"How To" for MA1101R

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This cute file is provided for the students in MA1101R. I hope it is helpful to "kill" the Final Examination. Good luck! By the way, corrections and suggestions are always welcome (Email: xiangsun@nus.edu.sg).

User' guide: For each how-to question, I will list the most powerful methods, related examples (chosen from Examples in the textbook and Exercises in the tutorial questions) and remarks if necessary.

Besides, I am grateful for helpful commends and suggestion from my three friends.

1 Chapter 1

Question 1. *How to identify a row-echelon form (REF) and a reduced row-echelon form (RREF)?*

Answer. [Page 7] A matrix is said to be in row-echelon form (REF) if it has properties (1–2):

(1) If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

(2) In any two successive nonzero rows, the first nonzero number in the lower row occurs farther to the right than the first nonzero number in the higher row.

A matrix is said to be in reduced row-echelon form (RREF) if it is has properties (1–4):

(3) The leading entry of every nonzero row is 1.

(4) In each pivot column, except the pivot point, all other entries are zeros.

Properties (3) and (4) are the differences between a REF and a RREF.
✠ Example: Exercises 1.21 and 1.22.
♣ Remark: In the textbook, the REF and RREF are defined for the augmented matrix, but they can be generalized for any arbitrary matrix. □

Question 2. *Given a matrix, how to a REF or a RREF for it?*

Answer. [Page 10] If we want to get a REF, we should apply Gaussian elimination. If we want to get a RREF, then we should apply Gauss-Jordan elimination.
✠ Example: Example 1.4.3.
♣ Remark: In the textbook, Gaussian elimination and Gauss-Jordan elimination are defined for the augmented matrix, but they can be generalized for any arbitrary matrix, too. □

Question 3. *How to tell the number of solutions of linear system from REF?*

Answer. [Page 14–15] There are the following three cases:

- A linear system has no solution if and only if the last column of its REF of the augmented matrix is a pivot column, i.e. there is a row with non-zero last entry but zero elsewhere.
- A linear system has exactly one solution if and only if except the last column, every column of a REF of the augmented matrix is a pivot column. That is, A linear system has exactly one solution if and only if it is consistent and (# variables) = (# nonzero rows).

- A linear system has infinitely many solutions if and only if apart from the last column, a REF of the augmented matrix has at least one more non-pivot column. That is, A linear system has exactly one solution if and only if it is consistent and (# variables) > (# nonzero rows).

  In this case, its general solution has (# variables) − (# nonzero rows) arbitrary parameter(s).

✠ Example: Example 1.4.9, Exercises 1.18 and 1.23. □

Question 4. *If a linear system is consistent, how to find a general solution of it?*

Answer. We need follow this process:

1. Transfer the linear system to the related augmented matrix;
2. Apply Gaussian elimination (resp. Gauss-Jordan elimination) to obtain a REF (resp. a RREF) of the augmented matrix;
3. Identify the pivot columns and non-pivot columns;
4. For any i, if the i-th column is a non-pivot column, then take the i-th variable to be a parameter.
5. Express the left variable(s) as a expression of the parameter(s).

✠ Example: Example 1.4.6. □

2 Chapter 2

Question 5. *How to identify whether an $m \times n$ matrix $\boldsymbol{A}$ is invertible?*

Answer. If $\boldsymbol{A}$ is not square, it can not be invertible. If $\boldsymbol{A}$ is square, we have the following four methods.

- First method: [Page 56] $\boldsymbol{A}$ is invertible iff $\det(\boldsymbol{A}) \neq 0$. So after computing the determinant of $\boldsymbol{A}$, it is easy to identify the invertibility of $\boldsymbol{A}$.
- Second method: [Page 161] $\boldsymbol{A}$ is invertible iff $\operatorname{rank}(\boldsymbol{A}) = n$. So we may apply Gaussian elimination to obtain a REF for $\boldsymbol{A}$, then find the $\operatorname{rank}(\boldsymbol{A})$ and identify whether $\operatorname{rank}(\boldsymbol{A}) = n$.
- Third method: [Page 38] If we can find a matrix $\boldsymbol{B}$, such that $\boldsymbol{AB} = \boldsymbol{I}$ or $\boldsymbol{BA} = \boldsymbol{I}$, then by definition $\boldsymbol{A}$ is invertible.

