# Solution to Tutorial 9[*]

2011/2012 Semester I MA4264 Game Theory

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**Exercise 1.** *A buyer and a seller have valuations $v_b$ and $v_s$. It is common knowledge that there are gains from trade (i.e., that $v_b > v_s$), but the size of the gains is private information, as follows: the seller's valuation is uniformly distributed on $[0, 1]$; the buyer's valuation $v_b = k \cdot v_s$, where $k > 1$ is common knowledge; the seller knows $v_s$ (and hence $v_b$) but the buyer does not know $v_b$ (or $v_s$). Suppose the buyer makes a single offer, $p$, which the seller either accepts or rejects. What is the perfect Bayesian equilibrium when $k < 2$? When $k > 2$?*

*Solution.* The extensive-form representation of this game is as follows:

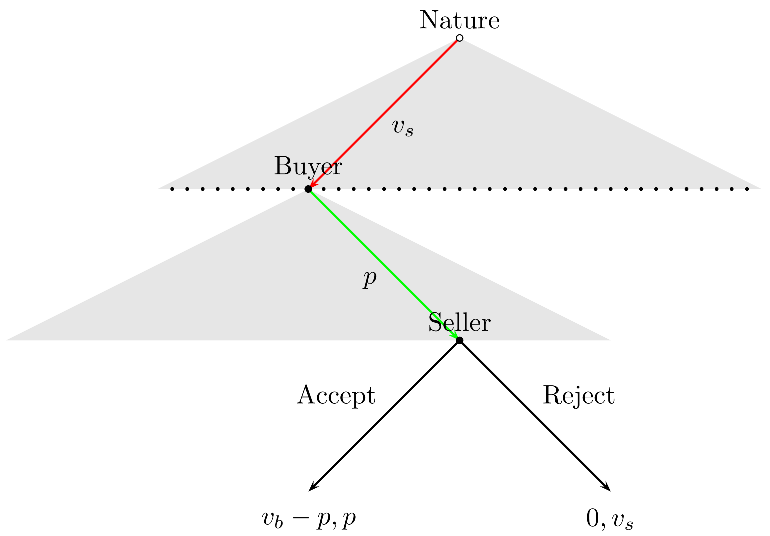


Figure 1

Clearly, the buyer has no incentive to offer $p > 1$, since the seller will accept $p \geq v_s$ and $v_s$ is uniformly distributed on $[0, 1]$.

- By backwards induction, the seller's best response is

$$s_s^*(v_s \mid p) = \begin{cases} \text{accept}, & \text{if } v_s \leq p \\ \text{reject}, & \text{if } v_s < p \end{cases}.$$

[*]Corrections are always welcome.

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Note that we assume seller will accept if $v_s = p$. This will not affect the our analysis of the game since the probability is zero for $v_s = p$.

- The buyer's maximization problem is:

$$\max_{0 \le p \le 1} \mathbb{E}[v_b - p \mid v_s \le p].$$

Since $v_b = kv_s$, the buyer's maximization problem is:

$$\max_{0 \le p \le 1} \int_0^p (kv_s - p)\,\mathrm{d}v_s = \max_{0 \le p \le 1} (k/2 - 1)p^2.$$

Therefore, the maximizer is

$$p^* = \begin{cases} 1, & \text{if } k > 2 \\ 0, & \text{if } k < 2 \end{cases}.$$

- Each information set of buyer is reached, so buyer's belief is a uniform distribution on $[0, 1]$.

