Social and Economic Networks

Games on Networks

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Outline

1. Games on networks
2. Strategic complements and strategic substitutes
3. Properties of equilibrium
   - Strategic complements
   - Strategic substitutes
4. General action sets
   - A local public goods model
   - A linear quadratic model

Section 1

Games on networks

Game theoretic reasoning

- To understand how the structure of social networks influences behavior, beyond diffusion and learning.

  Depending in more complicated ways on what neighbors are doing.
  - An individual only wants to buy a product or make an investment when his or her neighbors do not.
  - If an individual is choosing a piece of software or some other product and wants it to be compatible with a majority of neighbors.
- This interactive considerations require game theoretic reasoning, adapted and extended to a network setting.

Canonical setting

- Each player i chooses action x_i in $\{0, 1\}$.
  - 1: buy a book/invest a new technology/learn a language.
  - 0: does not buy a book/invest technology/learn a language.
- Payoff will depend on
  - how many neighbors choose each action.
  - how many neighbors a player has.
- Consider the case

$$u_{d_i}(x_i, m_{N_i}),$$

  depending only on d_i and m_{N_i} (the number of neighbors of i choosing action 1).

Canonical setting: Limitations

- Only two actions.
- Only care about the number of friends taking the action, not the identities of them.
- Treat friends equally in terms of who is taking the action.
- It only matters how many friends they have.

Example 1: Simple complement

- Agent i is willing to choose 1 if and only if at least t neighbors do.
- Payoff of action 0:

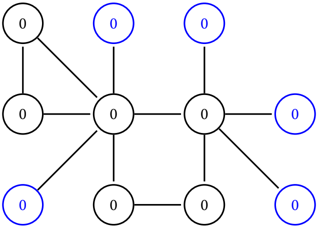
$$u_{d_i}(0, m_{N_i}) = 0.$$

- Payoff of action 1:

$$u_{d_i}(1, m_{N_i}) = -t + m_{N_i},$$

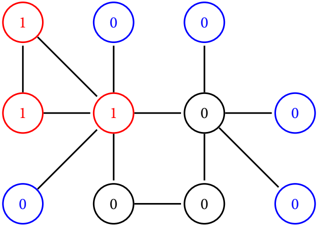
where m_{N_i} is the number of neighbors of i choosing action 1.

Example 1: Simple complement (Cont.)



An agent is willing to take action 1 if and only if at least 2 neighbors do.

Example 1: Simple complement (Cont.)



An agent is willing to take action 1 if and only if at least 2 neighbors do.

Example 2: Best shot

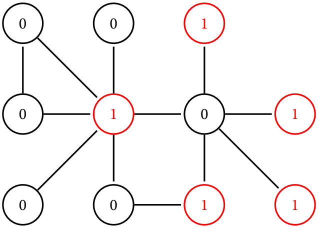
- Agent i is willing to choose 1 if and only if no neighbors do.
- For example, one of my friends buys the book, I do not buy the book because now I can borrow it from them.
- Payoff of action 0:

$$u_{d_i}(0, m_{N_i}) = \begin{cases} 1, & \text{if } m_{N_i} > 0, \\ 0, & \text{if } m_{N_i} = 0. \end{cases}$$

- Payoff of action 1:

$$u_{d_i}(1, m_{N_i}) = 1 - c.$$

Example 2: Best shot (Cont.)



An agent is willing to take action 1 if and only if at least 2 neighbors do.

Section 2

Strategic complements and strategic substitutes

Strategic complements/substitutes

- Strategic complements:

  as more of my friends take the action, it is a more attractive action to me.
- Strategic substitutes:

  as more of my friends take the action, it is a less attractive action for me to take.

Strategic complements

- Strategic complements: for all d, $m \geq m'$, differences are increasing:

$$u_d(1, m) - u_d(0, m) \geq u_d(1, m') - u_d(0, m').$$

* As the number of friends who take the action increases, the payoff to taking the action compared to not taking the action has gone up.
* So the difference between taking the action and not taking it, it is more attractive than it was before.

- Positive relationship.

Strategic substitutes

- Strategic substitutes: for all d, $m \geq m'$, differences are decreasing:

$$u_d(1, m) - u_d(0, m) \leq u_d(1, m') - u_d(0, m').$$

- As more people take the action, as we move from m' to m, it becomes less attractive to take the action.
- Negative relationship.

