Networks in the Real World

- A network is a set of items (nodes or vertices) connected by edges or links.
- Systems taking the form of networks abound in the world.
- **Types of Networks:**
  - Social and economic networks: A set of people or groups of people with some pattern of contacts or interactions between them.
    - Facebook, friendship networks, business relations between companies, intermarriages between families, labor markets
    - **Questions:** Degree of connectedness, homophily, small-world effects
  - Information networks: Connections of "information" objects.
    - Network of citations between academic papers, World Wide Web (network of Web pages containing information with links from one page to other), semantic (how words or concepts link to each other)
    - **Questions:** Ranking, navigation

Networks in the Real World (Continued)

- **Types of Networks:**
  - Technological networks: Designed typically for distribution of a commodity or service.
    - Infrastructure networks: e.g., Internet (connections of routers or administrative domains), power grid, transportation networks (road, rail, airline, mail)
    - Temporary networks: e.g., ad hoc communication networks, sensor networks, autonomous vehicles
    - **Questions:** Does network structure support performance? Fragility? Cascading failures?
  - Biological networks: A number of biological systems can also be represented as networks.
    - Food web, protein interaction network, network of metabolic pathways

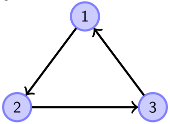
Network Study

- Historical study of networks:
  - Mathematical graph theory: One of the pillars of discrete mathematics
    - Started with Euler's celebrated 1735 solution of the Königsberg bridge problem.
  - Networks also studied extensively in sociology.
    - Typical studies involve circulation of questionnaires, leading to small networks of interactions.
- Recent years witnessed a substantial change in network research.
  - From analysis of single small graphs (10-100 nodes) to statistical properties of large scale networks (million-billion nodes).
  - Motivated by availability of computers and computer networks that allow us to gather and analyze large scale data.
- New Analytical Approach:
  - Find statistical properties that characterize the structure of these networks and ways to measure them
  - Create models of networks
  - Predict behavior of networks on the basis of measured structural properties and models

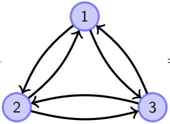
Graphs—1

- We represent a network by a graph (N, g), which consists of a set of nodes $N = \{1, \ldots, n\}$ and an $n \times n$ matrix $g = [g_{ij}]_{i,j \in N}$ (referred to as an adjacency matrix), where $g_{ij} \in \{0, 1\}$ represents the availability of an edge from node i to node j.
    - The edge weight $g_{ij} > 0$ can also take on non-binary values, representing the intensity of the interaction, in which case we refer to (N, g) as a weighted graph.
- We refer to a graph as a directed graph (or digraph) if $g_{ij} \neq g_{ji}$ and an undirected graph if $g_{ij} = g_{ji}$ for all $i, j \in N$.

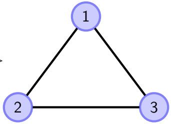
Example 1: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow$



Example 2: $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow$

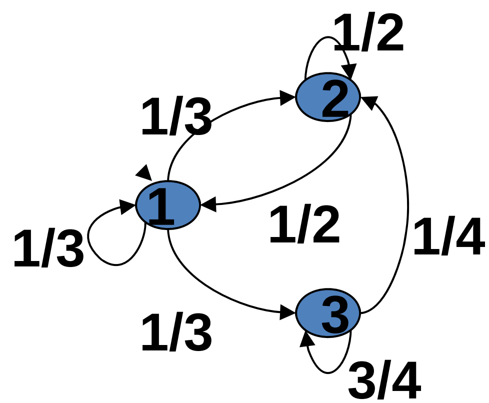


$\Rightarrow$



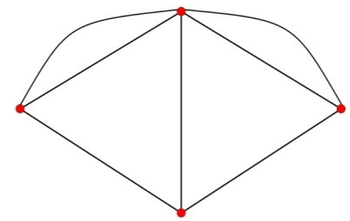
Weighted Directed Network:

$$g = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

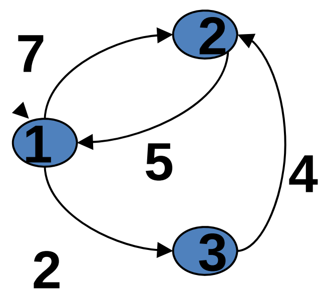


``row stochastic''

Weighted Directed Network:



$$g = \begin{pmatrix} 0 & 7 & 2 \\ 5 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}$$



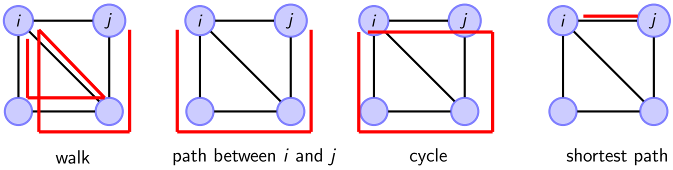
Graphs—2

- Another representation of a graph is given by (N, E), where E is the set of edges in the network.
    - *For directed graphs:* E is the set of "directed" edges, i.e., $(i, j) \in E$.
    - *For undirected graphs:* E is the set of "undirected" edges, i.e., $\{i, j\} \in E$.
- In Example 1, $E_d = \{(1, 2), (2, 3), (3, 1)\}$
- In Example 2, $E_u = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$
- When are directed/undirected graphs applicable?
    - Citation networks: directed
    - Friendship networks: undirected
- We will use the terms network and graph interchangeably.
- We will sometimes use the notation $(i, j) \in g$ (or $\{i, j\} \in g$) to denote $g_{ij} = 1$.

Walks, Paths, and Cycles—1

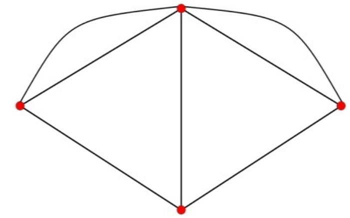
- We consider "sequences of edges" to capture indirect interactions.
- For an undirected graph (N, g):
  - A walk is a sequence of edges $\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{K-1}, i_K\}$.
  - A path between nodes i and j is a sequence of edges $\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{K-1}, i_K\}$ such that $i_1 = i$ and $i_K = j$, *and each node in the sequence* $i_1, \ldots, i_K$ *is distinct*.
  - A cycle is a path with a final edge to the initial node.
  - A geodesic between nodes i and j is a "shortest path" (i.e., with minimum number of edges) between these nodes.
- A path is a walk where there are no repeated nodes.
- The length of a walk (or a path) is the number of edges on that walk (or path).
- For directed graphs, the same definitions hold with directed edges (in which case we say "a path from node i to node j").

Walks, Paths, and Cycles—2

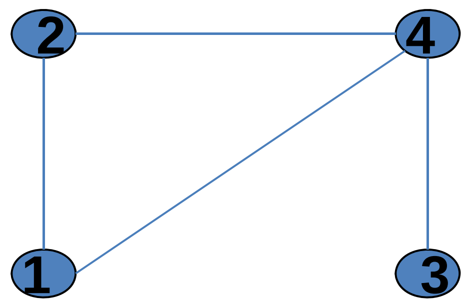


- *Note:* Under the convention $g_{ii} = 0$, the matrix g^2 tells us number of walks of length 2 between any two nodes:
  - $(g \times g)_{ij} = \sum_k g_{ik} g_{kj}$
  - Similarly, g^k tells us number of walks of length k.