To summarize, the perfect Bayesian equilibrium is:

$$s_b^* = p^* = \begin{cases} 1, & \text{if } k > 2 \\ 0, & \text{if } k < 2 \end{cases},$$

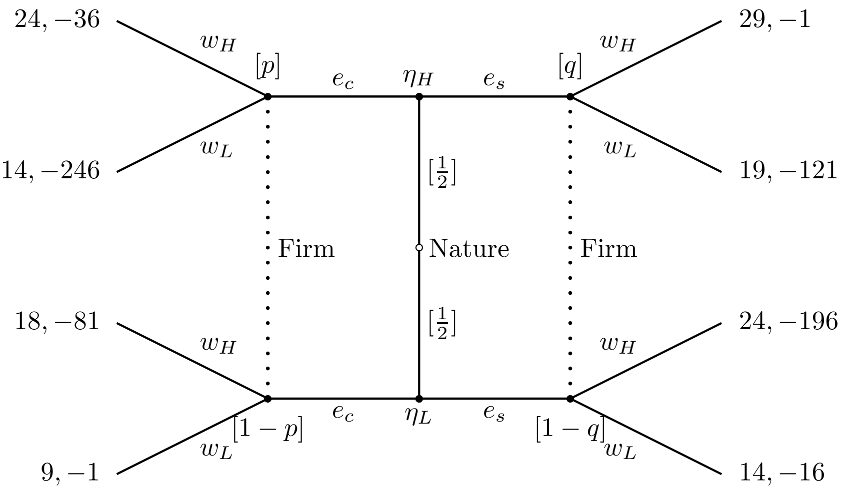
and for $v_s \in [0, 1]$,

$$s_s^*(v_s \mid p) = \begin{cases} \text{accept}, & \text{if } v_s < p \\ \text{accept or reject}, & \text{if } v_s = p \\ \text{reject}, & \text{if } v_s < p \end{cases},$$

the buyer's belief about the seller's valuation is a uniform distribution on $[0, 1]$. □

**Exercise 2.** *Find the Bayesian equilibria for the first case of the job-market signaling games in which the output is changed to (i) $y(\eta, e) = 3\eta + e$, and (ii) $y(\eta, e) = 4\eta$.*

*Solution.* (i) Assume $y(\eta, e) = 3\eta + e$. The extensive-form representation is as follows:




The normal-form representation is as follows:

- $T = \{\eta_H, \eta_L\}$, $M = \{e_c, e_s\}$, $A = \{w_H, w_L\}$.
- Payoff table:

| | | Firm | | | |
|---|---|---|---|---|---|
| | | $w_H w_H$ | $w_H w_L$ | $w_L w_H$ | $w_L w_L$ |
| Worker | $e_c e_c$ | $21, -117/2$ | $21, -117/2$ | $23/2, -247/2$ | $23/2, -247/2$ |
| | $e_c e_s$ | $24, -116$ | $19, -26$ | $19, -221$ | $14, -131$ |
| | $e_s e_c$ | $47/2, -41$ | $37/2, -101$ | $19, -1$ | $14, -61$ |
| | $e_s e_s$ | $53/2, -197/2$ | $33/2, -137/2$ | $53/2, -197/2$ | $33/2, -137/2$ |

Therefore, there are two pure-strategy Nash equilibria $(e_c e_c, w_h w_L)$ and $(e_s e_s, w_L w_L)$.

For $(e_c e_c, w_H w_L)$:

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since $(e_c e_c, w_H w_L)$ is a Nash equilibrium)
- Requirement 2R: $q \le \frac{3}{5}$.
- Requirement 3: $p = \frac{1}{2}$, $q \in [0, 1]$.

Thus, $(e_c e_c, w_H w_L)$ with $p = \frac{1}{2}$ and $q \le \frac{3}{5}$ is a perfect Bayesian equilibrium.

For $(e_s e_s, w_L w_L)$:

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.