  ✠ Example: Examples 2.3.3 and 2.3.9.
- Fourth method: [Page 46] $\boldsymbol{A}$ is invertible iff the RREF of $\boldsymbol{A}$ is an identity matrix.

  ✠ Example: Example 2.4.9 and Exercise 2.36.

✠ Example: Exercise 2.37.
♣ Remark: Actually the second method and the fourth method are same. □

Question 6. *How to find the inverse of an invertible matrix $\boldsymbol{A}$?*

Answer. There are the following three methods:

- First method: [Page 47]
  1. Consider the $n \times 2n$ matrix $(\boldsymbol{A} \mid \boldsymbol{I})$.
  2. By Gauss-Jordan elimination, we will have a sequence elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$, such that $\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}$, that is,

  $$\boldsymbol{A}^{-1} = \boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1.$$

  ✠ Example: Example 2.4.7.

- Second method: [Page 59] When $\boldsymbol{A}$ is sparse (non-zero entries is few), we may apply $\mathrm{adj}(\boldsymbol{A})$ to compute $\boldsymbol{A}^{-1}$:

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\,\mathrm{adj}(\boldsymbol{A}).$$

  ✠ Example: Example 2.5.31.

- Third method: [Page 38] If we can find a matrix $\boldsymbol{B}$, such that $\boldsymbol{AB} = \boldsymbol{I}$ or $\boldsymbol{BA} = \boldsymbol{I}$, then by definition $\boldsymbol{A}$ is invertible.

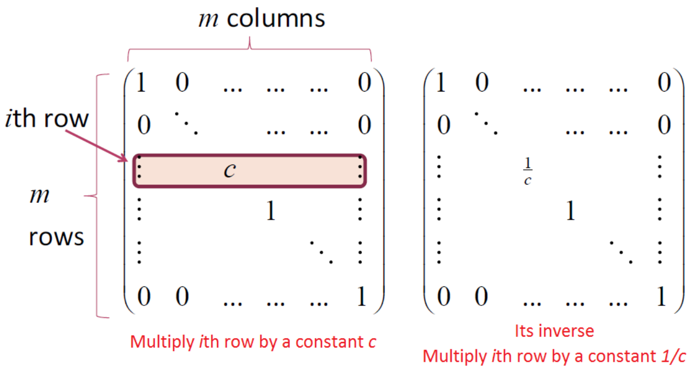
  ✠ Example: Exercises 2.20, 2.21 and 2.22.

□

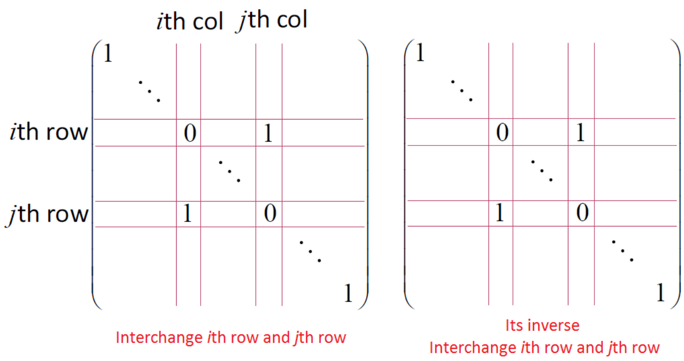
Question 7. *How to convert elementary row operations to elementary matrices, and vice versa?*

Answer. [Page 42–45] There are the following three cases:

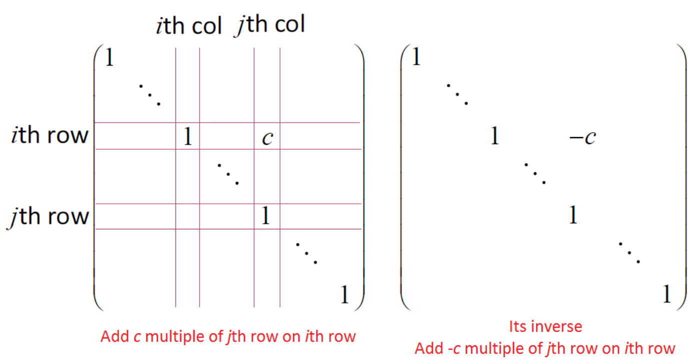
- Multiply a row by a constant:

