Introduction

1. Matching theory, a name referring to several loosely related research areas concerning matching, allocation, and exchange of indivisible resources, such as jobs, school seats, houses, *etc.*, lies at the intersection of game theory, social choice theory, and mechanism design.

2. Labor markets: the case of American hospital-intern markets:

   - Medical students in many countries work as residents (interns) at hospitals.
   - In the U.S. more than 20,000 medical students and 4,000 hospitals are matched through a clearinghouse, called NRMP (National Resident Matching Program).
   - Doctors and hospitals submit preference rankings to the clearinghouse, and the clearinghouse uses a specified rule (computer program) to decide who works where.
   - Some markets succeeded while others failed. What is a "good way" to match doctors and hospitals?

3. Kidney exchange:

- Kidney Exchange is a preferred method to save kidney-disease patients.
- There are lots of kidney shortages, and willing donor may be incompatible with the donor.
- Kidney Exchange tries to solve this by matching donor-patient pairs.
- What is a "good way" to match donor-patient pairs?

4. School choice:

- In many countries, especially in the past, children were automatically sent to a school in their neighborhoods.
- Recently, more and more cities in the United States and in other countries employ school choice programs: school authorities take into account preferences of children and their parents.

5. Targets: Efficiency, fairness, incentives.

Marriage and college admission

6. A marriage problem is a triple $\Gamma = \langle M, W, \succsim \rangle$, where

   - M is a finite set of men,
   - W is a finite set of women,
   - $\succsim = (\succsim_i)_{i \in M \cup W}$ is a list of preferences. Here
     - $\succsim_m$ denotes the preference of man m over $W \cup \{m\}$,
     - $\succsim_w$ denotes the preference of woman w over $M \cup \{w\}$,
     - $\succ_i$ denotes the strict preference derived from $\succsim_i$ for each $i \in M \cup W$.

7. For man m:

   - $w \succ_m w'$ means that man m prefers woman w to woman w'.
   - $w \succ_m m$ means that man m prefers woman w to remaining single.
   - $m \succ_m w$ means that woman w is unacceptable to man m.

We use similar notation for women.

8. If an individual is not indifferent between any two distinct acceptable alternatives, he has strict preferences. Unless otherwise mentioned all preferences are strict.

9. A matching in a marriage problem $\Gamma = \langle M, W, \succsim \rangle$ is a function $\mu\colon\ M \cup W \to M \cup W$ such that

   - for all $m \in M$, if $\mu(m) \neq m$ then $\mu(m) \in W$,
   - for all $w \in W$, if $\mu(w) \neq w$ then $\mu(w) \in M$,
   - for all $m \in M$ and $w \in W$, $\mu(m) = w$ if and only if $\mu(w) = m$.

   We refer to $\mu(i)$ as the mate of i, and $\mu(i) = i$ means that agent i remains single under the matching μ.

10. A matching will sometimes be represented as a set of matched pairs. Thus, for example, the matching

$$\mu = \begin{bmatrix} w_4 & w_1 & w_2 & w_3 & (m_5) \\ m_1 & m_2 & m_3 & m_4 & m_5 \end{bmatrix}$$

has m_1 married to w_4 and m_5 remaining single.

11. For two matchings μ and ν, an individual i prefers μ to ν if and only if i prefers $\mu(i)$ to $\nu(i)$.

    Let $\mu \succ_M \nu$ if $\mu(m) \succsim_m \nu(m)$ for all $m \in M$, and $\mu(m) \succ_m \nu(m)$ for at least one man m.

    Let $\mu \succsim_M \nu$ denote that either $\mu \succ_M \nu$ or that all men are indifferent between μ and ν.

12. A matching μ is Pareto efficient if there is no other matching ν such that

    - $\nu(i) \succsim_i \mu(i)$ for all $i \in M \cup W$,
    - $\nu(i_0) \succ_{i_0} \mu(i_0)$ for some $i_0 \in M \cup W$.

13. A matching μ is blocked by an individual $i \in M \cup W$ if $i \succ_i \mu(i)$.

    A matching is individually rational if it is not blocked by any individual.

14. A matching μ is blocked by a pair $(m, w) \in M \cup W$ if they both prefer each other to

their partners under μ, *i.e.*,

$$w \succ_m \mu(m) \text{ and } m \succ_w \mu(w).$$

15. A matching μ is stable if it is not blocked by any individual or any pair.

16. Example: There are three men and three women, with the following preferences:

m_1	m_2	m_3	w_1	w_2	w_3
w_2	w_1	w_1	m_1	m_3	m_1
w_1	w_3	w_2	m_3	m_1	m_3
w_3	w_2	w_3	m_2	m_2	m_2

Table 1

All possible matchings are individually rational, since all pairs (m, w) are mutually acceptable.