Counting Walks:



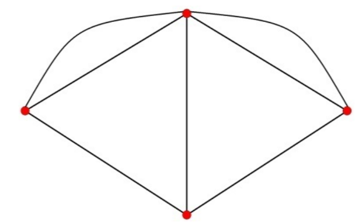
$$g = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$



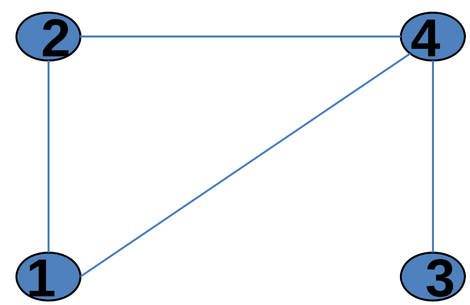
$$g^2 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{pmatrix}$$

number of walks of length 2 from i to j

Counting Walks:



$$g= \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$



$$g^3 = \begin{pmatrix} 2 & 3 & 1 & 4 \\ 3 & 2 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 4 & 4 & 3 & 2 \end{pmatrix}$$

number of walks of length 3 from i to j

Connectivity and Components

- An undirected graph is connected if every two nodes in the network are connected by some path in the network.
- Components of a graph (or network) are the distinct maximally connected subgraphs.
- A directed graph is
  - connected if the underlying undirected graph is connected (i.e., ignoring the directions of edges).
  - strongly connected if each node can reach every other node by a "directed path".

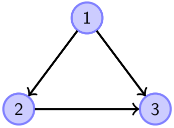
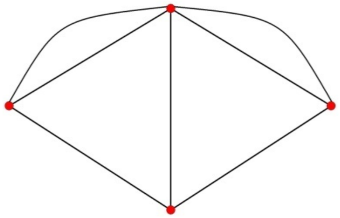
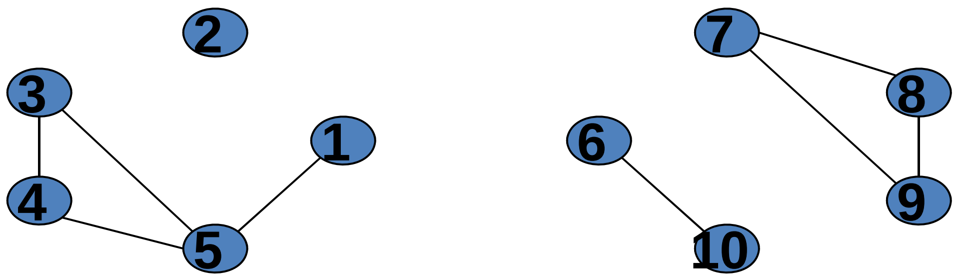


Figure: A directed graph that is connected but not strongly connected

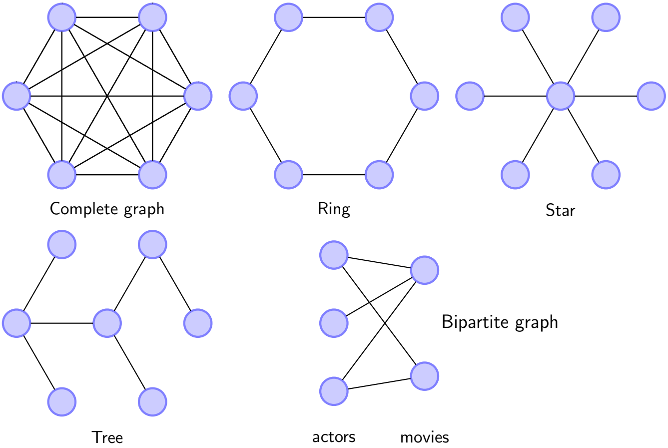
A network with four components:





Trees, Stars, Rings, Complete and Bipartite Graphs

- A tree is a connected (undirected) graph with no cycles.
  - A connected graph is a tree if and only if it has $n-1$ edges.
  - In a tree, there is a unique path between any two nodes.



Neighborhood and Degree of a Node

- The neighborhood of node i is the set of nodes that i is connected to.
- For undirected graphs:
  - The degree of node i is the number of edges that involve i (i.e., cardinality of his neighborhood).
- For directed graphs:
  - Node i's in-degree is $\sum_j g_{ji}$.
  - Node i's out-degree is $\sum_j g_{ij}$.

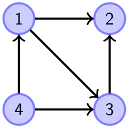
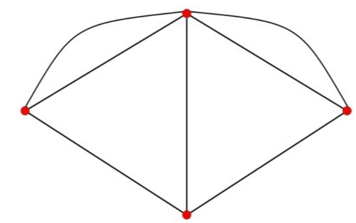


Figure: Node 1 has in-degree 1 and out-degree 2

Properties of Networks

- While a small network can be visualized directly by its graph (N, g), larger networks can be more difficult to envision and describe.
- Therefore, we define a set of **summary statistics** or **quantitative performance measures** to describe and compare networks (*focus on undirected graphs*):
  - Diameter and average path length
  - Clustering
  - Centrality
  - Degree distributions
- A Simple Random Graph Model—Erdös-Renyi model
  - We use the notation $G(n, p)$ to denote the undirected Erdös-Renyi graph.
  - Every edge is formed with probability $p \in (0, 1)$ **independently** of every other edge.
  - Expected degree of a node i is $\mathbb{E}[d_i] = (n-1)p$
  - Expected number of edges is $\mathbb{E}[\text{number of edges}] = \frac{n(n-1)}{2}\, p$

Simplifying the Complexity



- Global patterns of networks
  - degree distributions, path lengths...
- Segregation Patterns
  - node types and homophily
- Local Patterns
  - Clustering, Transitivity, Support…
- Positions in networks
  - Neighborhoods, Centrality, Influence...

Diameter and Average Path Length

- Let $l(i,j)$ denote the length of the shortest path (or geodesic) between node i and j (or the distance between i and j).
- The diameter of a network is the largest distance between any two nodes in the network:

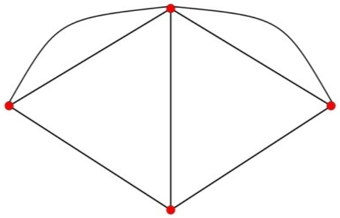
$$\text{diameter} = \max_{i,j} l(i,j)$$

- The average path length is the average distance between any two nodes in the network:

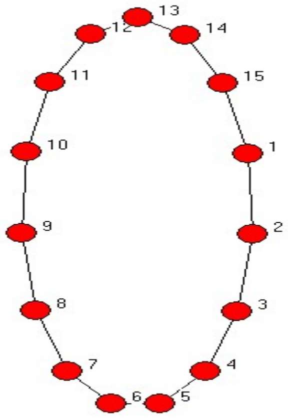
$$\text{average path length} = \frac{\sum_{i \geq j} l(i,j)}{\frac{n(n-1)}{2}}$$

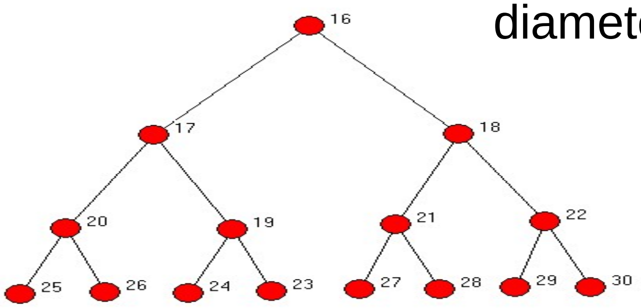
- Average path length is bounded from above by the diameter; in some cases, it can be much shorter than the diameter.
- If the network is not connected, one often checks the diameter and the average path length in the largest component.

