Solution to Tutorial 9

2012/2013 Semester I MA4264 Game Theory

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Exercise 1. *We consider a game between two software developers, who sell operating systems (OS) for personal computers (PC). Simultaneously, each software developer i offers "bribe" b_i to the PC maker. (The bribes are in the form of contracts.) Looking at the offered bribes b_1 and b_2, the PC maker accepts the highest bribe (and tosses a fair coin to choose between them if they happen to be equal), and he rejects the other. If a software developers offer is rejected, it goes out of business, and gets 0 profit. Let i^* denote the software developer whose bribe is accepted. Then, i^* pays the bribe b_{i^*} , and the PC maker develops its PC compatible only with the OS of i^*. Then in the next stage, i^* becomes the monopolist in the market for operating systems. In this market the price is given by*

$$P = 1 - Q,$$

where P is the price of the OS and Q is the supply for the OS. The marginal cost of producing the OS for each software developer i is c_i. The costs c_1 and c_2 are independently and identically distributed with the uniform distribution on $[0, 1]$. The software developer i knows its own marginal costs, but the other developer does not know. Each software developer tries to maximize its own expected profit. Everything described so far is common knowledge.

(i) What quantity a software developer i would produce if it becomes monopolist? What would be its profit?

(ii) Compute a Bayesian Nash equilibrium in which each software developers bribe is in the form of $b_i = a_i + e_i(1 - c_i)^2$.

Solution. Leave as Question 2 of Assignment 4. □

Exercise 2. *Find the Bayesian equilibria for the first case of the job-market signaling games in which the output is changed to (i) $y(\eta, e) = 3\eta + e$, and (ii) $y(\eta, e) = 4\eta$.*

Solution. (i) Assume $y(\eta, e) = 3\eta + e$. The extensive-form representation is as follows:

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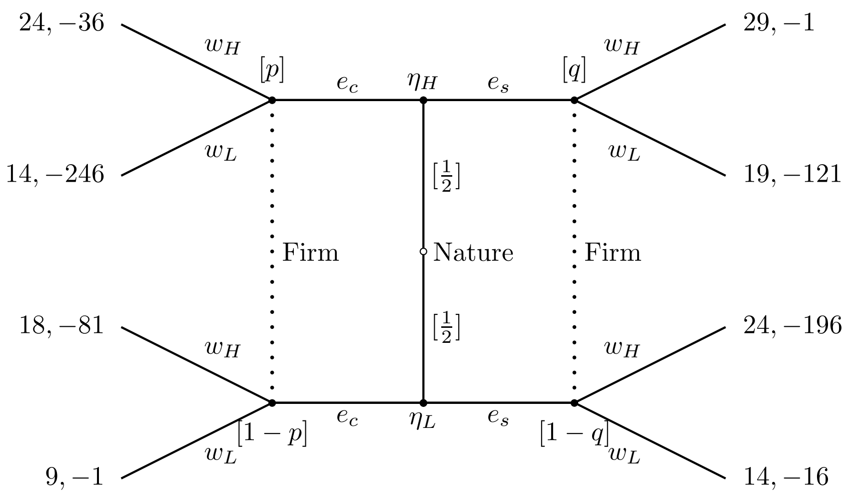


The normal-form representation is as follows:

- $T = \{\eta_H, \eta_L\}$, $M = \{e_c, e_s\}$, $A = \{w_H, w_L\}$.
- Payoff table:

		Firm			
		w_Hw_H	w_Hw_L	w_Lw_H	w_Lw_L
Worker	e_ce_c	21, −117/2	21, −117/2	23/2, −247/2	23/2, −247/2
	e_ce_s	24, −116	19, −26	19, −221	14, −131
	e_se_c	47/2, −41	37/2, −101	19, −1	14, −61
	e_se_s	53/2, −197/2	33/2, −137/2	53/2, −197/2	33/2, −137/2

Therefore, there are two pure-strategy Nash equilibria (e_ce_c, w_Hw_L) and (e_se_s, w_Lw_L).

For (e_ce_c, w_Hw_L):

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since (e_ce_c, w_Hw_L) is a Nash equilibrium)
- Requirement 2R: $q \leq \frac{3}{5}$.
- Requirement 3: $p = \frac{1}{2}$, $q \in [0, 1]$.

Thus, (e_ce_c, w_Hw_L) with $p = \frac{1}{2}$ and $q \leq \frac{3}{5}$ is a perfect Bayesian equilibrium.