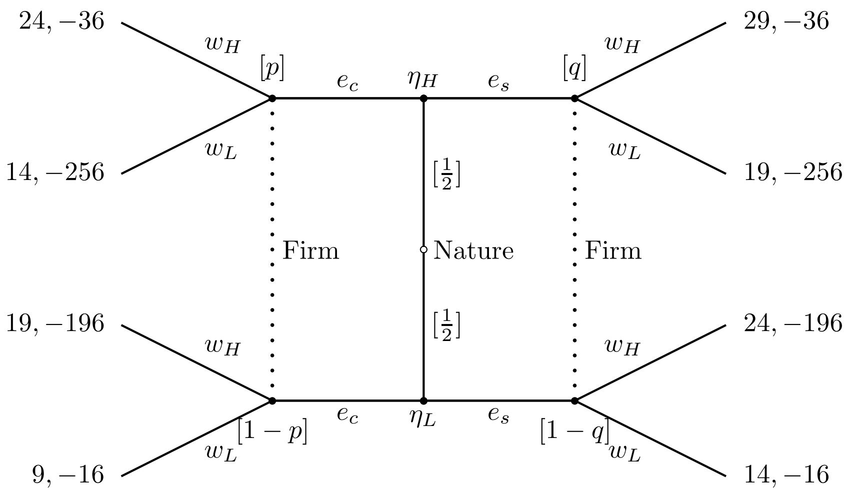
- Requirement 2S: Holds automatically. (since $(e_se_s, w_Lw_L)$ is a Nash equilibrium)
- Requirement 2R: $p \leq \frac{4}{15}$.
- Requirement 3: $p \in [0, 1]$, $q = \frac{1}{2}$.

Thus, $(e_ce_c, w_Hw_L)$ with $p \leq \frac{4}{15}$ and $q = \frac{1}{2}$ is a perfect Bayesian equilibrium.

To summarize, there are three pure-strategy perfect Bayesian equilibria:

- $(e_ce_c, w_Hw_L)$ with $p = \frac{1}{2}$ and $q \leq \frac{3}{5}$;
- $(e_ce_c, w_Hw_L)$ with $p \leq \frac{4}{15}$ and $q = \frac{1}{2}$.

(ii) Assume $y(\eta, e) = 4\eta$. The extensive-form representation is as follows:




The normal-form representation is as follows:

- $T = \{\eta_H, \eta_L\}$, $M = \{e_c, e_s\}$, $A = \{w_H, w_L\}$.
- Payoff table:

| | | Firm | | | |
|---|---|---|---|---|---|
| | | $w_Hw_H$ | $w_Hw_L$ | $w_Lw_H$ | $w_Lw_L$ |
| Worker | $e_ce_c$ | $43/2, -116$ | $43/2, -116$ | $23/2, -136$ | $23/2, -136$ |
| | $e_ce_s$ | $24, -116$ | $19, -26$ | $19, -226$ | $14, -136$ |
| | $e_se_c$ | $24, -116$ | $19, -226$ | $19, -26$ | $14, -136$ |
| | $e_se_s$ | $53/2, -116$ | $33/2, -136$ | $53/2, -116$ | $33/2, -136$ |

Therefore, there are three pure-strategy Nash equilibria $(e_ce_c, w_hw_L)$, $(e_se_s, w_Hw_H)$ and $(e_se_s, w_Lw_H)$.

For $(e_ce_c, w_Hw_L)$:

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since $(e_c e_c, w_H w_L)$ is a Nash equilibrium)
- Requirement 2R: $q \leq \frac{9}{20}$.
- Requirement 3: $p = \frac{1}{2}$, $q \in [0, 1]$.

Thus, $(e_c e_c, w_H w_L)$ with $p = \frac{1}{2}$ and $q \leq \frac{9}{20}$ is a perfect Bayesian equilibrium.

For $(e_s e_s, w_H w_H)$:

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since $(e_s e_s, w_H w_H)$ is a Nash equilibrium)
- Requirement 2R: $p \leq \frac{9}{20}$.
- Requirement 3: $p \in [0, 1]$, $q = \frac{1}{2}$.

Thus, $(e_s e_s, w_H w_H)$ with $p \leq \frac{9}{20}$ and $q = \frac{1}{2}$ is a perfect Bayesian equilibrium.

For $(e_s e_s, w_L w_H)$:

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since $(e_s e_s, w_L w_H)$ is a Nash equilibrium)
- Requirement 2R: $p \geq \frac{9}{20}$.
- Requirement 3: $p \in [0, 1]$, $q = \frac{1}{2}$.