Diameter:



K levels has n = $2^{K+1}-1$ nodes
so, K = $\log_2(n+1) -1$
diameter is 2K

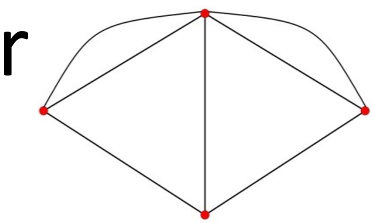




diameter is either
n/2 or (n-1)/2

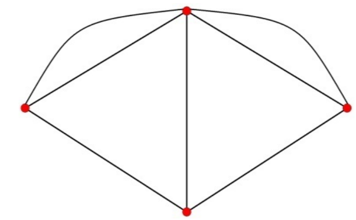
diameter is on order of
2 $\log_2(n+1)$

Small average path length and diameter



- Milgram (1967) letter experiments
  - median 5 for the 25% that made it
- Co-Authorship studies
  - Grossman (2002) Math mean 7.6, max 27,
  - Newman (2001) Physics mean 5.9, max 20
  - Goyal et al (2004) Economics mean 9.5, max 29
- WWW
  - Adamic, Pitkow (1999) – mean 3.1 (85.4% possible of 50M pages)
- Facebook
  - Backstrom et al (2012) – mean 4.74 (721 million users)

Sequences of Networks



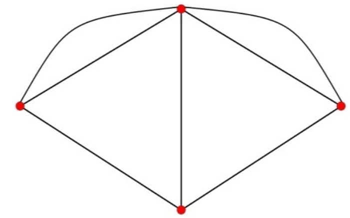
- Links are dense enough so that network is connected almost surely:

$$d(n) \geq (1+\varepsilon) \log(n) \text{ some } \varepsilon>0$$

- $d(n)/n \rightarrow 0$:

  network is not too complete

Theorem on Network Structure



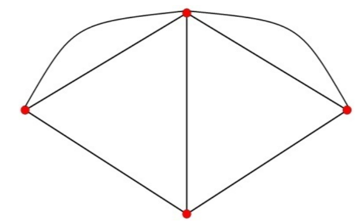
If $d(n) \geq (1+\varepsilon) \log(n)$ some $\varepsilon>0$ and $d(n)/n \to 0$

Then for large n, average path length and diameter are approximately proportional to $\log(n)/\log(d)$

(Proven for increasingly general models:

Erdos-Renyi 59 - Moon and Moser 1966, Bollobas 1981; Chung and Lu 01; Jackson 08; ...)

Theorem on Network Structure

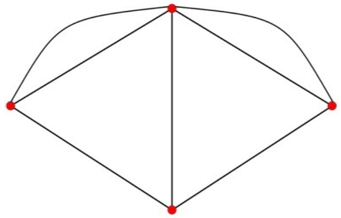


If $d(n) \geq (1+\varepsilon)\log(n)$ some $\varepsilon>0$ and $d(n)/n \rightarrow 0$

$$\frac{\text{AvgDist}(n)}{\log(n)/\log(d(n))} \rightarrow^{P} 1$$

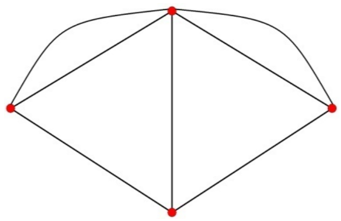
same for diameter

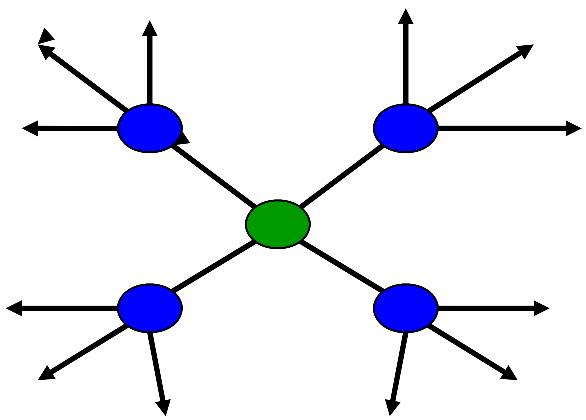
Diameter



- Bounds can be difficult – theorems are narrow, but intuition is easy
- Let's start with an easy calculation --
- Cayley Tree: each node besides leaves has degree d

Ideas:



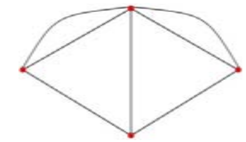


1 step: Reach d nodes,

then d(d-1),

then $d(d-1)^2$, $d(d-1)^3$, ...

After ℓ steps, totals roughly d^{ℓ}



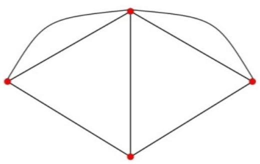
- Moving out ℓ links from root in each direction reaches d + d(d-1) + ... $d(d-1)^{\ell -1}$ nodes

- This is $d((d-1)^{\ell} -1)/(d-2)$ nodes: roughly $(d-1)^{\ell}$

- To reach n-1, need roughly $(d-1)^{\ell} = n$

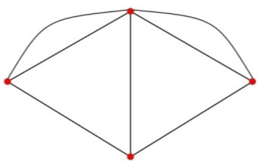
- or ℓ on the order of log(n)/log(d)

What if not a tree, but Erdos-Renyi random graph?



- all have same degree – really are random
  - show that fraction of nodes that have nearly average degree is going to 1

- $E[d] > (1+\varepsilon)\log(n)$



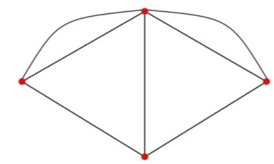
- **Chernoff Bounds:**

X is binomial variable then

$Pr(E[X]/3 \leq X \leq 3E[X]) \geq 1 - e^{-E[X]}$

http://en.wikipedia.org/wiki/Chernoff_bound

- **Chernoff Bounds: Links binomial implies**

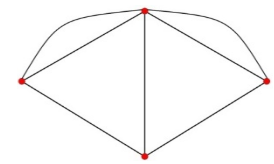


Probability that node has degree close to average:

$Pr(d/3 \leq d_i \leq 3d) \geq 1 - e^{-d}$

$Pr (d/3 \leq \text{all degrees} \leq 3d) \geq (1 - e^{-d})^n$

(missing steps: degrees not quite ind.)



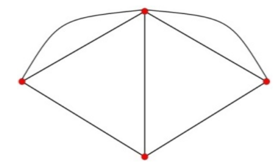
- **Chernoff Bounds:**

Pr ($d/3 \leq$ all degrees $\leq 3d$) $\geq (1 - e^{-d})^n$

- If $d > (1+\varepsilon)\log(n)$ then

Pr ($d/3 \leq$ all degrees $\leq 3d$) $> (1 - 1/n^{1+\varepsilon})^n$

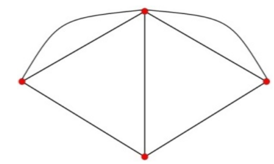
$\rightarrow \exp(-n^{-\varepsilon}) \rightarrow 1$



- **So:**

- If $d > (1+\varepsilon)\log(n)$ then

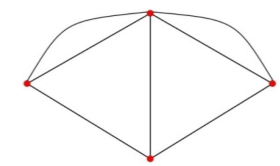
  $\Pr(\, d/3 \leq \text{all degrees} \leq 3d \,) \to 1$



- **Thus:**

- If $d > (1+\varepsilon)\log(n)$ then with prob $\to$ 1:

$$\log(n)/\log(3d) < \ell < \log(n)/\log(d/3)$$



- **Avg distance and diameter:**
- Large d: log(3d) & log(d/3) tend to log(d)
- $\log(n)/\log(3d) < \ell < \log(n)/\log(d/3)$
- $\log(n)/\log(d) \approx \ell$

Clustering

- Measures the extent to which my friends are friends with one another.
- This clustering measure is represented by the overall clustering coefficient $Cl(g)$, given by

$$Cl(g) = \frac{3 \times \text{number of triangles in the network}}{\text{number of connected triples of nodes}},$$

where a "connected triple" refers to a node with edges to an unordered pair of nodes.