For (e_se_s, w_Lw_L):

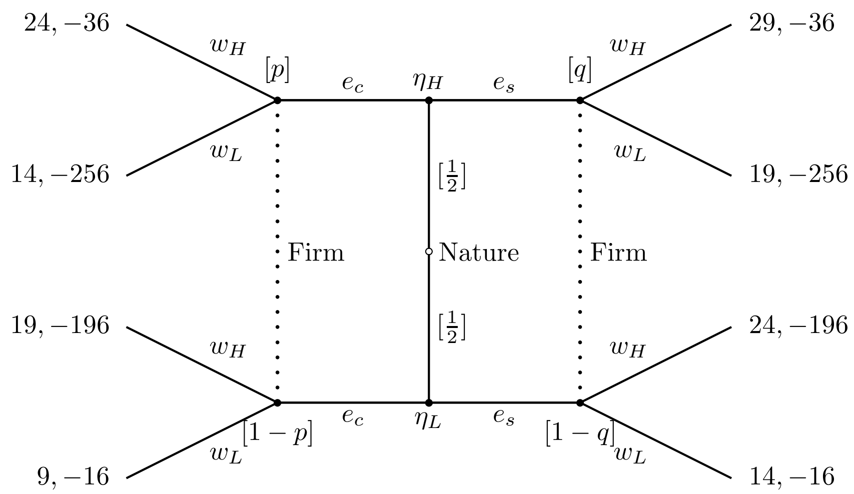
- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since (e_se_s, w_Lw_L) is a Nash equilibrium)
- Requirement 2R: $p \leq \frac{4}{15}$.
- Requirement 3: $p \in [0, 1]$, $q = \frac{1}{2}$.

Thus, (e_ce_c, w_Hw_L) with $p \le \frac{4}{15}$ and $q = \frac{1}{2}$ is a perfect Bayesian equilibrium.

To summarize, there are three pure-strategy perfect Bayesian equilibria:

- (e_ce_c, w_Hw_L) with $p = \frac{1}{2}$ and $q \le \frac{3}{5}$;
- (e_ce_c, w_Hw_L) with $p \le \frac{4}{15}$ and $q = \frac{1}{2}$.

(ii) Assume $y(\eta, e) = 4\eta$. The extensive-form representation is as follows:



The normal-form representation is as follows:

- $T = \{\eta_H, \eta_L\}$, $M = \{e_c, e_s\}$, $A = \{w_H, w_L\}$.
- Payoff table:

		Firm			
		w_Hw_H	w_Hw_L	w_Lw_H	w_Lw_L
Worker	e_ce_c	$43/2, -116$	$43/2, -116$	$23/2, -136$	$23/2, -136$
	e_ce_s	$24, -116$	$19, -26$	$19, -226$	$14, -136$
	e_se_c	$24, -116$	$19, -226$	$19, -26$	$14, -136$
	e_se_s	$53/2, -116$	$33/2, -136$	$53/2, -116$	$33/2, -136$

Therefore, there are three pure-strategy Nash equilibria (e_ce_c, w_hw_L), (e_se_s, w_Hw_H) and (e_se_s, w_Lw_H).

For (e_ce_c, w_Hw_L):

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since (e_ce_c, w_Hw_L) is a Nash equilibrium)
- Requirement 2R: $q \le \frac{9}{20}$.
- Requirement 3: $p = \frac{1}{2}$, $q \in [0, 1]$.

Thus, (e_ce_c, w_Hw_L) with $p = \frac{1}{2}$ and $q \le \frac{9}{20}$ is a perfect Bayesian equilibrium.

For (e_se_s, w_Hw_H):

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since (e_se_s, w_Hw_H) is a Nash equilibrium)
- Requirement 2R: $p \leq \frac{9}{20}$.
- Requirement 3: $p \in [0,1]$, $q = \frac{1}{2}$.

Thus, (e_se_s, w_Hw_H) with $p \leq \frac{9}{20}$ and $q = \frac{1}{2}$ is a perfect Bayesian equilibrium.

For (e_se_s, w_Lw_H):

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since (e_se_s, w_Lw_H) is a Nash equilibrium)
- Requirement 2R: $p \geq \frac{9}{20}$.
- Requirement 3: $p \in [0,1]$, $q = \frac{1}{2}$.