The matching μ given below is unstable, since (m_1, w_2) is a blocking pair.

$$\mu = \begin{bmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{bmatrix}.$$

The matching μ' is stable.

$$\mu' = \begin{bmatrix} w_1 & w_2 & w_3 \\ m_1 & m_3 & m_2 \end{bmatrix}.$$

17. Men-proposing deferred acceptance algorithm.

Step 1: Each man m proposes to his first choice (if he has any acceptable choices). Each woman rejects any offer except the best acceptable proposal and "holds" the most-preferred acceptable proposal (if any). Note that she does not accept him yet, but keeps him on a string to allow for the possibility that someone better may come along later.

Step k: Any man who was rejected at Step $k-1$ makes a new proposal to his most-preferred acceptable potential mate who has not yet rejected him (If no acceptable choices remain, he makes no proposal). Each woman receiving proposals chooses her most-preferred acceptable proposal from the group consisting of the new proposers and the man on her string, if any. She rejects all the rest and again keeps the best-preferred in suspense.

End: The algorithm terminates when there are no more rejections. Each woman is matched with the man she has been holding in the last step. Any woman who has not been holding an offer or any man who was rejected by all acceptable women remains single.

18. Theorem on stability (Theorem 1 in Gale and Shapley (1962)): The men-proposing deferred acceptance algorithm gives a stable matching for each marriage problem.

19. Theorem on optimality (Theorem 2 in Gale and Shapley (1962)): The matching determined by men-proposing deferred acceptance algorithm is at least good as any other stable matching for all men.

20. Rural hospital theorem (Theorem in McVitie and Wilson (1970), Theorem 1 in Gale and Sotomayor (1985)): The set of individuals who are matched is the same for all stable matchings.

21. A (direct) mechanism φ is a systematic procedure that determines a matching for each marriage problem $\langle M, W, \succsim \rangle$. Note that M, W and $\succsim$ are all allowed to vary.

22. A mechanism φ is stable if it is always selects a stable matching.

    A mechanism φ is Pareto efficient if it is always selects a Pareto efficient matching.

    A mechanism φ is individually rational if it is always selects an individually rational matching.

23. A mechanism φ is strategy-proof if for any M and W, for each $i \in M \cup W$, for each $\succsim_i, \succsim_i' \in \mathcal{P}_i$, for each $\succsim_{-i} \in \mathcal{P}_{-i}$,

$$\varphi[\succsim_{-i}, \succsim_i](i) \succsim_i \varphi[\succsim_{-i}, \succsim_i'](i).$$

24. Impossibility theorem (Theorem 3 in Roth (1982b)): There exists no mechanism that is both stable and strategy-proof.

25. Theorem (Theorem 9 in Dubins and Freedman (1981), Theorem 5 in Roth (1982b)): Truth-

telling is a weakly dominant strategy for any man under the man-optimal stable mechanism. Similarly truth-telling is a weakly dominant strategy for any woman under the woman-optimal stable mechanism.

26. Definition: A college admissions problem $\Gamma = \langle S, C, q, \succ \rangle$ consists of:

- a finite set of students S,
- a finite set of colleges C,
- a quota vector $q = (q_c)_{c \in C}$ such that $q_c \in \mathbb{Z}_+$ is the quota of college c,
- a preference profile for students $\succ_S = (\succ_s)_{s \in S}$ such that $\succ_s$ is a strict preference over colleges and remaining unmatched, denoting the strict preference of student s,
- a preference profile for colleges $\succ_C = (\succ_c)_{c \in C}$ such that $\succ_c$ is a strict preference over students and remaining unmatched, denoting the strict preference of college c.

In this chapter, we will use $\emptyset$ to denote "unmatched."

27. Definition: A matching is the outcome of a problem, and is defined by a function $\mu : C \cup S \to 2^S \cup 2^C$ such that

- for each student $s \in S$, $\mu(s) \in 2^C$ with $|\mu(s)| \leq 1$,
- for each college $c \in C$, $\mu(c) \in 2^S$ with $|\mu(c)| \leq q_c$,
- $\mu(s) = c$ if and only if $s \in \mu(c)$.

28. Definition: A matching μ is blocked by a college $c \in C$ if there exists $s \in \mu(c)$ such that $\emptyset \succ_c s$.

    A matching μ is blocked by a student $s \in S$ if $\emptyset \succ_s \mu(s)$.

    A matching is individually rational if it is not blocked by any college or student.

29. Definition: A matching μ is blocked by a pair $(c, s) \in C \times S$ if

    - $c \succ_s \mu(s)$, and
    - – either there exists $s' \in \mu(c)$ such that $s \succ_c s'$, or
      – $|\mu(c)| < q_c$ and $s \succ_c \emptyset$.