  - Note that $0 \leq Cl(g) \leq 1$.
  - $Cl(g)$ measures the fraction of triples that have their third edge filled in to complete the triangle.
  - Also referred to as network transitivity: measures the extent that a friend of my friend is also my friend.

Clustering (Continued)

- Another measure of clustering is defined on an individual node basis: The individual clustering for a node i is

$$Cl_i(g) = \frac{\text{number of triangles connected to vertex } i}{\text{number of triples centered at } i}.$$

- The average clustering coefficient is $Cl^{Avg}(g) = \frac{1}{n}\sum_i Cl_i(g)$.

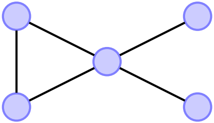
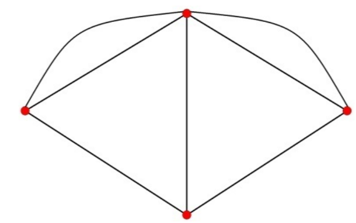


Figure: The overall clustering coefficient for this network is 3/8. The individual clustering for the nodes are 1, 1, 1/6, 0, and 0.

- What is the individual clustering for a node in the Erdös-Renyi model?

Clustering

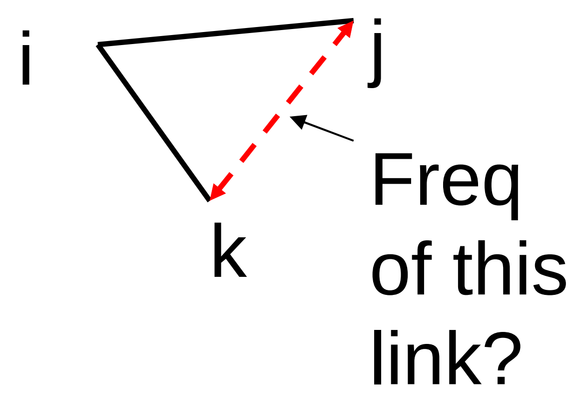


- What fraction of my friends are friends of each other?

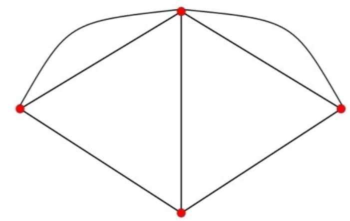
- $Cl_i(g) = \#\{ kj \text{ in } g \mid k, j \text{ in } N_i(g)\} \;/\; \#\{ kj \mid k, j \text{ in } N_i(g)\}$

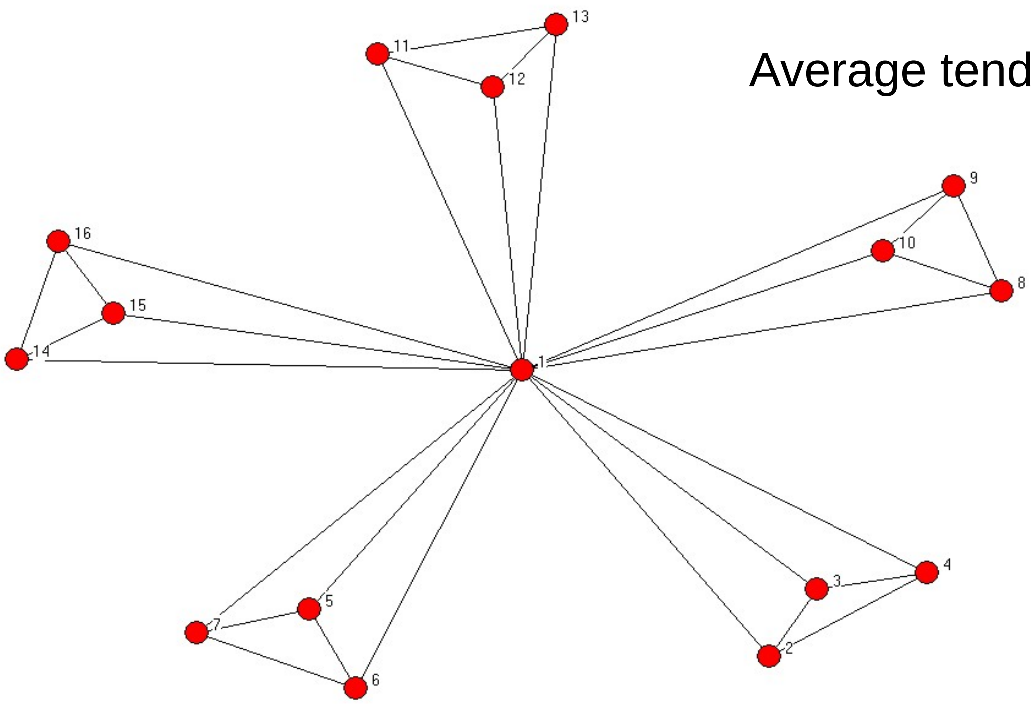
- Average clustering:

$$Cl^{avg}(g) = \sum_i Cl_i(g) / n$$



Differences in Clustering

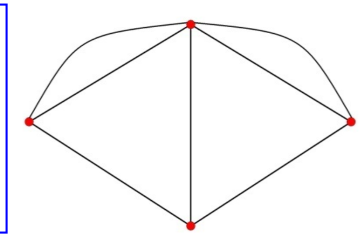




Average tends to 1

Overall tends to 0

Centrality,
Four different things to measure:

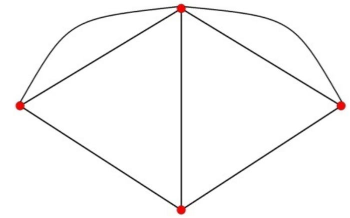


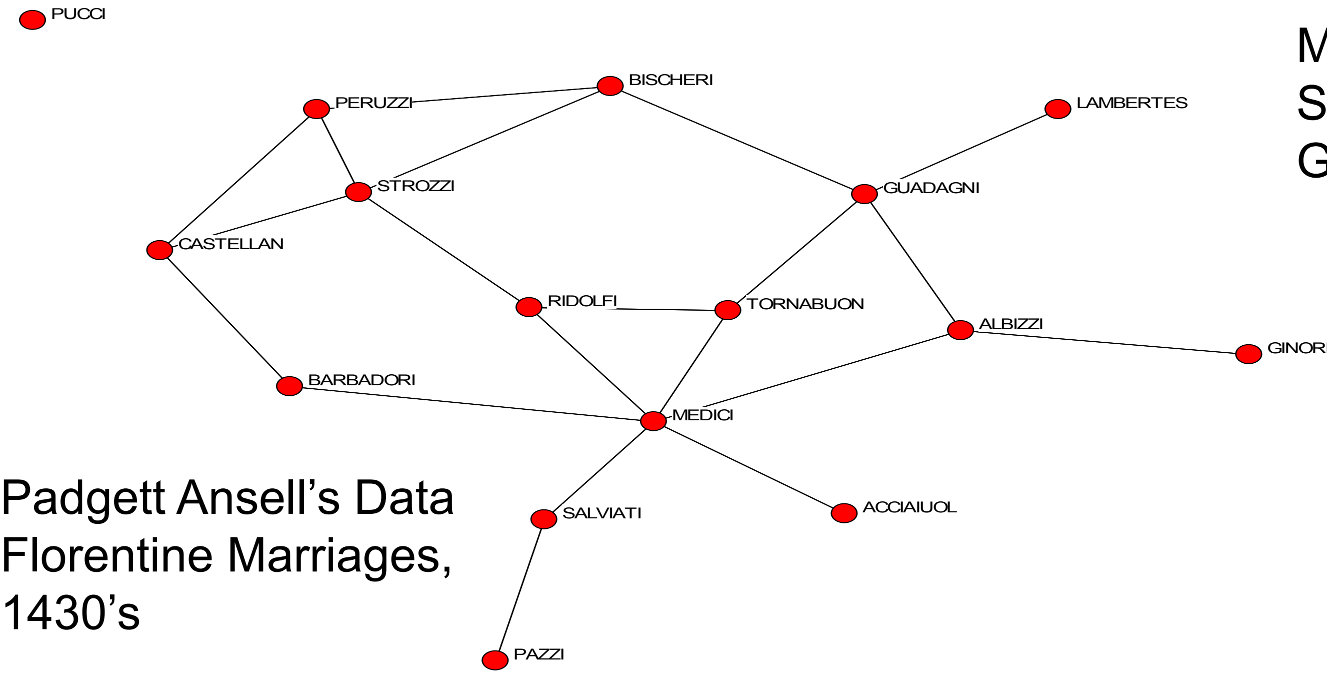
- Degree – connectedness
- Closeness, Decay – ease of reaching other nodes
- Betweenness – role as an intermediary, connector
- Influence, Prestige, Eigenvectors –
  ``not what you know, but who you know..''

Centrality

- A micro measure that captures the importance of a node's position in the network.
- Different measures of centrality
  - Degree centrality: for node i,
  $$d_i(g)/n-1, \quad \text{where } d_i(g) \text{ is the degree of node } i$$
  - Closeness centrality: Tracks how close a given node is to any other node: for node i, one such measure is
  $$\frac{n-1}{\sum_{j\neq i} l(i,j)}, \quad \text{where } l(i,j) \text{ is the distance between } i \text{ and } j$$
  - Betweenness centrality: Captures how well situated a node is in terms of paths that it lies on (see the Florentine marriages example from the previous lecture).

Degree Centrality

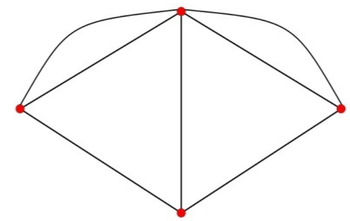




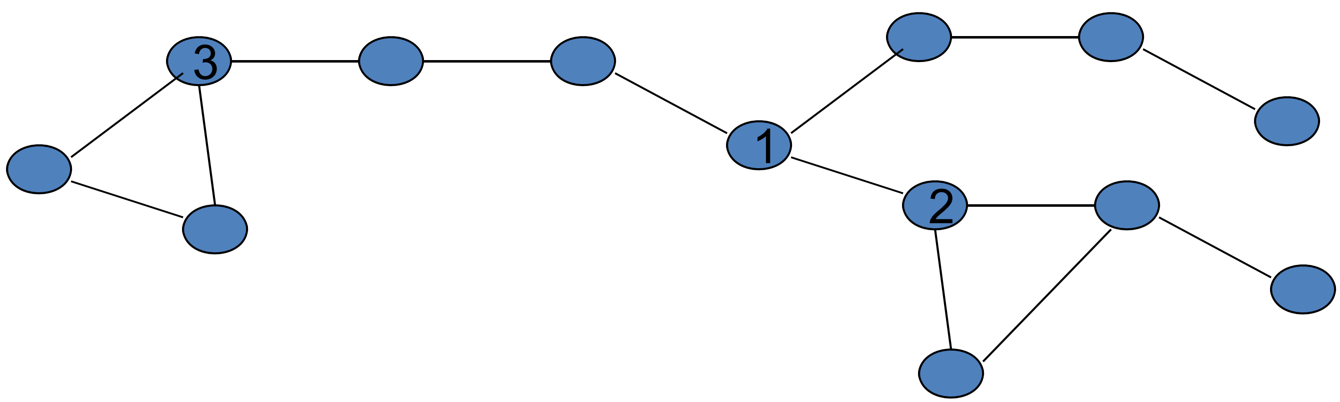
Medici = 6
Strozzi = 4
Guadagni = 4

Padgett Ansell's Data
Florentine Marriages,
1430's

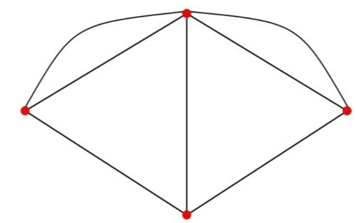
Degree Centrality



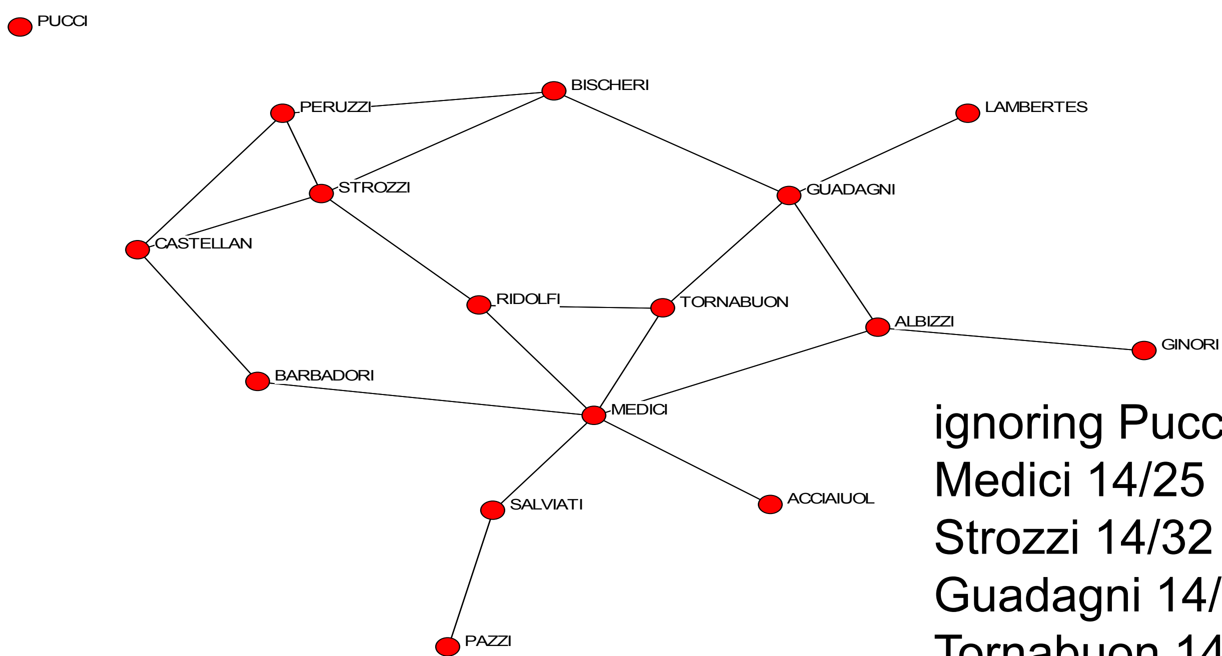
- Node 3 is considered as ``central'' as 1 and 2