Thus, (e_se_s, w_Lw_H) with $p \geq \frac{9}{20}$ and $q = \frac{1}{2}$ is a perfect Bayesian equilibrium.

To summarize, there are three pure-strategy perfect Bayesian equilibria:

- (e_ce_c, w_Hw_L) with $p = \frac{1}{2}$ and $q \leq \frac{9}{20}$;
- (e_se_s, w_Hw_H) with $p \leq \frac{9}{20}$ and $q = \frac{1}{2}$;
- (e_se_s, w_Lw_H) with $p \geq \frac{9}{20}$ and $q = \frac{1}{2}$.

□

Exercise 3. *Consider the job-market signaling game where $c(\eta, e)$ and $y(\eta, e)$ are general functions and w is chosen from the action space $[0, \infty)$.*

(i) For each of the separating strategies (e_c, e_s) and (e_s, e_c), write down conditions on c and y under which the separating perfect Bayesian equilibria exist.

(ii) Find concrete and reasonable examples of $c(\eta, e)$ and $y(\eta, e)$ which satisfy the conditions you present in (i).

Solution. (i) Suppose in a perfect Bayesian equilibrium, e_ce_s is worker's strategy. Then by Bayes' rule, we have $p = 1$ and $q = 0$.

For firm, given message e_c, his maximization problem is

$$\max_{0 \leq w} -[w - y(\eta_H, e_c)]^2,$$

and hence the best choice is $w_c^* = y(\eta_H, e_c)$. Similarly, given message e_s, firm's best choice is $w_s^* = y(\eta_L, e_s)$.

For worker, given firm's strategy (w_c^*, w_s^*), when η_H occurs, e_c is the best response, that is,

$$y(\eta_H, e_c) - c(\eta_H, e_c) \geq y(\eta_L, e_s) - e(\eta_H, e_s).$$

Similarly, when η_L occurs, we have

$$y(\eta_H, e_c) - c(\eta_L, e_c) \leq y(\eta_L, e_s) - c(\eta_L, e_s).$$

Thus,

$$c(\eta_L, e_c) - c(\eta_L, e_s) \geq y(\eta_H, e_c) - y(\eta_L, e_s) \geq c(\eta_H, e_c) - c(\eta_H, e_s).$$

(ii) Exercise.

□

Exercise 4. *Suppose the HAL Corporation is a monopolist in the Cleveland market for mainframe computers. We will suppose that the market is a "natural monopoly", meaning that only one firm can survive in the long run. HAL faces only one potential competitor, DEC. In the first period, HAL moves first and chooses one of two prices for its computers: High or Low. DEC moves second and decides whether to enter the market or not. Here are the first-period profits of the two firms: In the second period, three things can occur:*

		DEC	
		Enter	StayOut
HAL	High	$0, 0$	$5, 0$
	Low	$0, 0$	$1, 0$

(a) DEC did not enter in the first period. Then HAL retains its monopoly forever and earns monopoly profits of $125 - C$*, where* C *is its costs. DEC earns zero profits.*

(b) DEC entered in the first period and has the lower costs. HAL leaves the market and DEC gets the monopoly forever, earning the monopoly profits of $100 = 125 - 25$*, where* 25 *are its costs, which is common knowledge. HAL earns zero profits.*

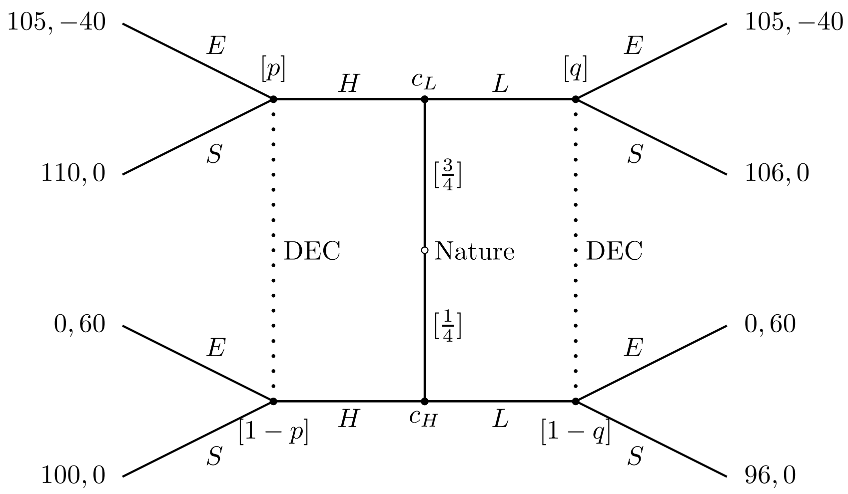
(c) DEC entered in the first period and HAL has the lower costs. In this case, DEC drops out of the market, HAL retains its monopoly forever, and it earns monopoly profits of $125 - C$*, where* C *is its costs. DEC earns zero profits.*