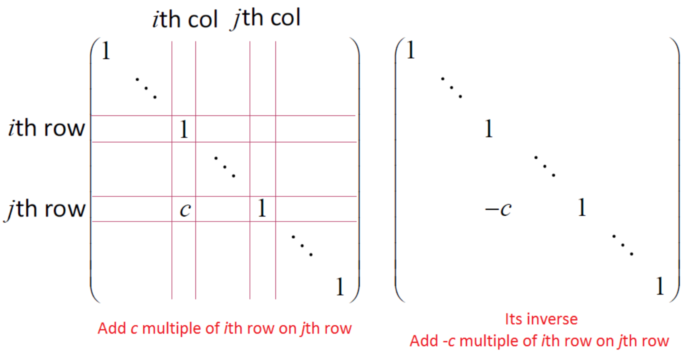


- Interchange two rows:



- Add a multiple of a row by a constant:





□

Question 8. *How to express Gaussian elimination as product of elementary matrices?*

Answer. [Page 46] Let $\boldsymbol{R}$ be a REF of $\boldsymbol{A}$ obtained by Gaussian elimination, then $\boldsymbol{R}$ is obtained from $\boldsymbol{A}$ by elementary row operations. For each of these elementary row operations, we write down the corresponding elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \dots, \boldsymbol{E}_k$:

$$\boldsymbol{A} \xrightarrow{\boldsymbol{E}_1} \xrightarrow{\boldsymbol{E}_2} \cdots \xrightarrow{\boldsymbol{E}_k} \boldsymbol{R}.$$

Then Gaussian elimination will be expressed as

$$\boldsymbol{E}_k \boldsymbol{E}_{k-1} \cdots \boldsymbol{E}_1.$$

✠ Example: Example 2.4.4 and Exercise 2.26. □

Question 9. *How to compute determinant of a matrix $\boldsymbol{A}$ using various methods (and not just cofactor expansion)?*

Answer. We have the following three methods:

- First method: [Page 50–52] Apply the definition or cofactor expansion.

  ✠ Example: Examples 2.5.4 and 2.5.12.

- Second method: [Page 50–51] For the 2×2 and 3×3 matrices, we have the following formulas:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi.$$

- Third method: [Page 53–56] Apply Gauss elimination or Gauss-Jordan elimination to get a REF or a RREF $\boldsymbol{R}$ for $\boldsymbol{A}$, that is, we have a sequence of elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \dots, \boldsymbol{E}_k$, such that

$$\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{R}.$$

  Then we have

$$\det(\boldsymbol{A}) = \det(\boldsymbol{E}_k)^{-1} \cdots \det(\boldsymbol{E}_2)^{-1} \det(\boldsymbol{E}_1)^{-1} \det(\boldsymbol{R}),$$

  where $\boldsymbol{R}$ is an upper-triangular matrix whose determinant is the product of the diagonal entries.

  ✠ Example: Example 2.5.21 and Exercise 2.42.

□

Question 10. *How does a row (column) operation change the determinant?*

Answer. [Page 53–55] There are the following three cases:

- If $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by multiplying one row of $\boldsymbol{A}$ by a constant k, then $\det(\boldsymbol{B}) = k \det(\boldsymbol{A})$;

- If $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by interchanging two rows of $\boldsymbol{A}$, then $\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$;
- If $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by adding a multiple of one row of $\boldsymbol{A}$ to another row, then $\det(\boldsymbol{B}) = \det(\boldsymbol{A})$.

□

Question 11. *How are invertibility of a matrix and the homogeneous system related?*

Answer. [Page 46] A matrix $\boldsymbol{A}$ is invertible iff the linear homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
✠ Example: Exercise 2.35. □

Question 12. *How are invertibility and determinant of a matrix related?*

Answer. [Page 56] A matrix $\boldsymbol{A}$ is invertible iff $\det(\boldsymbol{A}) \neq 0$.
✠ Example: Example 2.5.25. □

3 Chapter 3

Question 13. *How to write down implicit and explicit set notation for lines and planes in $\mathbb{R}^3$?*

Answer. [Page 77] The implicit form for a line in $\mathbb{R}^3$ is

$$\{(x, y, z) \mid a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2\},$$

where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are real constants, a_1, b_1, c_1 are not all zero, and a_2, b_2, c_2 are not all zero.