30. Definition: A matching is stable if it is not blocked by any agent or pair.

31. Given a college admissions problem $\langle S, C, q, \succ \rangle$, the related marriage problem is constructed as follows:

- "Divide" each college c_ℓ into q_{c_ℓ} separate pieces $c_\ell^1, c_\ell^2, \ldots, c_\ell^{q_{c_\ell}}$, where each piece has a capacity of one; and let each piece have the same preferences over S as college c has. (Since college preferences are responsive, $\succ_c$ is consistent with a unique ranking of students.)

  C^*: The resulting set of college "pieces" (or seats).

- For any student s, extend her preference to C^* by replacing each college c_ℓ in her original preference $\succ_s$ with the block $c_\ell^1, c_\ell^2, \ldots, c_\ell^{q_{c_\ell}}$ in that order.

32. Student-proposing deferred acceptance algorithm.

Step 1: Each student proposes to her top-choice individually rational college (if she has one). Each college c rejects any individually irrational proposal and, if more than q_c individually rational proposals are received, "holds" the most preferred q_c of them and rejects the rest.

Step k: Any student who was rejected at the previous step makes a new proposal to her most preferred individually rational college that hasn't yet rejected her (if there is one). Each college c "holds" at most q_c best student proposals to date, and rejects the rest.

End: The algorithm terminates after a step where no rejections are made by matching each college to the students (if any) whose proposals it is "holding."

33. Theorem on stability (Theorem 1 in Gale and Shapley (1962)): The student- and college-proposing deferred acceptance algorithm give stable matchings for each college admissions model.

34. Theorem: The college-proposing deferred acceptance algorithm produces a matching that gives each college c_ℓ its k_ℓ highest ranked achievable students.

35. Theorem: The student-optimal stable matching is weakly Pareto efficient for the students.

36. Example: The college-optimal stable matching need not be even weakly Pareto optimal for the colleges.

37. Example: The college-optimal stable matching need not be even weakly Pareto optimal for the colleges.

38. Theorem: The set of students admitted and seats filled is the same at every stable matching.

39. Theorem (Theorem 1 in Roth (1986)): Any college that does not fill its quota at some stable matching is assigned precisely the same set of students at every stable matching.

40. A mechanism φ is strategy-proof if for each $i \in S \cup C$, for each $\succsim_i, \succsim_i' \in \mathcal{P}_i$, for each $\succsim_{-i} \in \mathcal{P}_{-i}$,

$$\varphi[\succsim_{-i}, \succsim_i, q](i) \succsim_i \varphi[\succsim_{-i}, \succsim_i', q](i).$$

41. Theorem (Theorem 3 in Roth (1982b)): There exists no mechanism that is stable and strategy-proof.

42. Theorem (Theorem 5 in Roth (1982b)): Truth-telling is a weakly dominant strategy for all students under the student-optimal stable mechanism.

43. Theorem (Proposition 2 in Roth (1985a)): There exists no stable mechanism where truth-telling is a weakly dominant strategy for all colleges.

Housing market and house allocation

44. Housing market model is introduced by Shapley and Scarf (1974). Each agent owns a house, and a housing market is an exchange (with indivisible objects) where agents have the opinion to trade their house in order to get a better one.

45. Definition: Formally, a housing market is a triple $\langle A, H, \succ, e \rangle$ such that

    - $A = \{a_1, a_2, \ldots, a_n\}$ is a set of agents,
    - H is a set of houses such that $|A| = |H|$,
    - $\succ = (\succ_a)_{a \in A}$ is a strict preference profile such that for each agent $a \in A$, $\succ_a$ is a strict preference over houses. Let $\mathcal{P}_a$ be the set of preferences of agent a. The induced weak preference of agent a is denoted by $\succsim_a$ and for any $h, g \in H$, $h \succsim_a g$ if and only if $h \succ_a g$ or $h = g$.

- $e\colon A \to H$ is an initial endowment matching, that is, $h_i \triangleq h_{a_i} \triangleq e(a_i)$ is the initial endowment of agent i.

46. Definition: In a housing market $\langle A, H, \succ, e\rangle$, a matching (allocation) is a bijection $\mu\colon A \to H$. Here $\mu(a)$ is the assigned house of agent a under matching μ. Let $\mathcal{M}$ be the set of matchings.

47. Definition: A (deterministic direct) mechanism is a procedure that assigns a matching for each housing market $\langle A, H, \succ, e\rangle$.

    For the fixed sets of agents A and houses H, a mechanism becomes a function

$$\varphi\colon \times_{a\in A} \mathcal{P}_a \to \mathcal{M}.$$

48. Definition: A matching μ is individually rational if for each agent $a \in A$,

$$\mu(a) \succsim_a h_a = e(a),$$

that is, each agent is assigned a house at least as good as her own occupied house.

A mechanism is individually rational if it always selects an individually rational matching for each housing market.

49. Definition: A matching μ is Pareto efficient if there is no other matching ν such that

- $\nu(a) \succsim_a \mu(a)$ for all $a \in A$, and
- $\nu(a_0) \succ_{a_0} \mu(a_0)$ for some $a_0 \in A$.