DEC's payoff from playing this game equals 0 if it decides to stay out, and it equals the sum of its profits in the two periods minus entry costs of 40 *if it decides to enter. HAL's payoff equals the sum of its profits in the two periods. HAL's costs,* C*, can be either* 30 *(high) or* 20 *(low). This cost information is private information. DEC only knows that* $\text{Prob}(C = 20) = 0.75$ *and* $\text{Prob}(C = 30) = 0.25$*.*

Formulate the problem as a signaling game and find all perfect Bayesian equilibria.

Solution. The signaling game is as follows:

- $T = \{c_L = 20, c_H = 30\}$, $M = \{H(igh), L(ow)\}$, and $A = \{E(nter), S(tayout)\}$.
- The extensive-form representation is as follows:



The normal-form representation is:

HAL \ DEC	EE	ES	SE	SS
HH	$78.75, -15$	$78.75, -15$	$107.5, 0$	$107.5, 0$
HL	$78.75, -15$	$102.75, -30$	$82.5, 15$	$106.5, 0$
LH	$78.75, -15$	$79.5, 15$	$103.75, -30$	$104.5, 0$
LL	$78.75, -15$	$103.5, 0$	$78.75, -15$	$103.5, 0$

There are three pure-strategy Nash equilibria (HH, SE), (HH, SS) and (LL, ES).
For (HH, SE):

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since (HH, SE) is a Nash equilibrium)
- Requirement 2R: $q \leq 0.6$.
- Requirement 3: $p = \frac{3}{4}$, $q \in [0, 1]$.

Thus, (HH, SE) with $p = \frac{3}{4}$ and $q \leq 0.6$ is a perfect Bayesian equilibrium.
For (HH, SS):

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since (HH, SS) is a Nash equilibrium)
- Requirement 2R: $q \geq 0.6$.
- Requirement 3: $p = \frac{3}{4}$, $q \in [0, 1]$.

Thus, (HH, SS) with $p = \frac{3}{4}$ and $q \geq 0.6$ is a perfect Bayesian equilibrium.
For (LL, ES):

- Requirement 1: Assume the believes on left information set and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.

- Requirement 2S: Holds automatically. (since (LL, ES) is a Nash equilibrium)
- Requirement 2R: $p \leq 0.6$.
- Requirement 3: $p \in [0, 1]$, $q = \frac{3}{4}$.

Thus, (LL, ES) with $p \leq 0.6$ and $q = \frac{3}{4}$ is a perfect Bayesian equilibrium.

To summarize, there are three pure-strategy perfect Bayesian equilibria:

- (HH, SE) with $p = \frac{3}{4}$ and $q \leq 0.6$;
- (HH, SS) with $p = \frac{3}{4}$ and $q \geq 0.6$;
- (LL, ES) with $p \leq 0.6$ and $q = \frac{3}{4}$.

□

Exercise 5. *There are two Players in the game: Judge and Plaintiff. The Plaintiff has been injured. Severity of the injury, denoted by v, is the Plaintiff's private information. The Judge does not know v and believes that v is uniformly distributed on $\{0, 1, \ldots, 9\}$ (so that the probability that $v = i$ is $\frac{1}{10}$ for any $i \in \{0, 1, \ldots, 9\}$). The Plaintiff can verifiably reveal v to the Judge without any cost, in which case the Judge will know v. The order of the events is as follows. First, the Plaintiff decides whether to reveal v or not. Then, the Judge rewards a compensation R which can be any nonnegative real number. The payoff of the Plaintiff is $R - v$, and the payoff of the Judge is $-(v - R)^2$. Everything described so far is common knowledge. Find a perfect Bayesian equilibrium.*

Solution. The signaling game is as follows: types $T = \{0, 1, \ldots, 9\}$; signals $M = \{R, N\}$, where R is "Reveal" and N is "Not Reveal"; actions $A = \mathbb{R}_+$.