The explicit form for a line in $\mathbb{R}^3$ is

$$\{(a_0, b_0, c_0) + t(a, b, c) \mid t \in \mathbb{R}\},$$

where a_0, b_0, c_0, a, b, c are real constants and a, b, c are not all zero.

The implicit form for a plane in $\mathbb{R}^3$ is

$$\{(x, y, z) \mid ax + by + cz = d\},$$

where a, b, c, d are real constants, and a, b, c are not all zero.

The explicit form for a plane in $\mathbb{R}^3$ is

$$\begin{cases} \{(\frac{d-bs-ct}{a}, s, t) \mid s, t \in \mathbb{R}\}, & \text{if } a \neq 0; \\ \{(s, \frac{d-as-ct}{b}, t) \mid s, t \in \mathbb{R}\}, & \text{if } b \neq 0; \\ \{(s, t, \frac{d-as-bt}{c}) \mid s, t \in \mathbb{R}\}, & \text{if } c \neq 0. \end{cases}$$

□

Question 14. *How to find a line in $\mathbb{R}^2$ or $\mathbb{R}^3$ when you are given the direction vector of the line and a point on the line?*

Answer. [Page 77] In $\mathbb{R}^2$, given a pint on the line (a_0, b_0), and the direction (a, b), then the line is

$$\{(a_0, b_0) + t(a, b) \mid t \in \mathbb{R}\}.$$

In $\mathbb{R}^3$, given a pint on the line (a_0, b_0, c_0), and the direction (a, b, c), then the line is

$$\{(a_0, b_0, c_0) + t(a, b, c) \mid t \in \mathbb{R}\}.$$

If we want to get an implicit form for the line $\{(a_0, b_0, c_0) + t(a, b, c) \mid t \in \mathbb{R}\}$, we should remove the parameter t, and construct 2 equations in terms of x, y, z. Indeed, let $x = a_0 + ta$, $y = b_0 + tb$ and $z = c_0 + tc$. Now we need to remove t. From $x = a_0 + ta$ and $y = b_0 + tb$, we will obtain

$$bx - ay = a_0b - b_0a. \ (1).$$

Similarly, from $y = b_0 + tb$ and $z = c_0 + tc$, we will have

$$cy - bz = b_0 c - c_0 b \ (2).$$

Then the set

$$\{(x, y, z) \mid bx - ay = a_0 b - b_0 a, cy - bz = b_0 c - c_0 b\}$$

is the implicit form for the line. □

Question 15. *How to find the equation of a plane in $\mathbb{R}^3$ when you are given three points on the plane?*

Answer. [Page 77] Assume the plane is represented by the equation $ax + by + cz = d$, where a, b, c are not all zero. Since there are three points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ on the plane, they satisfy the equation $ax + by + cz = d$. Then consider the following linear system in terms of a, b, c, d:

$$\begin{cases} x_1 a + y_1 b + z_1 c - d = 0 \\ x_2 a + y_2 b + z_2 c - d = 0 \\ x_3 a + y_3 b + z_3 c - d = 0 \end{cases}$$

We may have a general solution for a, b, c, d (should be determined by 1 parameter), and take a special one, then we will get an equation for plane.
♣ Remark: if the three points are on a line, the plane can not be determined uniquely. □

Question 16. *How to determine whether a vector $\boldsymbol{u}$ is a linear combination of a given set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$?*

Answer. [Page 78–79] Assume $\boldsymbol{u} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{u}_k$. Then we will have a linear system in terms of $c_1, c_2, \ldots, c_k$:

$$\begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{u},$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k, \boldsymbol{u}$ are column vectors.