Thus, $(e_s e_s, w_L w_H)$ with $p \geq \frac{9}{20}$ and $q = \frac{1}{2}$ is a perfect Bayesian equilibrium.

To summarize, there are three pure-strategy perfect Bayesian equilibria:

- $(e_c e_c, w_H w_L)$ with $p = \frac{1}{2}$ and $q \leq \frac{9}{20}$;
- $(e_s e_s, w_H w_H)$ with $p \leq \frac{9}{20}$ and $q = \frac{1}{2}$;
- $(e_s e_s, w_L w_H)$ with $p \geq \frac{9}{20}$ and $q = \frac{1}{2}$.

□

**Exercise 3.** *Consider the job-market signaling game where $c(\eta, e)$ and $y(\eta, e)$ are general functions and $w$ is chosen from the action space $[0, \infty)$.*

*(i) For each of the separating strategies $(e_c, e_s)$ and $(e_s, e_c)$, write down conditions on $c$ and $y$ under which the separating perfect Bayesian equilibria exist.*

(ii) *Find concrete and reasonable examples of $c(\eta, e)$ and $y(\eta, e)$ which satisfy the conditions you present in (i).*

*Solution.* (i) Suppose in a perfect Bayesian equilibrium, $e_c e_s$ is worker's strategy. Then by Bayes' rule, we have $p = 1$ and $q = 0$.

For firm, given message $e_c$, his maximization problem is

$$\max_{0 \le w} -[w - y(\eta_H, e_c)]^2,$$

and hence the best choice is $w_c^* = y(\eta_H, e_c)$. Similarly, given message $e_s$, firm's best choice is $w_s^* = y(\eta_L, e_s)$.

For worker, given firm's strategy $(w_c^*, w_s^*)$, when $\eta_H$ occurs, $e_c$ is the best response, that is,

$$y(\eta_H, e_c) - c(\eta_H, e_c) \ge y(\eta_L, e_s) - e(\eta_H, e_s).$$

Similarly, when $\eta_L$ occurs, we have

$$y(\eta_H, e_c) - c(\eta_L, e_c) \le y(\eta_L, e_s) - c(\eta_L, e_s).$$

Thus,

$$c(\eta_L, e_c) - c(\eta_L, e_s) \ge y(\eta_H, e_c) - y(\eta_L, e_s) \ge c(\eta_H, e_c) - c(\eta_H, e_s).$$

(ii) Exercise.

□

**Exercise 4.** *Suppose the HAL Corporation is a monopolist in the Cleveland market for mainframe computers. We will suppose that the market is a "natural monopoly", meaning that only one firm can survive in the long run. HAL faces only one potential competitor, DEC. In the first period, HAL moves first and chooses one of two prices for its computers: High or Low. DEC moves second and decides whether to enter the market or not. Here are the first-period profits of the two firms: In the second*

| | | DEC | |
|---|---|---|---|
| | | Enter | StayOut |
| HAL | High | 0, 0 | 5, 0 |
| | Low | 0, 0 | 1, 0 |

*period, three things can occur:*

(a) *DEC did not enter in the first period. Then HAL retains its monopoly forever and earns monopoly profits of $125 - C$, where $C$ is its costs. DEC earns zero profits.*

(b) *DEC entered in the first period and has the lower costs. HAL leaves the market and DEC gets the monopoly forever, earning the monopoly profits of $100 = 125 - 25$, where $25$ are its costs, which is common knowledge. HAL earns zero profits.*