From the extensive-form representation, there are 10 subgames, and Judge has 11 information sets $I_0, I_1, \ldots, I_9$, where for $v = 0, 1 \ldots, 9$, I_v denotes that Plaintiff reveals v to Judge, and I_{10} denotes the case that Plaintiff does not reveal the value.

Plaintiff's strategy space is

$$S = \{s = (s_0, s_1, \ldots, s_9) \mid s_v = R \text{ or } N, v = 0, 1, \ldots, 9\}.$$

For a particular strategy of Plaintiff $s = (s_0, s_1, \ldots, s_9)$, s_v is the action of Plaintiff when she/he faces injury v.

Judge's strategy space is

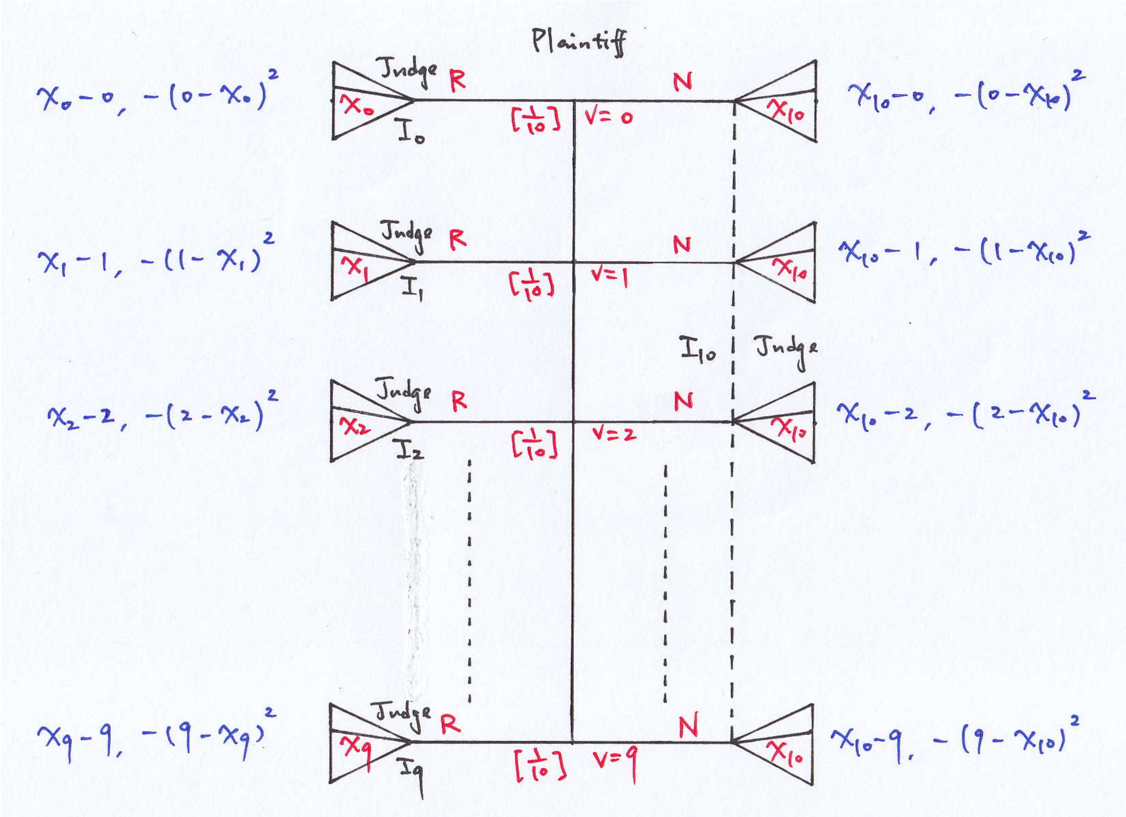
$$Q = \{q = (x_0, x_1, \ldots, x_9, x_{10}) \mid x_v \geq 0, v = 0, 1, \ldots, 9, 10\}.$$

For a particular strategy of Judge $q = (x_0, x_1, \ldots, x_9, x_{10})$, x_v is the action of Judge at the information set I_v.

Given any strategy s of Plaintiff, let $s^{-1}(N) = \{v \colon s(v) = N\}$, which denotes the set of Plaintiff's types at which the value is not revealed to Judge.

Claim 1: In any perfect Bayesian equilibrium (s^*, q^*, p^*), if Plaintiff chooses R when $v = 0$, that is $s_0^* = R$, then Judge's action on information set I_{10} should be 0, that is, $x_{10}^* = 0$.