A mechanism is Pareto efficient if it always selects a Pareto efficient matching for each housing market.

50. Definition: Given a market $\langle A, H, \succ, e \rangle$ and a coalition $B \subseteq A$, a matching μ is a B-matching if for all $a \in B$, $\mu(a) = h_b$ for some $b \in B$.

51. Definition: A matching μ is in the core if there exists no coalition of agents $B \subseteq A$ such that some B-matching $\nu \in \mathcal{M}$ weakly dominates μ, that is,

- $\nu(a) \succsim_a \mu(a)$ for all $a \in B$, and

- $\nu(a_0) \succ_{a_0} \mu(a_0)$ for some $a_0 \in B$.

52. Theorem (Theorem in Shapley and Scarf (1974)): The core of a housing market is nonempty.

53. Top trading cycles algorithm.

    Step 1: Each agent points to the owner of his favorite house.

    Due to the finiteness of agents, there exists at least one cycle (including self-cycles). Moreover, cycles do not intersect.

    Each agent in a cycle is assigned the house of the agent he points to and removed from the market.

    If there is at least one remaining agent, proceed with the next step.

    Step k: Each remaining agent points to the owner of his favorite house among the remaining houses.

    Each agent in a cycle is assigned the house of the agent he points to and removed from the market.

If there is at least one remaining agent, proceed with the next step.

End: No agents remain. It is clear that the algorithm will terminate within finite steps. Let Step t denote the last step.

The mechanism determined by top trading cycles algorithm is denoted by φ^{TTC}.

54. Theorem (Theorem 2 in Roth and Postlewaite (1977)): If the preference of each agent is strict, the core of a housing market has exactly one matching.

55. Definition: A mechanism φ is strategy-proof if for each housing market $\langle A, H, \succ, e\rangle$, for each $a \in A$, and for each $\succ'_a$, we have

$$\varphi[\succ](a) \succsim_a \varphi[\succ_{-a}, \succ'_a](a).$$

56. Theorem (Theorem in Roth (1982a)): The core mechanism φ^{TTC} is strategy-proof.

57. Theorem (Theorem 1 in Ma (1994)): The core mechanism φ^{TTC} is the only mechanism that is individually rational, Pareto efficient, and strategy-proof.

58. The house allocation problem is introduced by Hylland and Zeckhauser (1979). In this problem, there is a group of agents and houses. Each agent shall be allocated a house by a central planner using her preferences over the houses.

59. Formally, a house allocation problem is a triple $\langle A, H, \succ \rangle$ such that

- $A = \{a_1, a_2, \ldots, a_n\}$ is a set of agents,
- $H = \{h_1, h_2, \ldots, h_n\}$ is a set of houses,
- $\succ = (\succ_a)_{a \in A}$ is a strict preference profile such that for each agent $a \in A$, $\succ_a$ is a strict preference over houses. Let $\mathcal{P}_a$ be the set of preferences of agent a. The induced weak preference of agent a is denoted by $\succsim_a$ and for any $h, g \in H$, $h \succsim_a g$ if and only if $h \succ_a g$ or $h = g$.

60. Definition: In a house allocation problem $\langle A, H, \succ \rangle$, a matching (allocation) is a bijection $\mu \colon A \to H$. Here $\mu(a)$ is the assigned house of agent a under matching μ. Let $\mathcal{M}$ be the set of matchings.

61. Definition: A (deterministic direct) mechanism is a procedure that assigns a matching for each house allocation problem $\langle A, H, \succ \rangle$.

For the fixed sets of agents A and houses H, a mechanism becomes a function

$$\varphi\colon\ \times_{a \in A} \mathcal{P}_a \to \mathcal{M}.$$

62. Definition: A matching μ is Pareto efficient if there is no other matching ν such that

- $\nu(a) \succsim_a \mu(a)$ for all $a \in A$, and
- $\nu(a_0) \succ_{a_0} \mu(a_0)$ for some $a_0 \in A$.

Let $\mathscr{E}$ denote the set of all Pareto efficient matchings.

A mechanism is Pareto efficient if it always selects a Pareto efficient matching for each house allocation.

63. An ordering $f\colon \{1, 2, \ldots, n\} \to A$ is a one-to-one and onto function. Each ordering induces the following simple mechanism, which is especially plausible if there is a natural

hierarchy of agents. Let $\mathcal{F}$ be the set of all orderings.

Simple serial dictatorship induced by an ordering f, denoted by φ^f.

Step 1: The highest priority agent $f(1)$ is assigned her top choice house under $\succ_{f(1)}$.

Step k: The k-th highest priority agent $f(k)$ is assigned her top choice house under $\succ_{f(k)}$ among the remaining houses.