Since we have that $\boldsymbol{u}$ is a linear combination iff the linear system is consistent, it suffices to find whether the linear system is consistent.
✠ Example: Example 3.2.2. □

Question 17. *How to express a vector $\boldsymbol{u}$ as a linear combination of a given set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$?*

Answer. Following the process in the last question, if the linear system is consistent, and we have found a solution $(x_1, x_2, \ldots, x_k)$, then we have

$$\boldsymbol{u} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_k\boldsymbol{u}_k.$$

✠ Example: Example 3.2.2. □

Question 18. *How to show a linear span* span(S_1) *is contained in another one* span(S_2)*, where $S_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$, $S_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$?*

Answer. [Page 84–85] Since $\text{span}(S_1) \subset \text{span}(S_2)$ iff each $\boldsymbol{u}_i$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$, it suffices to show that the each column of last m columns in a REF of the following matrix is not a pivot column:

$$(\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m \mid \boldsymbol{u}_1 \mid \boldsymbol{u}_2 \mid \boldsymbol{u}_k),$$

where $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are column vectors.
✠ Example: Example 3.2.12 and Exercise 3.16. □

Question 19. *How to show that a set V is a subspace of $\mathbb{R}^n$?*

Answer. Firstly, we should show that V is a subset of $\mathbb{R}^n$. Then there are the following three methods:

- First method: [Page 79–80] If we can find a span set S for V, the V is a subspace of $\mathbb{R}^n$ by definition directly.

  ✠ Example: Example 3.2.6, Exercises 3.10, 5.6.

- Second method: [Page 107] V is a subspace iff $V \neq \emptyset$ and for any $\boldsymbol{u}, \boldsymbol{v} \in V$, and $a, b \in \mathbb{R}$, $a\boldsymbol{u} + b\boldsymbol{v} \in V$.

  ✠ Example: Exercises 3.10, 3.11, 3.12, 3.22 and 5.6.

- Third method: [Page 82] If V is the solution set for a homogeneous linear system, then V is a subspace.

  ✠ Example: Exercises 3.11, 3.12 and 5.6.

□

Question 20. *How to show that a set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is linearly (in)dependent?*

Answer. [Page 86–87] Apply working definition. The equation

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \boldsymbol{0}$$

gives us a linear system in terms of $c_1, c_2, \ldots, c_k$:

$$\begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{0},$$

where $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k, \boldsymbol{0}$ are column vectors.

By Gaussian elimination, we can identify whether this linear system has only the trivial solution, and we have the fact that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is linearly independent iff this linear system has only the trivial solution.

✠ Example: Example 3.3.3. □

Question 21. *How to show that a set S is a basis for a vector space V?*

Answer. [Page 90–97] We have the following four conditions:

- Condition (1): $S \subset V$;
- Condition (2-1): S is linearly independent;
- Condition (2-2): $V = \mathrm{span}(S)$;
- Condition (2-3): $|S| = \dim(V)$.

If condition (1) and any 2 of conditions (2-1), (2-2) and (2-3) hold, then S is a basis for V.

♣ Remark: if we know $\dim(V)$, then we choose conditions (1), (2-1) and (2-3) to check; if we do not know $\dim(V)$, we need to check conditions (1), (2-1), and (2-2).

✠ Example: Examples 3.3.4, 3.5.7, Exercises 3.25, 3.31 and 3.37. □

Question 22. *How to find a basis for a vector space V?*

Answer. [Page 91–96]

1. Find a span set $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ for V, where $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are column vectors;
2. Apply Gaussian elimination to the matrix $\begin{pmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \\ \vdots \\ \boldsymbol{u}_k^T \end{pmatrix}$, get a REF R.

3. Let S' be the set of non-zero rows in R, then S' is a basis for V.

♣ Remark: S' is not necessarily unique.

✠ Example: Examples 3.3.7, 3.5.4 and Exercise 4.11. □

Question 23. *Given a basis* $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ *for a vector space* V *and a vector* $\boldsymbol{u} \in V$*, how to find coordinate vectors* $[\boldsymbol{u}]_S$ *and* $(\boldsymbol{u})_S$*?*

Answer. [Page 93–94] Let $\boldsymbol{u} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$. By solving the following linear system

$$\begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{u},$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k, \boldsymbol{u}$ are column vectors, we will find a solution $(x_1, x_2, \ldots, x_k)$. Then the coordinate vectors are

$$[\boldsymbol{u}]_S^T = (\boldsymbol{u})_S = (x_1, x_2, \ldots, x_k).$$

✠ Example: Example 3.4.7. □

Question 24. *How to compute dimension for a vector space?*

Answer. [Page 95–97] Following the process in the last question, S' is a basis for V. Then we have

$$\dim(V) = |S'|.$$

✠ Example: Example 3.5.4.