Closeness

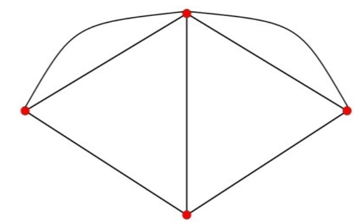


Closeness centrality: $(n-1) / \sum_j \ell(i,j)$



ignoring Pucci:
Medici 14/25
Strozzi 14/32
Guadagni 14/26
Tornabuon 14/29
Ridolfi 14/28

Decay Centrality



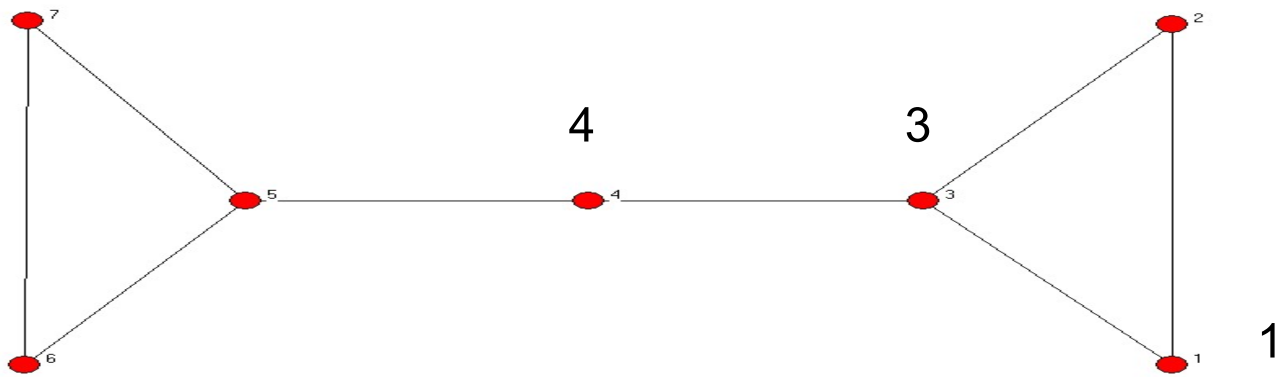
$$C_i^d(g) = \sum_{j \neq i} \delta^{\ell(i,j)}$$

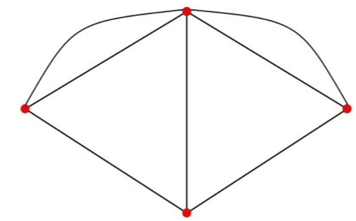
δ near 1 becomes component size

δ near 0 becomes degree

δ in between decaying distance measure

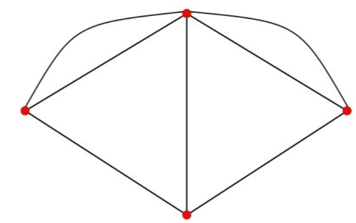
– weights distance exponentially





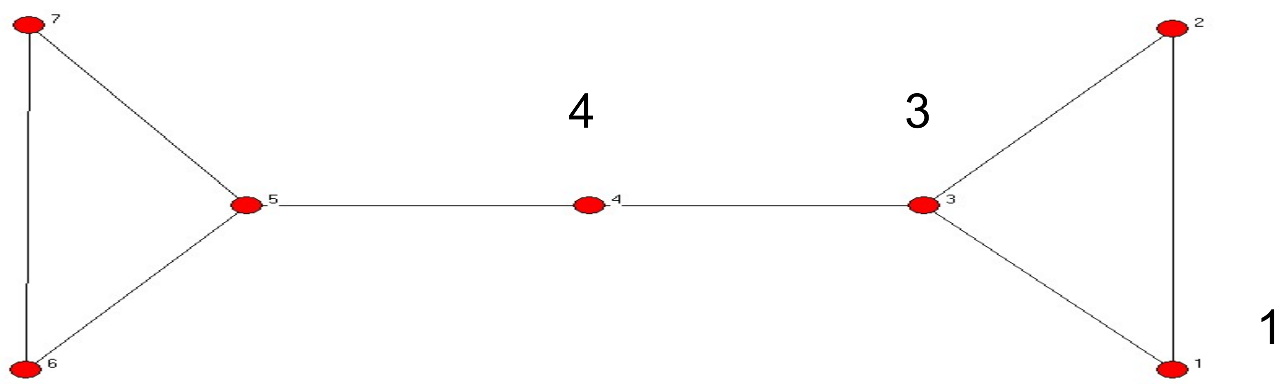
	Node 1	Node 3	Node 4
Degree	.33	.50	.33
Closeness	.40	.55	.60
Decay δ = .5	1.5	2.0	2.0
Decay δ = .75	3.1	3.7	3.8
Decay δ = .25	.59	.84	.75

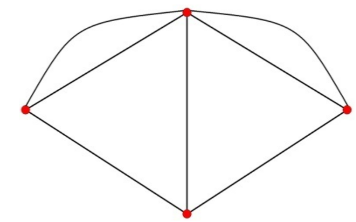
Normalize: Decay Centrality



$$C_i^d(g) = \sum_{j \neq i} \delta^{\ell(i,j)} \,/\, ((n-1)\,\delta)$$

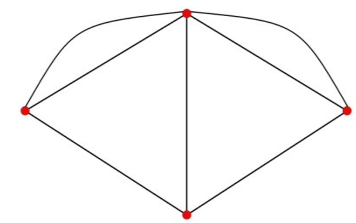
- $(n-1)\,\delta$ is the lowest decay possible