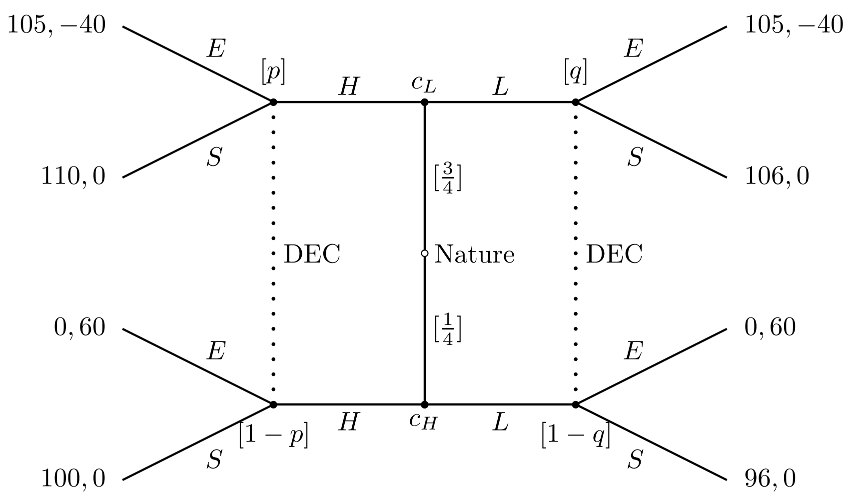
(c) *DEC entered in the first period and HAL has the lower costs. In this case, DEC drops out of the market, HAL retains its monopoly forever, and it earns monopoly profits of* $125 - C$*, where* $C$ *is its costs. DEC earns zero profits.*

*DEC's payoff from playing this game equals 0 if it decides to stay out, and it equals the sum of its profits in the two periods minus entry costs of* $40$ *if it decides to enter. HAL's payoff equals the sum of its profits in the two periods. HAL's costs,* $C$*, can be either* $30$ *(high) or* $20$ *(low). This cost information is private information. DEC only knows that* $\text{Prob}(C = 20) = 0.75$ *and* $\text{Prob}(C = 30) = 0.25$*.*

*Formulate the problem as a signaling game and find all perfect Bayesian equilibria.*

*Solution.* The signaling game is as follows:

- $T = \{c_L = 20, c_H = 30\}$, $M = \{H(igh), L(ow)\}$, and $A = \{E(nter), S(tayout)\}$.
- The extensive-form representation is as follows:




The normal-form representation is:

| HAL \ DEC | $EE$ | $ES$ | $SE$ | $SS$ |
|---|---|---|---|---|
| $HH$ | $78.75, -15$ | $78.75, -15$ | $107.5, 0$ | $107.5, 0$ |
| $HL$ | $78.75, -15$ | $102.75, -30$ | $82.5, 15$ | $106.5, 0$ |
| $LH$ | $78.75, -15$ | $79.5, 15$ | $103.75, -30$ | $104.5, 0$ |
| $LL$ | $78.75, -15$ | $103.5, 0$ | $78.75, -15$ | $103.5, 0$ |

There are three pure-strategy Nash equilibria $(HH, SE)$, $(HH, SS)$ and $(LL, ES)$. For $(HH, SE)$:

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.

- Requirement 2S: Holds automatically. (since $(HH, SE)$ is a Nash equilibrium)
- Requirement 2R: $q \leq 0.6$.
- Requirement 3: $p = \frac{3}{4}$, $q \in [0, 1]$.

Thus, $(HH, SE)$ with $p = \frac{3}{4}$ and $q \leq 0.6$ is a perfect Bayesian equilibrium.

For $(HH, SS)$:

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since $(HH, SS)$ is a Nash equilibrium)
- Requirement 2R: $q \geq 0.6$.
- Requirement 3: $p = \frac{3}{4}$, $q \in [0, 1]$.

Thus, $(HH, SS)$ with $p = \frac{3}{4}$ and $q \geq 0.6$ is a perfect Bayesian equilibrium.

For $(LL, ES)$:

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since $(LL, ES)$ is a Nash equilibrium)
- Requirement 2R: $p \leq 0.6$.
- Requirement 3: $p \in [0, 1]$, $q = \frac{3}{4}$.

