = 0$ and $q_1 = q_2 = 0$, $q_3 = 1$ is a perfect Bayesian equilibrium.

□

Exercise 5. *Two partners must dissolve their partnership. Partner 1 currently owns share s of the partnership, partner 2 owns share $1 - s$. the partners agree to play the following game: partner 1 names a price, p, for the whole partnership, and partner 2 then chooses either to buy 1's share for ps or to sell his or her share to 1 for $p(1 - s)$. Suppose it is common knowledge that the partners' valuations for owning the whole partnership are independently and uniformly distributed on $[0, 1]$, but that each partner's valuation is private information. What is the perfect Bayesian equilibrium?*

Solution. Figure 5 is the extensive-form representation.

- It is easy to see Partner 2's best response is

$$s_2^*(p, v_2) = \begin{cases} \text{buy}, & \text{if } v_2 \ge p; \\ \text{sell}, & \text{if } v_2 < p. \end{cases}$$

  Note that we assume Partner 2 will buy if $v_2 = p$. This will not affect the our analysis of the game since the probability is zero for $v_2 = p$.

- Given Partner 2's strategy s_2^*, Partner 1's payoff is

$$\pi_1 = \begin{cases} ps, & \text{if } v_2 \ge p \\ v_1 - (1 - s)p, & \text{if } v_2 < p \end{cases},$$

  and expected payoff is

$$\begin{aligned} \mathbb{E}[\pi_1] &= ps \times \text{Prob}\{v_2 \colon v_2 \ge p\} + (v_1 - (1 - s)p) \times \text{Prob}\{v_2 \colon v_2 < p\} \\ &= ps(1 - p) + (v_1 - (1 - s)p)p = (v_1 + s - p)p \end{aligned}$$

  By the first order condition, we have $p^*(v_1) = \frac{v_1 + s}{2}$.

- Each information set of Partner 1 is reached, so the belief on it should be determined by Bayes' rule, and hence, Partner 1's belief on each information set is a uniform distribution on $[0, 1]$.

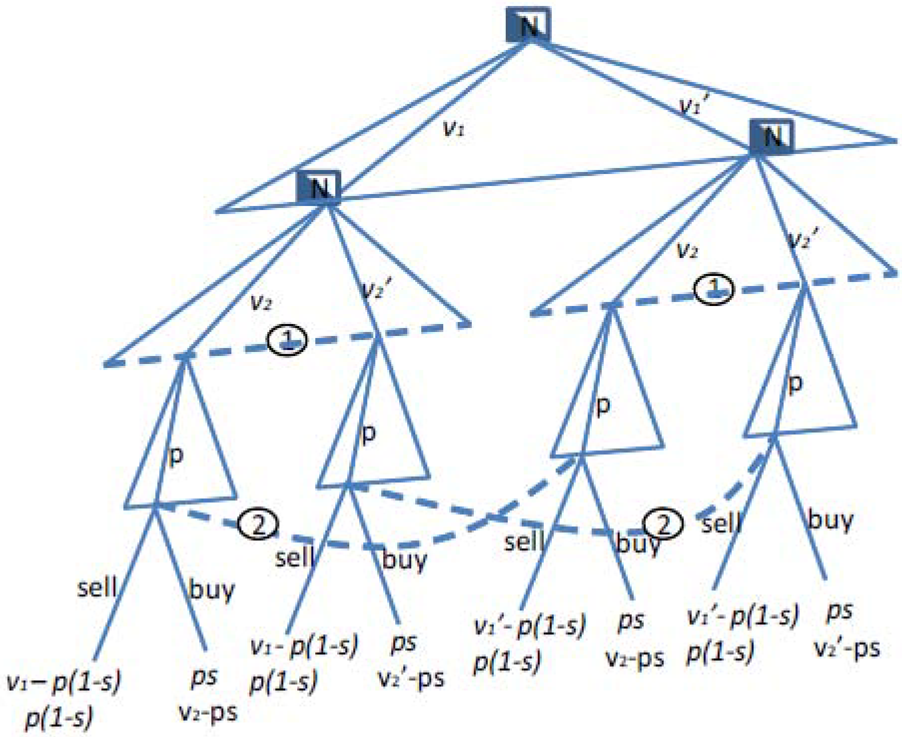


Figure 5

- Since $v_1 \in [0,1]$, $p^* \in [\frac{s}{2}, \frac{1+s}{2}]$, and hence Partner 2's beliefs should be as follows:
  - If $p \in [\frac{s}{2}, \frac{1+s}{2}]$, then the information set p is on the path, so Partner 2's belief about Partner 1's valuation is a uniform distribution on $[0,1]$;
  - Otherwise, the information set p is off the path, so Partner 2's belief could be arbitrary.