64. Proposition: Simple serial dictatorship induced by an ordering f, φ^f, is Pareto efficient.

65. Core from assigned endowments μ, denoted by φ^μ: For any house allocation problem $\langle A, H, \succ\rangle$, select the unique element of the core of the housing market $\langle A, H, \succ, \mu\rangle$ where each agent a's initial house is $\mu(a)$. That is,

$$\varphi^\mu = \varphi^{\mathrm{TTC}}[\mu].$$

66. Theorem (Lemma 1 in Abdulkadiroğlu and Sönmez (1998)): For any ordering f and any matching μ, the simple serial dictatorship induced by f and the core from assigned en-

dowments μ both yield Pareto efficient matchings. Moreover, for any Pareto efficient matching ν, there is a simple serial dictatorship and a core from assigned endowments that yield it.

67. Theorem (Theorem 1 in Abdulkadiroğlu and Sönmez (1998)): For any house allocation problem, the number of simple serial dictatorships selecting a Pareto efficient matching μ is the same as the number of cores from assigned endowments selecting μ. That is, for all $\nu \in \mathscr{E}$, we have $|\mathcal{M}^\nu| = |\mathcal{F}^\nu|$, where $\mathcal{M}^\nu = \{\mu \in \mathcal{M} \mid \varphi^\mu = \nu\}$ and $\mathcal{F}^\nu = \{f \in \mathcal{F} \mid \varphi^f = \nu\}$.

68. Let σ be a permutation (relabeling) of houses. Let $\succ^\sigma$ be the preference profile where each house h is renamed to $\sigma(h)$. That is, $g \succ_a^\sigma h$ if and only if $\sigma^{-1}(g) \succ_a \sigma^{-1}(h)$.

Definition: A mechanism φ is neutral if, for any permutation σ and $\succ$,

$$\varphi[\succ^\sigma](a) = \sigma\big(\varphi[\succ](a)\big) \text{ for all } a \in A.$$

69. Definition: A mechanism φ is non-bossy if for any $\succ$, $a \in A$ and $\succ'_a$,

$$\varphi[\succ](a) = \varphi[\succ'_a, \succ_{-a}](a) \text{ implies } \varphi[\succ] = \varphi[\succ'_a, \succ_{-a}].$$

70. Theorem (Theorem 1 in Svensson (1999)): A mechanism φ is strategy-proof, non-bossy and neutral mechanism if and only if it is a simple serial dictatorship.

71. Definition: A house allocation problem with existing tenants, denoted by $\langle A_E, A_N, H_O, H_V, \succ \rangle$, consists of

    - a finite set of existing tenants A_E,
    - a finite set of new applicants A_N,
    - a finite set of occupied houses $H_O = \{h_i \colon a_i \in A_E\}$,
    - a finite set of vacant houses H_V, and
    - a strict preference profile $\succ = (\succ_i)_{i \in A_E \cup A_N}$.

Let $A = A_E \cup A_N$ denote the set of all agents and $H = H_O \cup H_V \cup \{h_0\}$ denote the set of all houses plus the null house.

Agent i's strict preference $\succ_i$ is on H. Let $\mathcal{P}$ be the set of all strict preferences on H. Let $\succsim_i$ be agent i's induced weak preference. We assume that the null house h_0 is the last choice for each agent.

72. Definition: A matching $\mu\colon\ A \to H$ is an assignment of houses to agents such that

- every agent is assigned one house, and
- only the null house h_0 can be assigned to more than one agent.

For any agent $a \in A$, we refer to $\mu(a)$ as the assignment of agent i under μ. Let $\mathcal{M}$ be the set of all matchings.

73. Definition: A direct mechanism is a procedure that assigns a matching for each house allocation problem with existing tenants $\langle A_E, A_N, H_O, H_V, \succ\rangle$.

74. Definition: A matching is Pareto efficient if there is no other matching that makes all agents weakly better off and at least one agent strictly better off.

A mechanism is individually rational if it always selects a Pareto efficient matching for each house allocation problem with existing tenants.

75. Definition: A matching is individually rational if no existing tenant strictly prefers his endowment to his assignment.

    A mechanism is individually rational if it always selects an individually rational matching for each house allocation problem with existing tenants.

76. Definition: A mechanism φ is strategy-proof if for each house allocation problem with existing tenants $\langle A_E, A_N, H_O, H_V, \succ\rangle$, for each $a \in A$, for each $\succ'_a$, we have

$$\varphi[\succ](a) \succsim_a \varphi[\succ'_a, \succ_{-a}](a).$$

77. You request my house—I get your turn (YRMH-IGYT) algorithm, induced by a given ordering f:

    Phase 1: Assign the first agent her top choice, the second agent her top choice among the remaining houses, and so on, until someone demands the house of an existing tenant.

Phase 2: If at that point the existing tenant whose house is requested is already assigned another house, then do not disturb the procedure.

Otherwise, modify the remainder of the ordering by inserting this existing tenant before the requestor at the priority order and proceed with the Phase 1 through this existing tenant.