♣ Remark: Except $\mathbb{R}^n$ and $\{\mathbf{0}\}$, we can identify $\dim(V)$ only when we have found a basis for it. □

Question 25. *How to compute transition matrix from* $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ *to* $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$*, where* S *and* T *are bases for a vector space* V*?*

Answer. There are the following two methods:

- First method: [Page 100] By solving linear systems, we will find $[\boldsymbol{u}_1]_T, [\boldsymbol{u}_2]_T, \ldots, [\boldsymbol{u}_k]_T$. Then the transition matrix from S to T is

$$\boldsymbol{P} = ([\boldsymbol{u}_1]_T \; [\boldsymbol{u}_2]_T \; \cdots \; [\boldsymbol{u}_k]_T).$$

  ✠ Example: Example 3.6.3.

- Second method: [Page 102] If we know the transition matrix $\boldsymbol{Q}$ form T to S, then the transition $\boldsymbol{P}$ from S to T is $\boldsymbol{Q}^{-1}$

  ✠ Example: Example 3.6.5.

□

4 Chapter 4

Question 26. *How to find bases for row space or column space of* $\boldsymbol{A}$*?*

Answer. [Page 112–116] We may apply the similar method in Question 22:

Apply Gaussian elimination to $\boldsymbol{A}$ to obtain a REF $\boldsymbol{R}$ of $\boldsymbol{A}$. Then the non-zero rows of $\boldsymbol{R}$ form a basis for the row space of $\boldsymbol{A}$.

Besides, we will find a set of columns of $\boldsymbol{R}$ which forms a basis for the column space of $\boldsymbol{R}$ (choose the pivot columns), then the set of corresponding columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$.

✠ Example: Examples 4.1.12, 4.1.14, Exercise 4.11. □

Question 27. *How to extend a linearly independent set S to a basis for $\mathbb{R}^n$?*

Answer. [Page 116–117] We need follow this process:

1. Form a matrix $\boldsymbol{A}$ using the vector in S as rows;
2. Reduce $\boldsymbol{A}$ to a REF $\boldsymbol{R}$;
3. Identify the non-pivot columns in $\boldsymbol{R}$;
4. For each non-pivot column identified in Step 3, get a vector such that leading entry of the vector is at that column;
5. Now, $S \cup$ (the set of vectors obtained in Step 4) is a basis for $\mathbb{R}^n$.

✠ Example: Example 4.1.14. □

Question 28. *How to find the rank and nullity for a matrix $\boldsymbol{A}$?*

Answer. [Page 118–122] Apply Gaussian elimination for $\boldsymbol{A}$ to get a REF $\boldsymbol{R}$, then

$$\text{rank}(\boldsymbol{A}) = (\#\text{ leading entries in } \boldsymbol{R}),$$

and

$$\text{nullity}(\boldsymbol{A}) = \#\text{ columns of } \boldsymbol{A} - \text{rank}(\boldsymbol{A}).$$

✠ Example: Exercise 4.13. □

5 Chapter 6

Question 29. *How to find eigenvalues and eigenvectors of a matrix?*

Answer. There are the following three methods:

- First method: [Page 160] Solving the characteristic equation.

  ✠ Example: Examples 6.1.8, 6.1.11, and Exercise 6.3.

- Second method: [Page 158] By definition, find λ and $\boldsymbol{x}$, such that $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, then λ is an eigenvalue of $\boldsymbol{A}$.

  ✠ Example: Exercise 6.14.

- Third method: If we have such an equation

  $$\boldsymbol{A}^k + a_{k-1}\boldsymbol{A}^{k-1} + \cdots + a_1\boldsymbol{A} + a_0\boldsymbol{I} = \boldsymbol{0},$$

  then any solution for the equation $x^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0 = 0$ is an eigenvalue of $\boldsymbol{A}$.

  ✠ Example: Exercise 6.4.