Proof of Claim 1: Otherwise, Plaintiff can be better off by deviating from R to N: If Plaintiff chooses R when $v = 0$, then she/he will get 0 when $v = 0$; otherwise she/he will get $x_{10}^* > 0$. Therefore, such a strategy s^* can not be a strategy in a perfect Bayesian equilibrium, which is a contradiction.



Claim 2: In a perfect Bayesian equilibrium (s^*, q^*, p^*), if $(s^*)^{-1}(N) \neq \emptyset$, then Judge's strategy should be

$$q^* = \left(0, 1, \ldots, 9, \sum_{v \in s^{-1}(N)} \frac{v}{n}\right),$$

where $n = |s^{-1}(N)|$.

Motivation of Claim 2: Based on Judge's belief p^*, her/his optimal action x_{10}^* should be wighted payoff

$$0 \cdot p_0^* + 1 \cdot p_1^* + 2 \cdot p_2^* + \cdots + 9 \cdot p_9^*.$$

Given Plaintiff's strategy s^*, Judge's belief p^* on the information set I_{10} can be determined by Bayes' law.

Proof of Claim 2: (s^*, q^*) should be a subgame-perfect Nash equilibrium, and hence on the information set $I_v (v = 0, 1, \ldots, 9)$, Judge will choose optimal action based on her/his payoff $-(v - x_v)^2$. Therefore, Judge's action on the information set I_v should be v $(v = 0, 1, 2 \ldots, 9)$.

On the information set I_{10}, which is on the equilibrium path, only the branches v, where $v \in s^{-1}(N)$ can be reached. Thus, by Bayes' rule, Judge believes that these branches are reached with equal probability, $\frac{1}{n}$, where $n = |s^{-1}(N)|$. Thus, Judge will choose the optimal action based on her/his expected payoff, and the optimal action is the maximizer of the following maximization problem

$$\max_{x_{10} \geq 0} -\frac{1}{n} \sum_{v \in s^{-1}(N)} (v - x_{10})^2.$$

By first order condition, it is easy to find the unique maximizer $x_{10}^* = \frac{1}{n}\sum_{v \in s^{-1}(N)} v$.

Claim 3: In any perfect Bayesian equilibrium (s^*, q^*, p^*), Plaintiff's strategy s^* should be

$$(R, R, \ldots, R) \text{ or } (N, R, \ldots, R).$$

Proof of Claim 3: Case 1: assume $(s^*)^{-1}(N) = \{v_0\}$, where $v_0 \neq 0$. Given such a Plaintiff's strategy s^*, that is, $(s^*)^*(v_0) = N$, and $(s^*)^*(v) = R$ for others v, by Claim 2, Judge's best response is

$$q^* = (0, 1, 2, \ldots, 9, v_0).$$

However, s^* is not a best response for Plaintiff given Judge's strategy $q^*(s)$: when $v = 0$, Plaintiff can be better off if she/he chooses N rather then R: if she/he chooses R, she/he will get 0; otherwise, she/he will get $v_0 > 0$.

Case 2: assume $(s^*)^{-1}(N)$ contains at least 2 elements. Let $v_1 = \min(s^*)^{-1}(N)$, and $v_2 = \max(s^*)^{-1}(N)$. Note that,

$$v_1 < x_{10}^* = \frac{1}{n}\sum_{v \in s^{-1}(N)} v < v_2.$$

By Claim 2, Judge's best response is

$$q^* = \left(0, 1, 2, \ldots, 9, \sum_{v \in s^{-1}(N)} \frac{v}{n}\right).$$

However, s^* is not a best response for Plaintiff given Judges' strategy q^*: when the injury is v_2, Plaintiff can get a higher amount v_2 by revealing: if she/he chooses N, she/he will get $x_{10}^* - v_2 < 0$; otherwise she/he will get 0.

Case 2 implies that there is at most 1 type at which Plaintiff chooses N in a perfect Bayesian equilibrium; and Case 1 implies that this unique type can only be $v = 0$.

Based on Claim 3, we have the following two claims:

Claim 4:

$$s^* = (N, R, \ldots, R), \quad q^* = (0, 1, 2, \ldots, 9, 0)$$

with belief $(1, 0, \ldots, 0)$ on I_{10} is a perfect Bayesian equilibrium.

Proof of Claim 4: Routine.