Similarly, insert any existing tenant who is not already served just before the requestor in the priority order once her house is requested by an agent.

Phase 3: If at any point a cycle forms, it is formed by exclusively existing tenants and each of them requests the house of the tenant who is next in the cycle. A cycle is an ordered list $(h_1, a_1, \ldots, h_k, a_k)$ of occupied houses and existing tenants where agent a_1 demands the house a_2, h_2, agent a_2 demands the house of agent a_3, h_3, ..., agent a_k demands the house of a_1, h_1.

In such case, remove all agents in the cycle by assigning them the house they demand and proceed similarly.

School choice

78. A school choice problem is a five-tuple $\langle I, S, q, P, \succsim \rangle$, where

- $I = \{i_1, i_2, \ldots, i_n\}$ is a finite set of students,
- $S = \{s_1, s_2, \ldots, s_m\}$ is a finite set of schools,
- $q \triangleq (q_s)_{s \in S}$ is a quota profiles for schools where $q_s \in \mathbb{Z}_+$ is the quota of school s,
- $P \triangleq (P_i)_{i \in I}$ is a strong preference profile for students where P_i is a strict preference relation over $S \cup \{\emptyset\}$, denoting the strict preference relation of student i,
- $\succsim \triangleq (\succsim_s)_{s \in S}$ is a weak priority profile for schools where $\succsim_s$ is a weak priority relation over $I \cup \{\emptyset\}$, denoting the weak priority of school s.

Here $\emptyset$ represents remaining unmatched. For each $i \in I$, let R_i be the symmetric extension of P_i, that is, sR_is' if and only if sP_is' or $s = s'$.

79. In school choice problem, the priorities of schools are exogenous, that is, students are strategic agents but schools are simply objects to be consumed. So a school choice problem is a one-sided matching problem. It is one difference between the school choice problem and the college admission problem.

If each school has a strong priority relation $\succ_s$, then it is clear that a school choice problem naturally associates with an isomorphic college admission problem by letting each school s's preference relation be its priority relation $\succ_s$.

80. In a school choice problem $\langle I, S, q, P, \succsim \rangle$, a matching is a function $\mu \colon I \to S \cup \{\emptyset\}$ such that for each school s, $|\mu^{-1}(s)| \leq q_s$.

    Let $\mathcal{M}$ denote the set of all matchings.

81. In a school choice problem $\langle I, S, q, P, \succsim \rangle$, let $\mathcal{P}$ denote the sets of all the possible preferences for students. We allow only students to report preferences, and schools' priorities are exogenously given and publicly known.

    Then a mechanism $\varphi^{\succsim}$ or simply φ selects a matching $\varphi[P]$ for every $P \in \mathcal{P}^n$. Formally, φ is a function

$$\varphi \colon \mathcal{P}^n \to \mathcal{M}.$$

82. A matching μ' (Pareto) dominates μ if for all $i \in I$, $\mu'(i) R_i \mu(i)$, and for some $i' \in I$, $\mu'(i) P_i \mu(i)$.

A matching is Pareto efficient if it is not dominated.

A mechanism φ is Pareto efficient if $\varphi[P]$ is Pareto efficient for all $P \in \mathcal{P}^n$.

A mechanism φ dominates ψ if

- for all P, $\varphi[P](i) R_i \psi[P](i)$ for all i
- for some P, $\varphi[P](i) P_i \psi[P](i)$ for some i

83. A matching μ is individually rational if no student prefers being unmatched to her assignment.

    A mechanism φ is individually rational if $\varphi[P]$ is individually rational for all $P \in \mathcal{P}^n$.

84. A matching μ is non-wasteful if no student prefers a school with one or more empty seats to her assignment. That is, μ is non-wasteful if, whenever i prefers s to her assignment $\mu(i)$, $|\mu^{-1}(s)| = q_s$.

    A mechanism φ is non-wasteful if $\varphi[P]$ is non-wasteful for all $P \in \mathcal{P}^n$.

85. We say that student i desires school s at μ if $s P_i \mu(i)$.

A matching μ eliminates justified envy if no student i prefers the assignment of another student j while at the same time having higher priority at school $\mu(j)$.

A mechanism φ eliminates justified envy if $\varphi[P]$ eliminates justified envy for all $P \in \mathcal{P}^n$.

86. A mechanism φ is strategy-proof if no student can benefit from misreporting, *i.e.*, truth-telling is a weakly dominant strategy for all students under the mechanism φ. Formally,

$$\varphi[P_i, P_{-i}](i) R_i \varphi[P_i', P_{-i}](i), \text{ for all } i, P_i', P.$$

87. A mechanism φ is non-bossy if for any $P, i \in I$ and P_i',

$$\varphi[P](i) = \varphi[P_i', P_{-i}](i) \text{ implies } \varphi[P] = \varphi[P_i', P_{-i}].$$

Non-bossiness ensures that students can not be bossy, that is, change the matching for others, by reporting different preferences, without changing their own.