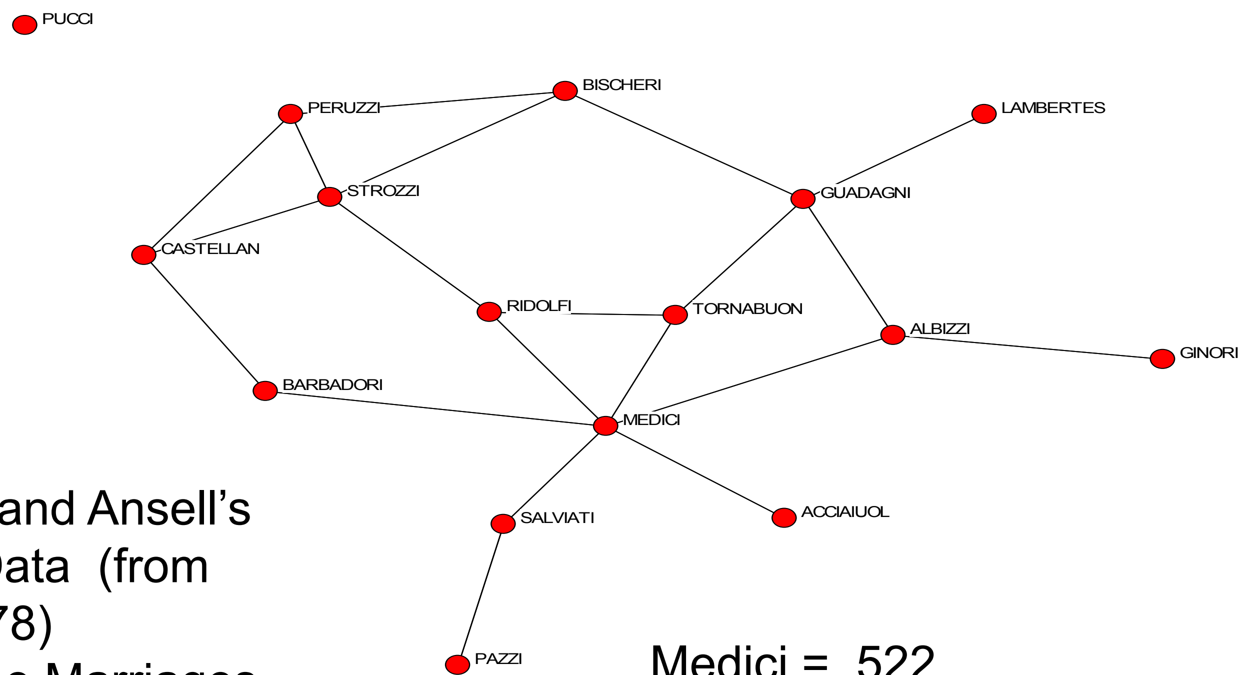




	Node 1	Node 3	Node 4
Degree	.33	.50	.33
Closeness	.40	.55	.60
N. Decay δ = .5	.50	.67	.67
N. Decay δ = .75	.69	.82	.84
N. Decay δ = .25	.39	.56	.50

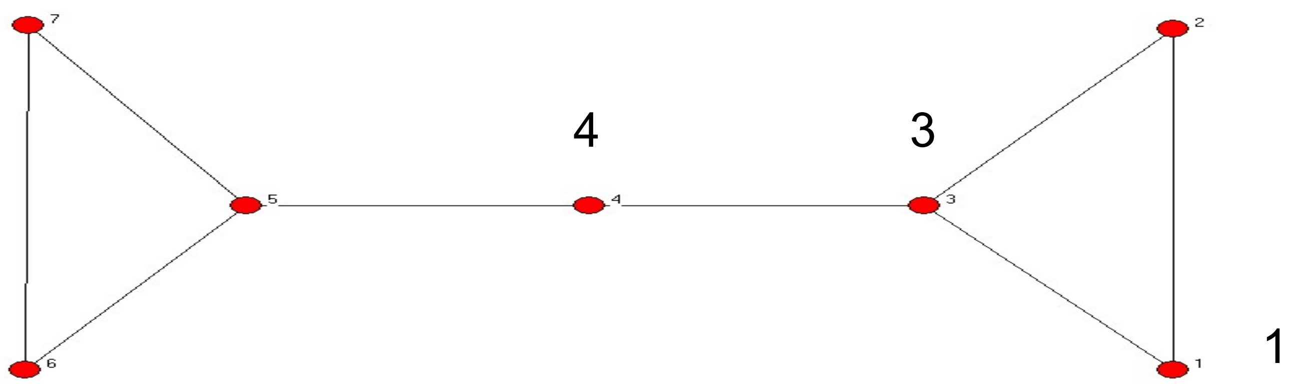
Betweenness Centrality

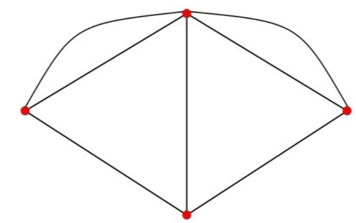




Padgett and Ansell's (1993) Data (from Kent 1978) Florentine Marriages, 1430's

Medici = .522
Strozzi = .103
Guadagni = .255





	Node 1	Node 3	Node 4
Degree	.33	.50	.33
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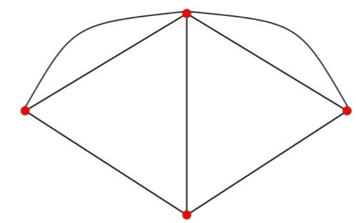
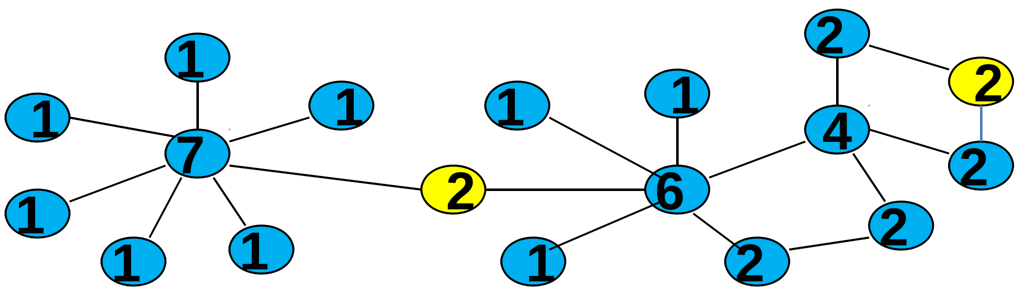
N. Decay δ = .5	.50	.67	.67
N. Decay δ = .75	.69	.82	.84
N. Decay δ = .25	.39	.56	.50
Betweenness	.00	.53	.60

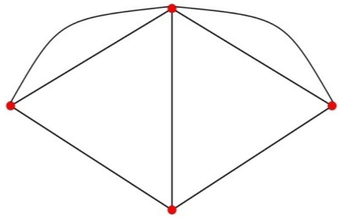
Degree Centrality?



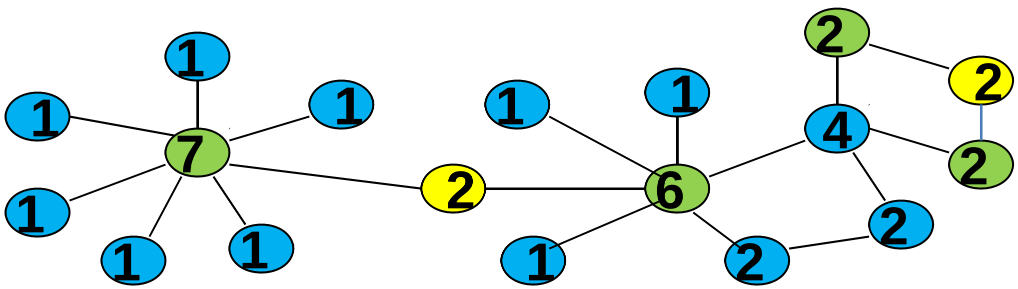
- Failure of degree centrality to capture reach of a node:



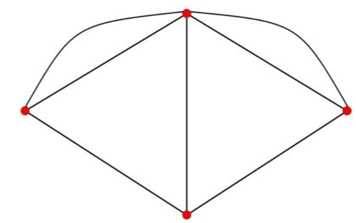
Degree Centrality?



- More reach if connected to a 6 and 7 than a 2 and 2?