Examples

- Strategic complements
  - education decisions
  - smoking and other behavior among peers
  - technology adoption
  - learn a language
  - cheating
- Strategic substitutes
  - information gathering
  - local public goods (shareable products)
  - competing firms (oligopoly with local markets)

Strategic complements/substitutes

- Complements:

  There is a threshold $t(d)$, such that i prefers 1 if $m_{N_i} > t(d)$ and 0 if $m_{N_i} < t(d)$.

- Substitutes:

  There is a threshold $t(d)$, such that i prefers 1 if $m_{N_i} < t(d)$ and 0 if $m_{N_i} > t(d)$.

Externalities

- Others' behaviors affect my utility/welfare.
- Others' behaviors affect the relative payoffs to my behaviors.
- Complements: Choice to take an action by my friends increases my relative payoff to taking that action.
- Substitutes: Choice to take an action by my friends decreases my relative payoff to taking that action.

Section 3

Properties of equilibrium

Equilibrium

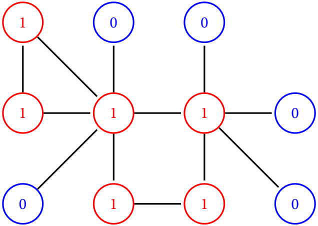
- Nash equilibrium:

  Every player's action is optimal for that player given the actions of others.

Subsection 1

Strategic complements

Example 1: Simple complement

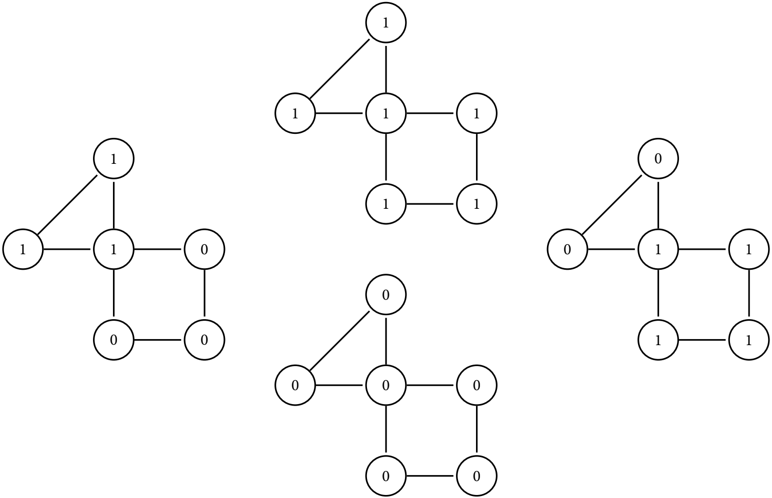


- An agent is willing to take action 1 if and only if at least 2 neighbors do.
- Each player takes the maximal action that she can in any equilibrium.

Complete lattice

- In a game of strategic complements, the set of pure strategy Nash equilibria has a nice structure: complete lattice.
- Complete lattice: for every subset of equilibria X,
  - there exists an equilibrium x' such that $x' \geq x$ for all $x \in X$,
  - there exists an equilibrium x'' such that $x'' \leq x$ for all $x \in X$.

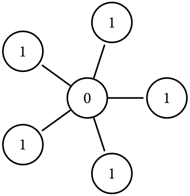
Complete lattice (Cont.)



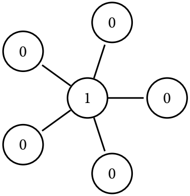
Subsection 2

Strategic substitutes

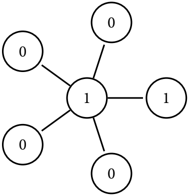
Example 2: Best shot



Equilibrium



Equilibrium



Not an equilibrium

Maximal independent set: each 1 has no 1's in its neighborhood, each 0 has at least one 1 in its neighborhood.

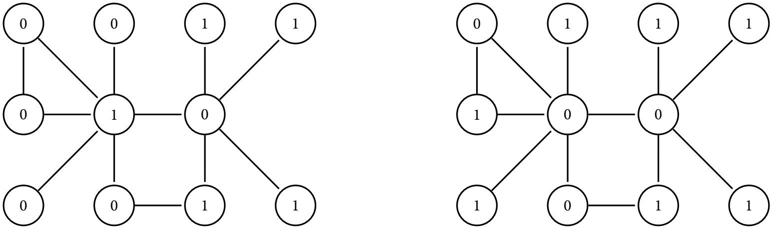
Maximal independent set

- Independent set:
  A set S of nodes such that no two nodes in S are linked.
- Maximal:
  Every node in N is either in S or linked to a node in S.