88. The Boston mechanism.[1]

1 For each school a priority ordering is exogenously determined. (In case of Boston, priorities depend on home address, whether the student has a sibling already attending a school, and a lottery number to break ties.)

2 Each student submits a preference ranking of the schools.

3 The final phase is the student assignment based on preferences and priorities:

Step 1: In Step 1 only the top choices of the students are considered. For each school, consider the students who have listed it as their top choice and assign seats of the school to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her top choice.

Step k: Consider the remaining students. In Step k only the kth choices of these students are considered. For each school still with available seats, consider the students who have listed it as their kth choice and assign the remaining seats to these students

[1] This name came from the fact that it was in use for school choice in Boston Public Schools before it was replaced by the student-proposing DA.

one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her kth choice.

End The algorithm terminates when no more students are assigned. At each step, every assignment is final.

89. The Boston mechanism assigns as many students as possible to their first choices based on their submitted preferences; next, as many students as possible to their second choices; and so on. The major drawback of this widely used mechanism is its lack of strategy-proofness.

90. Theorem: For any given $(P, \succ)$, DA produces a matching that is stable at $(P, \succ)$, which is also at least as good for every student as any other stable matching at $(P, \succ)$.

91. Theorem: Given fixed priorities $\succ$, DA is strategy-proof (for students).

92. Theorem (Theorem 3 in Alcalde and Barberà (1994)): DA is the unique stable and strategy-proof mechanism in school choice problem.

93. The major drawback of DA is its lack of efficiency.

94. Remark: DA is strategy-proof and stable, but not efficient. Are there mechanisms that improve the efficiency of students without sacrificing the other two properties?

- Stability will be lost for sure, since DA produces the student-optimal stable matching.
- Strategy-proofness will also be lost, due to the following impossibility result.

95. Theorem (Proposition 1 in Kesten (2010), Theorem 1 in Abdulkadiroğlu *et al.* (2009), Proposition 1 in Erdil (2014)): If φ is a strategy-proof and non-wasteful mechanism, then there is no strategy-proof mechanism that Pareto dominates φ.

96. Definition (Definition 1 in Ergin (2002)): Given a priority structure $\succ$ and quota profile q, a cycle is $a, b \in S, i, j, k \in I$ such that the following are satisfied:

(C) Cycle condition: $i \succ_a j \succ_a k \succ_b i$.

(S) Scarcity condition: There exist disjoint sets of students $I_a, I_b \subseteq I \setminus \{i, j, k\}$ such that $|I_a| = q_a - 1, |I_b| = q_b - 1, i' \succ_a j$ for every $i' \in I_a$, and $i'' \succ_b i$ for every $i'' \in I_b$.

A priority structure $\succ$ (or $(\succ, q)$) is acyclic if there exists no cycle.

97. Consider the school choice problem $\langle I, S, q, P, \succ \rangle$ in Example 93, where $I = \{i, j, k\}$, $S = \{a, b\}$, $q_a = q_b = 1$, and

i	j	k	a	b
b	a	a	i	k
a		b	j	i
			k	

Table 2

The matching produced by DA is

$$\mu = \begin{bmatrix} i & j & k \\ a & \emptyset & b \end{bmatrix}.$$

A mutually beneficial agreement between i and k would be to get schools a and b respectively by exercising their priority rights, and then to make an exchange so that finally i gets

b and k gets a.

However the final matching would violate the priority of j for a, contradicting the allocation on the basis of specified priorities.

Here the priority structure is not acyclic, since j may block a potential matching between i and k without affecting his own position, that is

$$i \succ_a j \succ_a k \succ_b i.$$

98. Theorem (Theorem 1 in Ergin (2002)): Given $\langle I, J, \succ, q\rangle$, the following are equivalent:

(i) $\succ$ is acyclic.

(ii) DA$^{\succ}$ is Pareto efficient.

(iii) DA$^{\succ}$ is group strategy-proof.

99. Example: Consider the school choice problem $\langle I, S, q, P, \succ\rangle$, where $I = \{i, j, k\}$, $S = \{s_1, s_2\}$, $q_{s_1} = q_{s_2} = 1$, and

i	j	k	s_1	s_2
s_2	s_1	s_1	i	k
s_1		s_2	j	i
			k	

Table 3

The matching produced by DA is

$$\begin{bmatrix} i & j & k \\ s_1 & \emptyset & s_2 \end{bmatrix},$$

and the procedure is

Step	1	2	3	End
s_1	$j, \cancel{k}$	j	$\cancel{j}, i$	i
s_2	i	$\cancel{i}, k$	k	k
$\emptyset$	k	i	j	j

Table 4

100. In Example 99, when the DA algorithm is applied to this problem, student j causes student k to be rejected from school s_1 and starts a chain of rejections that ends back at school s_1, forming a full cycle and causing student j himself to be rejected. There such a cycle has resulted in loss of efficiency

By applying to school s_1, student j "interrupts" a desirable settlement between students i and k without affecting her own placement and artificially introduces inefficiency into the outcome. The key idea behind the mechanism produced by Kesten (2010) is based on preventing students such as student j of this example from interrupting settlements among other students.