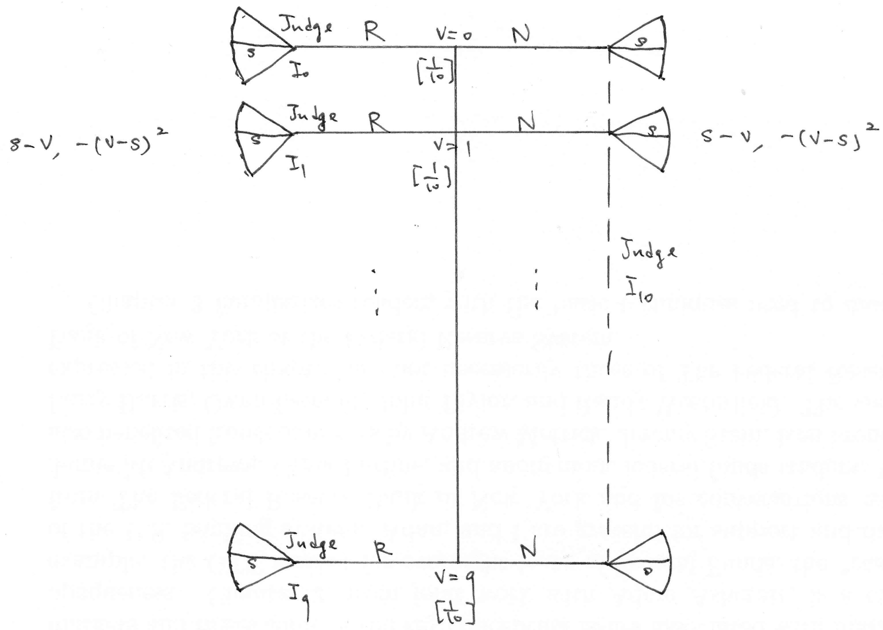
Thus, $(LL, ES)$ with $p \leq 0.6$ and $q = \frac{3}{4}$ is a perfect Bayesian equilibrium.

To summarize, there are three pure-strategy perfect Bayesian equilibria:

- $(HH, SE)$ with $p = \frac{3}{4}$ and $q \leq 0.6$;
- $(HH, SS)$ with $p = \frac{3}{4}$ and $q \geq 0.6$;
- $(LL, ES)$ with $p \leq 0.6$ and $q = \frac{3}{4}$.

□

**Exercise 5.** *There are two Players in the game: Judge and Plaintiff. The Plaintiff has been injured. Severity of the injury, denoted by $v$, is the Plaintiff's private information. The Judge does not know $v$ and believes that $v$ is uniformly distributed on $\{0, 1, \ldots, 9\}$ (so that the probability that $v = i$ is $\frac{1}{10}$ for any $i \in \{0, 1, \ldots, 9\}$). The Plaintiff can verifiably reveal $v$ to the Judge without any cost, in which case the Judge will know $v$. The order of the events is as follows. First, the Plaintiff decides whether to reveal $v$ or not. Then, the Judge rewards a compensation $R$ which can be any nonnegative real number. The payoff of the Plaintiff is $R - v$, and the payoff of the Judge is $-(v - R)^2$. Everything described so far is common knowledge. Find a perfect Bayesian equilibrium.*




*Solution.* The signaling game is as follows: $T = \{0, 1, \ldots, 9\}$; $M = \{R, N\}$, where $R$ is "Reveal" and $N$ is "Not Reveal"; $A = \mathbb{R}_+$.

From the extensive-form representation, there are 10 subgames, and Judge has 11 information sets $I_v$, $v = 0, 1, \ldots, 10$, where for each $v \leq 9$, $I_v$ denotes that Plaintiff reveals $v$ to Judge, and $I_{10}$ denotes the case that Plaintiff does not reveal the value.

Plaintiff's strategy space is

$$S = \{f \mid f \colon T \to M\} = \{R, N\}^{10}.$$

Judge's strategy space is

$$Q = \{(x_0, x_1, \ldots, x_9, x_{10}) \colon x_i \geq 0\},$$

where $x_i$ is the action at the information set $I_i$.

Given any strategy $s$ of Plaintiff, let $s^{-1}(N) = \{v \colon s(v) = N\}$, which denotes the set of Plaintiff's types at which the value is not revealed to Judge.