Equilibrium: Strategic substitutes

- Best shot game: pure strategy equilibria exist and are related to maximal independent sets.
- Others: pure strategy may not exist, but mixed will (with finite action spaces).
- Equilibria usually do not form a lattice.

Example: Best shot



- Invest if and only if no neighbors do (threshold is 1).
- Multiple equilibria.
- No lattice structure.

Find maximal independent sets

- Pick one person to be 1, and fill in all the 0's for her neighbors
- A bunch of people who are still left. Pick any one of those people who are still left. They are not one of the neighbors of the first person, so they still do not have any neighbors who have taken any action, put them as a 1.
- Now all of their neighbors would have to be 0's
- So forth.
- This algorithm will give us one of the maximum independent sets.
- Every possible order for picking nodes $\Rightarrow$ all the maximum independent sets.

Section 4

General action sets

Subsection 1

A local public goods model

A local public goods model

- Action set: $X_i = [0, \infty)$.
- Payoff:

$$f(x_i + \sum_{j \in N_i(g)} x_j) - cx_i,$$

  where f is a continuously differentiable, strictly concave function, and $c > 0$ is a cost parameter.
- Interesting case: $f'(0) > c > f'(x)$ for some large enough x.

A local public goods model: Equilibrium

- Let x^* be such that $f'(x^*) = c$.
- Any equilibrium must have at least x^* produced in each player's neighborhood (so that $x_i + \sum_{j \in N_i(g)} x_j \geq x^*$ for each i); otherwise a player could increase her payoff by increasing her action.
- A strategy profile $(x_1, \ldots, x_n)$ is an equilibrium if and only if the following holds for each i:
  - If $x_i > 0$, then $x_i + \sum_{j \in N_i(g)} x_j = x^*$; and
  - If $x_i = 0$, then $\sum_{j \in N_i(g)} x_j \geq x^*$.

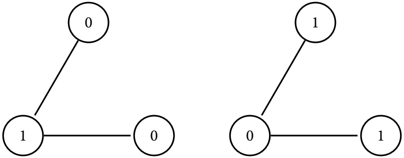
* In an equilibrium, a player only chooses a positive action if her neighbors produce less than x^* in aggregate.

A local public goods model: Equilibrium (Cont.)

- Two types of equilibria:
  - Distributed: $x_i \in (0, x^*)$ for some i.
  - Specialized: for each player i, either $x_i = 0$ or $x_i = x^*$.
    There are players who specialize in providing the information or public good, and others who free-ride on their neighbors.
- The specialized equilibria are precisely those where the players who specialize in providing the local public good at the level x^* form a maximal independent set and the remaining players choose an action of 0.

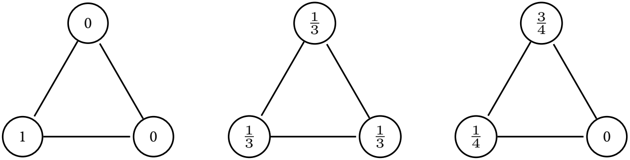
Example 1

$$x^* = 1.$$



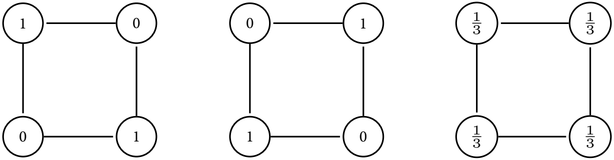
Example 2

$$x^* = 1.$$



Example 3

$$x^* = 1.$$



Stable equilibria

- The specialized equilibria are more robust than other equilibria.
- Only specialized equilibria (and in fact only a subset of them) satisfy the following notion of stability.
  - Start with a pure-strategy equilibrium profile $(x_1, \ldots, x_n)$.
  - Perturb it slightly by adding some small perturbation ϵ_i to each x_i, with a requirement that $x_i + \epsilon_i \geq 0$. Denote this by $x^1 = (x_1 + \epsilon_1, \ldots, x_n + \epsilon_n)$.
  - Consider the best-responses to x^1. Denote by x^2.
  - Iterate on the best responses, at each step examining the best responses x^k of the players to the previous step's strategies x^{k-1}.