101. Coming back to Example 99, suppose student j consents to give up her priority at school s_1, *i.e.*, if she is okay with accepting the the unfairness caused by matching k to s_1. Thus, school s_1 is to be removed from student j's preferences without affecting the relative ranking of the other schools in her preferences.

Note that, when we rerun DA, replacing the preferences of student j with her new preferences, there is no change in the placement of student j. But, because the previously mentioned cycle now disappears, students i and k each move one position up in their preferences. Moreover, the new matching is now Pareto-efficient. To be more detailed, the preference profiles become

i	j	k	s_1	s_2
s_2		s_1	i	k
s_1		s_2	j	i
			k	

Table 5

The matching produced by DA is

$$\begin{bmatrix} i & j & k \\ s_2 & \emptyset & s_1 \end{bmatrix},$$

and the procedure is

102. Definition: Given a problem to which DA is applied, let i be a student who is tentatively

Step	1	End
s_1	k	k
s_2	i	i
$\emptyset$	j	j

Table 6

placed at a school s at some Step t and rejected from it at some later Step t'. If there is at least one other student who is rejected from school s after Step t–1 and before Step t', that is, rejected at a Step $l \in \{t, t+1, \ldots, t'–1\}$, then we call student i an interrupter for school s and the pair (i, s) an interrupting pair of Step t'.

103. Lemma: If the outcome of DA is inefficient for a problem, then there exists one interrupting pair in DA. However, the converse is not necessarily true, *i.e.*, an interrupting pair does not always result in efficiency loss.

104. Efficiency-adjusted deferred acceptance mechanism (EADAM):

Round 0: Run DA for $(P, \succ)$.

Round $k \geq 1$:

(1) Find the last step of DA in Round $k-1$ in which a consenting interrupter is rejected from the school for which she is an interrupter.

(2) Identify all interrupting pairs of that step each of which contains a consenting interrupter.

(3) For each identified interrupting pair (i, s), remove school s from the preferences of student i without changing the relative order of the remaining schools. Do not make any changes in the preferences of the remaining students.

(4) Rerun DA with the new preference profile.

End: If there are no interrupting pairs, then stop.

When we say student i is an interrupter of Round t, this means that student i is identified as an interrupter during Round $t+1$ in DA that was run at the end of Round t.

105. Theorem (Theorem 1 in Kesten (2010)): The EADAM Pareto dominates the DA as well as any fair mechanism. If no student consents, the two mechanisms are equivalent. If all

students consent, then the EADAM outcome is Pareto-efficient. In the EADAM outcome all nonconsenting students' priorities are respected; however, there may be consenting students whose priorities for some schools are violated with their permission.

106. In a school choice problem $\langle I, S, q, P, \succ \rangle$ with a given matching μ, for each school s, let D_s be the highest $\succ_s$-priority students among those who desire s (*i.e.*, who prefer s to their assignments under μ).

107. Definition: A stable improvement cycle consists of distinct students $i_1, i_2, \ldots, i_n = i_0$ $(n \geq 2)$ such that for each $\ell = 0, 1, \ldots, n-1$,

(1) i_ℓ is matched to some school under μ;

(2) i_ℓ desires $\mu(i_{\ell+1})$; and

(3) $i_\ell \in D_{\mu(i_{\ell+1})}$.

108. Given a stable improvement cycle, define a new matching μ' by:

$$\mu'(j) = \begin{cases} \mu(j), & \text{if } j \notin \{i_1, i_2, \ldots, i_n\}; \\ \mu(i_{\ell+1}), & \text{if } j = i_\ell. \end{cases}$$

Note that the matching μ' continues to be stable and it Pareto dominates μ.

109. Theorem (Theorem 1 in Erdil and Ergin (2008)): Fix $\succ$ and P, and let μ be a stable matching. If μ is Pareto dominated by another stable matching ν, then it admits a stable improvement cycle.

110. Stable improvement cycles algorithm:

Step 0: Run DA algorithm and obtain a temporary matching μ^0.

Step $k \geq 1$:

(1) Find a stable improvement cycle for μ^{k-1}: for schools s and t, let $s \rightarrow t$ if some student $i \in D_t$ is matched to s under μ^{k-1}.

(2) If there are any cycles, select one. For each $s \to t$ in this cycle, select a student $i \in D_t$ with $\mu^{k-1}(i) = s$. Carry out this stable improvement cycle to obtain μ^k.

End: The algorithm stops when there is no cycle.

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