♣ Remark: If the $\boldsymbol{A}$ is an "abstract" matrix (For example, stochastic matrices), we can only apply the second method. □

Question 30. *How to find basis for eigenspace of a matrix?*

Answer. [Page 162–164] Given an eigenvalue λ of $\boldsymbol{A}$. By solving the linear system

$$(\lambda\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0},$$

we will have a general solution, for example, $\boldsymbol{x} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k$. That is, $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is a span set for the eigenspace E_λ. Then we could follow the process in Question 22, to find a set S', such that S' is a basis for E_λ.
✠ Example: Example 6.1.13. □

Question 31. *How is eigenvalue related to invertibility of matrix?*

Answer. The matrix $\boldsymbol{A}$ is invertible iff 0 is not an eigenvalue of $\boldsymbol{A}$. □

Question 32. *How to determine if a matrix $\boldsymbol{A}$ is diagonalizable?*

Answer. There are the following two methods:

- First method: [Page 171] If $\boldsymbol{A}$ has n distinct eigenvalues, then $\boldsymbol{A}$ is diadonalizable.
  ✠ Example: Example 6.2.12.
- Second method: [Page 167–169] Apply Algorithm 6.2.4.
  ✠ Example: Example 6.2.6 and Exercise 6.12.

□

Question 33. *How to diagonalize a matrix?*

Answer. [Page 167–169] Apply Algorithm 6.2.4.
✠ Example: Example 6.2.6 and Exercise 6.10. □

Question 34. *How to compute powers of matrix $\boldsymbol{A}$ using diagonalization?*

Answer. Assume there exists an invertible matrix $\boldsymbol{A}$, such that

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

Then for any positive integer n, we have

$$\begin{aligned}\boldsymbol{A}^k &= \underbrace{(\boldsymbol{PDP}^{-1})(\boldsymbol{PDP}^{-1})\cdots(\boldsymbol{PDP}^{-1})}_{k \text{ terms}} = \boldsymbol{PD}^k\boldsymbol{P}^{-1} \\ &= \boldsymbol{P}\begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix}\boldsymbol{P}^{-1}.\end{aligned}$$

✠ Example: Example 6.2.9. □

Question 35. *How to solve linear recurrence relation using diagonalization?*

Answer. [Page 170–172] We need follow this process:

1. Transfer the linear recurrence relation to the matrix form $\boldsymbol{A}$. For example, given a linear recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, then the related matrix form is

$$\begin{pmatrix} a_{n-1} \\ a_n \\ a_{n+1} \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} a_{n-2} \\ a_{n-1} \\ a_n \end{pmatrix},$$

   where $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

2. Applying algorithm 6.2.4, we will find an invertible matrix $\boldsymbol{P}$, such that $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{D}$, where $\boldsymbol{D} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$.

3. Thus

$$\begin{pmatrix} a_{n-1} \\ a_n \\ a_{n+1} \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} a_{n-2} \\ a_{n-1} \\ a_n \end{pmatrix} = \boldsymbol{A}^{n-1} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \boldsymbol{P} \begin{pmatrix} \lambda_1^{n-1} & & \\ & \lambda_2^{n-1} & \\ & & \lambda_3^{n-1} \end{pmatrix} \boldsymbol{P}^{-1} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}.$$

4. Then follow the process in the Example 6.2.12 on Page 172, to get the explicit form for a_n.

✠ Example: Example 6.2.12 and Exercise 6.18. □

6 Chapter 5

Question 36. *Given an orthogonal basis S or an orthonormal basis T for a vector space V and a vector $\boldsymbol{w} \in V$, how to find find $[\boldsymbol{w}]_S$, $(\boldsymbol{w})_S$, $[\boldsymbol{w}]_T$, and $(\boldsymbol{w})_T$.*

Answer. [Page 136] We discuss by cases:

- If $S = \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$ is an orthogonal basis for V, then

$$\boldsymbol{w} = \frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\boldsymbol{u}_1 + \frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}\boldsymbol{u}_2 + \cdots + \frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\boldsymbol{u}_k,$$

  and hence

$$(\boldsymbol{w})_S = [\boldsymbol{w}]_S^T = \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}, \frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}, \cdots, \frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right).$$