Claim 5:

$$s^* = (R, R, \ldots, R), \quad q^* = (0, 1, 2, \ldots, 9, 0)$$

with belief $(1, 0, \ldots, 0)$ on I_{10} is a perfect Bayesian equilibrium:

Proof of Claim 5: By Claims 1, 2 and 3, this strategy profile could be a strategy profile in a perfect Bayesian equilibrium.

Assume Judge's belief on the information set I_{10} is $(p_0^*, p_1^*, \ldots, p_9^*)$, then Judge's maximization problem is

$$\max_{x_{10} \geq 0} -p_0^*(x_{10} - 0)^2 - p_1^*(x_{10} - 1)^2 - \cdots - p_9^*(x_{10} - 9)^2.$$

Then the unique maximizer is $x_{10}^* = p_0^* \cdot 0 + p_1^* \cdot 1 + \cdots + p_9^* \cdot 9$. We have already known that $x_{10}^* = 0$, this implies $p_0^* = 1$ and $p_1^* = p_2^* = \cdots = p_9^* = 0$, that is, Judge believes that $v = 0$ with probability 1. □

Exercise 6. *Player 1 has two types, intelligent or dumb, with equal probability of each type. Player 1 may choose either to drop out of high school or finish high school. If he finishes high school, player 2 must decide whether or not to hire player 1. Player 1 knows his type, but player 2 does not. If player 1 drops out, both players get zeros. If player 1 finishes high school, but is not employed by player 2, player 2 gets nothing, and player 1 gets x if intelligent, and y if dumb, where $y > x > 0$, and $1 > x$, but y may be either larger or smaller than 1. If player 1 finishes high school and is employed, player 2 gets a if player 1 is intelligent and b if player 1 is dumb, where $a > b$. Here $a > 0$ but b may be either positive or negative. Player 1 gets $1 - x$ if intelligent and $1 - y$ if dumb.*

(a) For what values of a, b, x, y is there a perfect Bayesian equilibrium in which both types drop out?

(b) For what values of a, b, x, y is there a perfect Bayesian equilibrium which is separating.

Solution. Figure 1 is the extensive-form representation of this game.

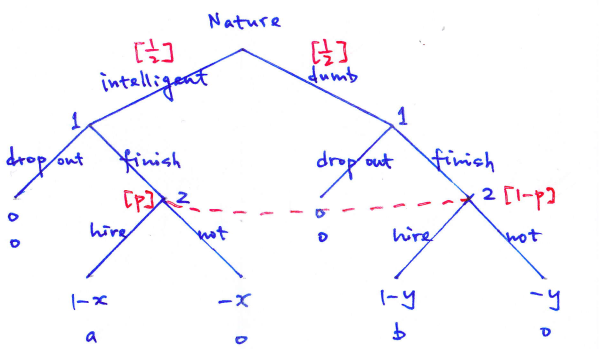


Figure 1

The normal-form representation is as follows:

- $S_1 = \{dd, df, fd, ff\}$, where d and f denote "drop out" and "finish", respectively. $S_2 = \{h, n\}$, where h and n denote "hire" and "not hire", respectively.
- Payoff table:

(a) Since $x < 1$, $\frac{1-x}{2} > 0$, and hence dd could not be a best response to h. Since $y > x > 0$, dd is a best response to n.

Player 2

		h	n
Player 1	dd	$0, 0$	$0, 0$
	df	$\frac{1-y}{2}, \frac{b}{2}$	$-\frac{y}{2}, 0$
	fd	$\frac{1-x}{2}, \frac{a}{2}$	$-\frac{x}{2}, 0$
	ff	$\frac{1-x}{2} + \frac{1-y}{2}, \frac{b}{2} + \frac{a}{2}$	$-\frac{x}{2} - \frac{y}{2}, 0$

Since Player 1 chooses dd, Player 2's information set will not be reached, and hence the belief could be arbitrary by Requirement 4. To support n is Player 2's best choice given his belief, b should be nonpositive, otherwise n is strictly dominated by h.

To summarize, we need $b \leq 0$.

(b) Since $x, y > 0$, each of df and fd can not be a best response to n. Since $y > x$, df can no be a best response to h.

fd to be a best response to h if and only if $y \geq 1$. By Bayes' rule, $p = 1$, and since $a > 0$, h is a best choice given this belief.

To summarize, we need $y \geq 1$.

$\square$

End of Solution to Tutorial 9