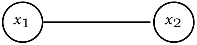
If there is some $\bar{\epsilon} > 0$ such that this process always converges back to $(x_1, \ldots, x_n)$ starting from any admissible perturbations such that $|\epsilon_i| < \bar{\epsilon}$ for all i, then the original equilibrium is said to be stable.

Stable equilibria (Cont.)

- The best responses to some x_{-i} take a simple form:
  - If $\sum_{j \in N_i(g)} x_j \geq x^*$, then the best response is 0.
  - Otherwise, the best response is $x^* - \sum_{j \in N_i(g)}$.
- The only stable equilibria, if they exist, are specialized equilibria such that each non-specialist player has at least two specialists in her neighborhood, and each specialist has no neighbors providing.

Stable equilibria (Cont.)

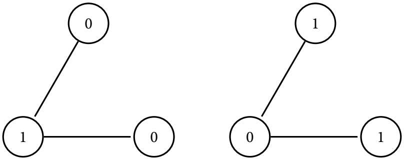
Consider an equilibrium (x_1, x_2).



- Clearly, $x_1 + x_2 = x^*$.
- Suppose that $x_2 > 0$.
- Consider a perturbation $(x_1 + \epsilon, x_2 - \epsilon)$.
- Then the best responses do not converge back to the original point.
- No equilibrium is stable.

Example 1

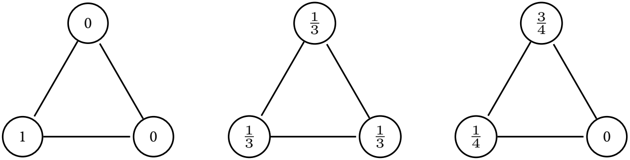
$$x^* = 1.$$



The right one is stable.

Example 2

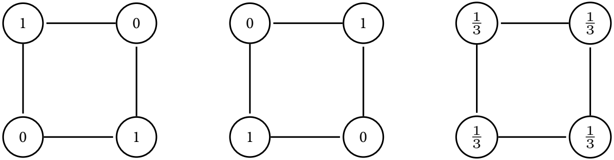
$$x^* = 1.$$



No stable equilibria.

Example 3

$$x^* = 1.$$



The left and middle ones are stable.

Subsection 2

A linear quadratic model

A linear quadratic model

- Action set: $X_i \in [0, \infty)$.
- payoff:

$$u_i(x_i, x_{-i}) = a_i x_i - \frac{b_i}{2} x_i^2 + \sum_{j \neq i} w_{ij} x_i x_j,$$

  where $a_i \geq 0$ and $b_i > 0$, and w_{ij} are weights that the player i places on j's action.
- If $w_{ij} > 0$, then i and j's activities are strategic complements.
- If $w_{ij} < 0$, then i and j's activities are strategic substitutes.

Equilibrium

- FOC:

$$x_i = \frac{a_i}{b_i} + \sum_{j \neq i} \frac{w_{ij}}{b_i} x_j.$$

- Let $g_{ij} = \frac{w_{ij}}{b_i}$ and $g_{ii} = 0$.
- Let $\alpha = (\frac{a_1}{b_1}, \cdots, \frac{a_n}{b_n})^T$.
- Then $x = \alpha + gx$, or

$$x = (I - g)^{-1} \alpha.$$

Equilibrium and centrality

- Consider a special case $a_i = a$ and $b_i = b$ for all i.
- The only heterogeneity in the society comes through the weights in the network of interactions, the w_{ij}'s.
- $$\begin{aligned} x &= (I - \tfrac{1}{b}w)^{-1}\tfrac{a}{b}\mathbf{1} = (I + \tfrac{1}{b}w + \tfrac{1}{b^2}w^2 + \cdots)\tfrac{a}{b}\mathbf{1} \\ &= \tfrac{a}{b}\mathbf{1} + \tfrac{a}{b}(\tfrac{1}{b}w + \tfrac{1}{b^2}w^2 + \cdots)\mathbf{1} \\ &= \tfrac{a}{b}\mathbf{1} + \tfrac{a}{b}P^{K2}(w, \tfrac{1}{b}). \end{aligned}$$
- Recall: Katz prestige-2 (Bonacich centrality)

$$P^{K2}(g, a) = ag\mathbf{1} + a^2g^2\mathbf{1} + \cdots = (\boldsymbol{I} - ag)^{-1}ag\mathbf{1}.$$

Comparative statistics

- Higher neighbors' actions, higher own action.
- Higher own action, higher neighbors' actions.
- For solution, need b to be large and/or w_{ij}'s to be small.