- If $T = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is an orthonormal basis for V, then

$$\boldsymbol{w} = (\boldsymbol{w}\cdot\boldsymbol{v}_1)\boldsymbol{v}_1 + (\boldsymbol{w}\cdot\boldsymbol{v}_2)\boldsymbol{v}_2 + \cdots + (\boldsymbol{w}\cdot\boldsymbol{v}_k)\boldsymbol{v}_k,$$

  and

$$(\boldsymbol{w})_T = [\boldsymbol{w}]_T^T = (\boldsymbol{w}\cdot\boldsymbol{v}_1, \boldsymbol{w}\cdot\boldsymbol{v}_2, \cdots, \boldsymbol{w}\cdot\boldsymbol{v}_k).$$

✠ Example: Example 5.2.9. □

Question 37. *How to find an orthogonal basis and an orthonormal basis for a vector space V?*

Answer. [Page 141] Following the process in the Question 22, we have a basis $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ for V. Then applying Gram-Schmidt process, we will obtain an orthogonal basis S' for V. At last, normalizing every vector in S' will give us an orthonormal basis.
✠ Example: Example 5.2.19. □

Question 38. *How to find the projection of a vector $\boldsymbol{w}$ onto a subspace V of $\mathbb{R}^n$?*

Answer. [Page 139] Following the process in the last Question, we may have an orthogonal basis $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ or an orthonormal basis $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$.

Then

$$\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}, \frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}, \cdots, \frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right),$$

and

$$(\boldsymbol{w}\cdot\boldsymbol{v}_1, \boldsymbol{w}\cdot\boldsymbol{v}_2, \cdots, \boldsymbol{w}\cdot\boldsymbol{v}_k)$$

are the projection of $\boldsymbol{w}$ onto V.
✠ Example: Examples 5.2.13 and 5.2.15. □

Question 39. *How to find the best approximate solution to inconsistent system $\boldsymbol{Ax} = \boldsymbol{b}$?*

Answer. [Page 142–145] Solving the linear system $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$.
✠ Example: Example 5.3.7. □

Question 40. *How to identify a matrix to be an orthogonal matrix?*

Answer. [Page 147] $\boldsymbol{A}$ is a square matrix, then the following statements are equivalent:

- $\boldsymbol{A}$ is orthogonal;
- $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$;
- $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$;
- the rows of $\boldsymbol{A}$ form an orthonormal basis for $\mathbb{R}^n$;
- the columns of $\boldsymbol{A}$ form an orthonormal basis for $\mathbb{R}^n$;
- $\|\boldsymbol{A}\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for any vector $\boldsymbol{x} \in \mathbb{R}^n$;
- $\boldsymbol{A}\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v}$ for any vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.

✠ Example: Example 5.4.5. □

Question 41. *How is orthogonal matrix related to orthonormal basis?*

Answer. [Page 147] $\boldsymbol{A}$ is an orthogonal matrix iff the columns (rows) of $\boldsymbol{A}$ form an orthonormal basis for $\mathbb{R}^n$
✠ Example: Exercise 5.33(cd). □

Question 42. *How is transition matrix related to orthogonal matrix?*

Answer. [Page 147] Let S and T be two orthonormal bases for a vector space, $\boldsymbol{P}$ the transition matrix from S to T. Then $\boldsymbol{P}$ is orthogonal. □

Question 43. *How to orthogonally diagonalize a symmetric matrix?*

Answer. [Page 175] Apply algorithm 6.3.5.
✠ Example: Example 6.3.7, Exercises 6.21 and 6.22. □

7 Chapter 7

Question 44. *How are linear transformations related to matrices?*

Answer. [Page 187] Apply the definition.
✠ Example: Exercises 7.12 and 7.13. □

Question 45. *How to show that a mapping $T\colon V \to W$ is a linear transformation?*

Answer. There are the following two methods:

- First method: [Page 187] If we can find the standard matrix for T, then T is a linear transformation.
- Second method: [Page 187] If $T(a\boldsymbol{u} + b\boldsymbol{v}) = aT(\boldsymbol{u}) + bT(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $a, b \in \mathbb{R}$, then T is a linear transformation.

✠ Example: Example 7.1.4. □