Therefore, the perfect Bayesian equilibrium is:

$$s_1^*(v_1) = p^* = \frac{v_1 + s}{2}, \quad s_2^*(v_2 \mid p) = \begin{cases} \text{buy}, & \text{if } v_2 \geq p \\ \text{sell}, & \text{if } v_2 < p \end{cases},$$

Partner 1's belief about the Partner 2's valuation is a uniform distribution on [0,1], and Partner 2's belief is given above. □

Exercise 6. *A buyer and a seller have valuations v_b and v_s. It is common knowledge that there are gains from trade (i.e., that $v_b > v_s$), but the size of the gains is private information, as follows: the seller's valuation is uniformly distributed on $[0,1]$; the buyer's valuation $v_b = k \cdot v_s$, where $k > 1$ is common knowledge; the seller knows v_s (and hence v_b) but the buyer does not know v_b (or v_s). Suppose the buyer makes a single offer, p, which the seller either accepts or rejects. What is the perfect Bayesian equilibrium when $k < 2$? When $k > 2$?*

Solution. The extensive-form representation of this game is as follows:

Clearly, the buyer has no incentive to offer $p > 1$, since the seller will accept $p \geq v_s$ and v_s is uniformly distributed on $[0,1]$.

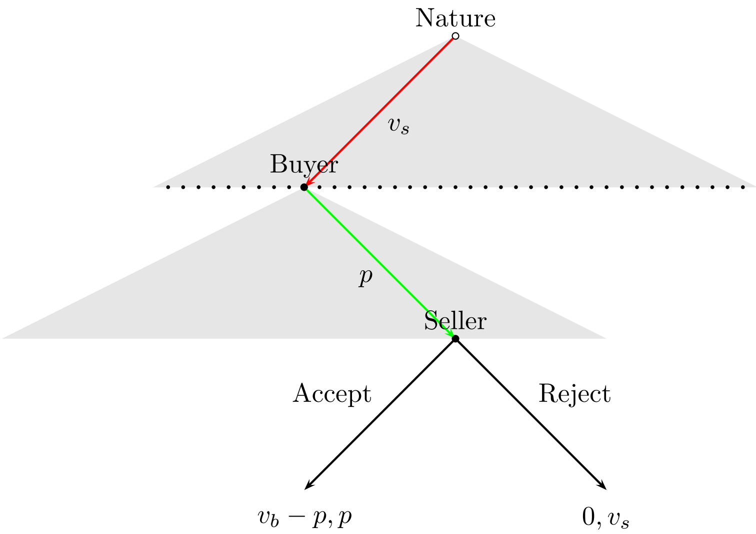


Figure 6

- By backwards induction, the seller's best response is
$$s_s^*(v_s \mid p) = \begin{cases} \text{accept}, & \text{if } v_s \leq p \\ \text{reject}, & \text{if } v_s > p \end{cases}.$$
Note that we assume seller will accept if $v_s = p$. This will not affect the our analysis of the game since the probability is zero for $v_s = p$.

- The buyer's maximization problem is:
$$\max_{0 \leq p \leq 1} \mathbb{E}[v_b - p \mid v_s \leq p].$$
Since $v_b = kv_s$, the buyer's maximization problem is:
$$\max_{0 \leq p \leq 1} \int_0^p (kv_s - p) \, \mathrm{d}v_s = \max_{0 \leq p \leq 1} (k/2 - 1)p^2.$$
Therefore, the maximizer is
$$p^* = \begin{cases} 1, & \text{if } k > 2 \\ 0, & \text{if } k < 2 \end{cases}.$$

- Each information set of buyer is reached, so buyer's belief is a uniform distribution on $[0, 1]$.

To summarize, the perfect Bayesian equilibrium is:
$$s_b^* = p^* = \begin{cases} 1, & \text{if } k > 2 \\ 0, & \text{if } k < 2 \end{cases},$$
and for $v_s \in [0, 1]$,
$$s_s^*(v_s \mid p) = \begin{cases} \text{accept}, & \text{if } v_s < p \\ \text{accept or reject}, & \text{if } v_s = p \\ \text{reject}, & \text{if } v_s < p \end{cases},$$
the buyer's belief about the seller's valuation is a uniform distribution on $[0, 1]$. □

End of Solution to Tutorial 8