**Claim 1**: Given any strategy $s$ of Plaintiff, if $s^{-1}(N) \neq \emptyset$, let $n = |s^{-1}(N)|$. Then in a perfect Bayesian equilibrium, Judge's strategy should be

$$q^*(s) = \left(0, 1, \ldots, 9, \sum_{v \in s^{-1}(N)} \frac{v}{n}\right).$$

By backwards induction, Judge should choose $v$ at the information set $I_v$ when $v \leq 9$.

At the information set $I_{10}$, which is on the equilibrium path, only the branches $v$, where $v \in s^{-1}(N)$ can be reached. Thus, by Bayes' rule, Judge believes that these branches are reached with equal probability, $\frac{1}{n}$, where $n = |s^{-1}(N)|$. Thus, Judge's maximization problem is

$$\max_{x \in \mathbb{R}_+} -\frac{1}{n} \sum_{v \in s^{-1}(N)} (x-v)^2.$$

It is easy to find the unique maximizer $x^*(s) = \frac{1}{n}\sum_{v \in s^{-1}(N)} v$.

**Claim 2**: In any subgame-perfect Nash equilibrium of this game, Plaintiff's strategy $s$ should satisfy the following condition

$$s^{-1}(N) = \{0\} \text{ or } \emptyset.$$

Case 1: assume $s^{-1}(N) = \{v_0\}$, where $v_0 \neq 0$. Given such a Plaintiff's strategy $s$, that is, $s(v_0) = N$, and $s(v) = R$ for others $v$. By Claim 1, Judge's best response is $q^*(s) = (0, 1, 2, \ldots, 9, v_0)$.

However, $s$ is not a best response for Plaintiff given Judge's strategy $q^*(s) = (0, 1, 2, \ldots, 9, v_0)$: at the type 0, Plaintiff can be better off if he chooses "Not Reveal" rather then "Reveal", since $v_0 > 0$.

Case 2: assume $s^{-1}(N)$ contains at least 2 elements. Let $v_1 = \min s^{-1}(N)$, and $v_2 = \max s^{-1}(N)$. Note that,

$$v_1 < x^*(s) = \frac{1}{n} \sum_{v \in s^{-1}(N)} v < v_2.$$

By Claim 2, Judge's best response is

$$q^*(s) = \left(0, 1, 2, \ldots, 9, \sum_{v \in s^{-1}(N)} \frac{v}{n}\right).$$

However, $s$ is not a best response for Plaintiff given Judges' strategy $q^*(s) = \left(0, 1, 2, \ldots, 9, \sum_{v \in s^{-1}(N)} \frac{v}{n}\right)$: at type $t_2$, Plaintiff can get a higher amount $v_2$ by revealing, since $v_2 > \sum_{v \in s^{-1}(N)} \frac{v}{n}$. $\square$

**Claim 3**:

$$s(v) \equiv R, \quad q = (0, 1, 2, \ldots, 9, 0)$$

with belief $(1, 0, \ldots, 0)$ on $I_{10}$ is a perfect Bayesian equilibrium: By Claims 1 and 2, this strategy profile is a subgame-perfect Nash equilibrium.

Assume Judge's belief at the information set $I_{10}$ is $(p_0, p_1, \ldots, p_9)$, then Judge's maximization problem is

$$\max_{x \in \mathbb{R}_+} -p_0(x-0)^2 - p_1(x-1)^2 - \cdots - p_9(x-9)^2.$$

Then the unique maximizer is $x^* = p_0 \cdot 0 + p_1 \cdot 1 + \cdots + p_9 \cdot 9$. We know that $x^* = 0$, this implies $p_0 = 1$ and $p_1 = p_2 = \cdots = p_9 = 0$, that is, Judge believes that $v = 0$ with probability 1.

**Claim 4:**

$$s(v) = \begin{cases} R, & \text{if } v > 0 \\ N, & \text{if } v = 0 \end{cases}, \quad q = (0, 1, 2, \ldots, 9, 0)$$

with belief $(1, 0, \ldots, 0)$ on $I_{10}$ is a perfect Bayesian equilibrium. (Exercise)