Eigenvector Centrality

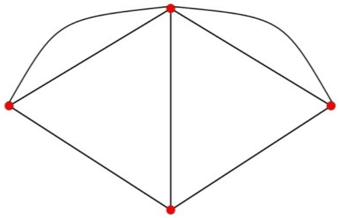


- Centrality is proportional to the sum of neighbors' centralities

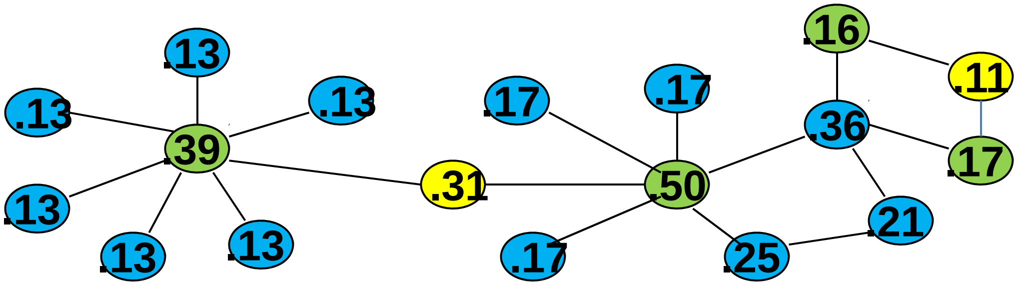
C_i proportional to $\sum_{j:\text{ friend of } i} C_j$

$C_i = a\sum_j g_{ij} C_j$

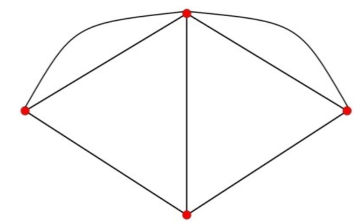
Eigenvector Centrality



Now distinguishes more ``influential'' nodes



Prestige, Influence, Eigenvector-based Centrality



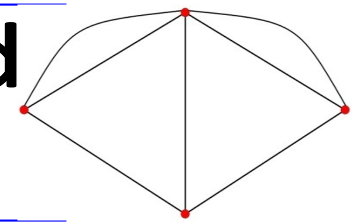
- Get value from connections to others, but proportional to their value

- Self-referential concept

$$C_i^e(g) = a \sum_j g_{ij} C_j^e(g)$$

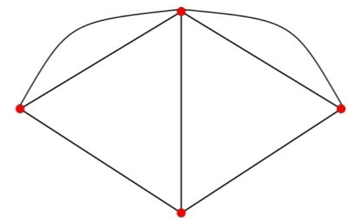
– centrality is proportional to the summed centralities of neighbors

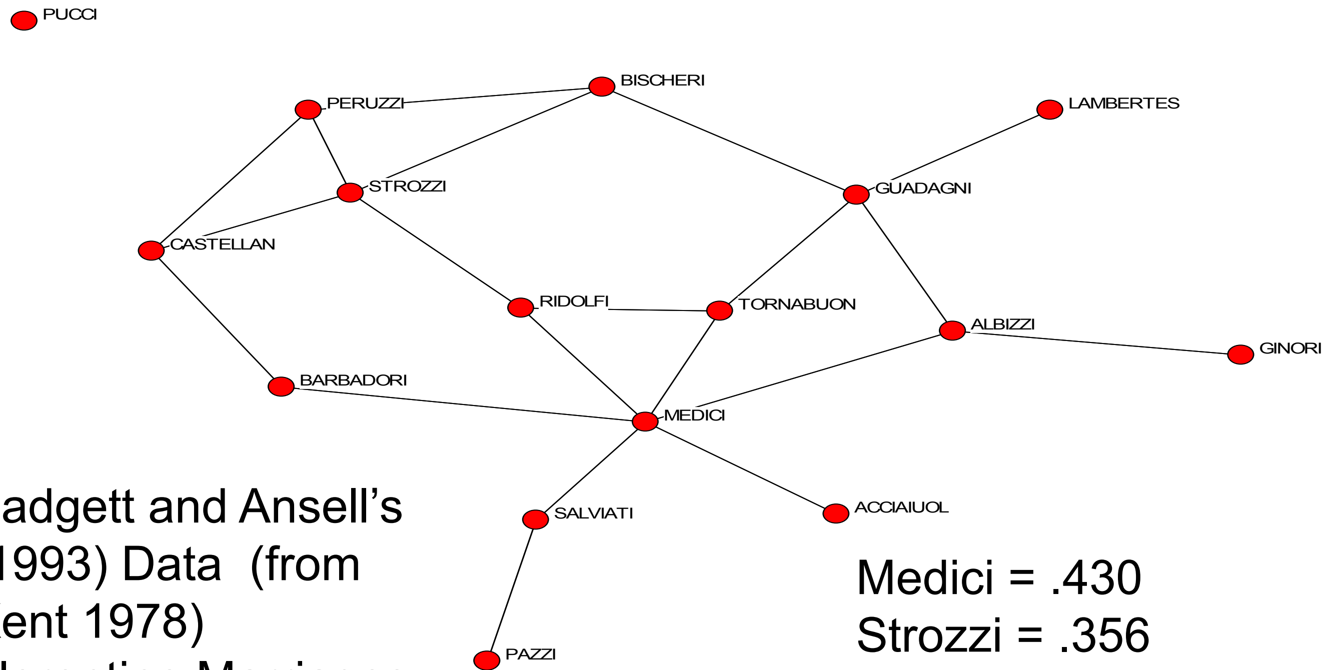
Prestige, Influence, Eigenvector-based Centrality



- $C_i^e(g) = a \sum_j g_{ij} C_j^e(g)$ $C^e(g) = a\, g\, C^e(g)$
  - $C^e(g)$ is an eigenvector - many possible solutions
  - Look for one with largest eigenvalue – will be nonnegative (Perron-Frobenius Theorem)
  - normalize entries to sum to one

Eigenvector Centrality

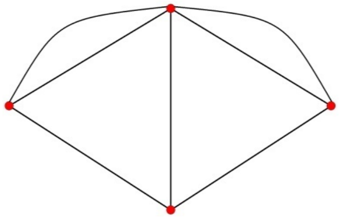




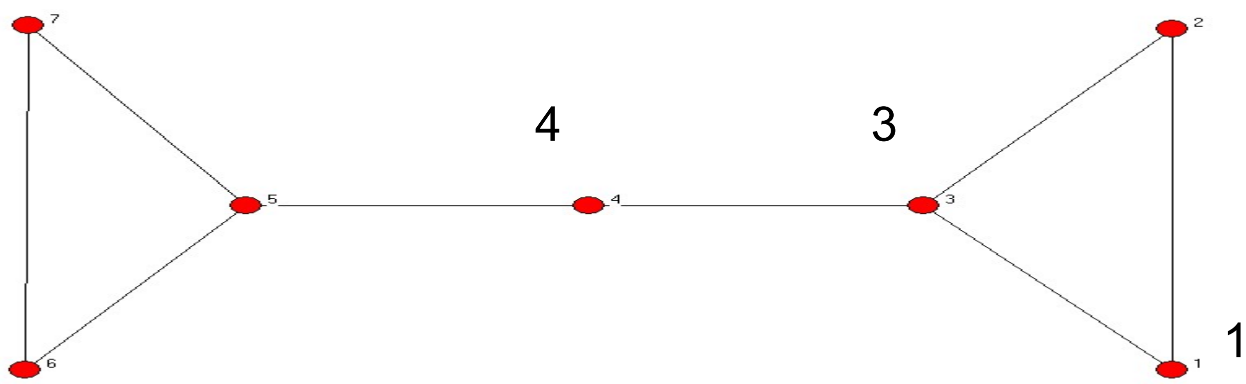
Padgett and Ansell's (1993) Data (from Kent 1978) Florentine Marriages, 1430's

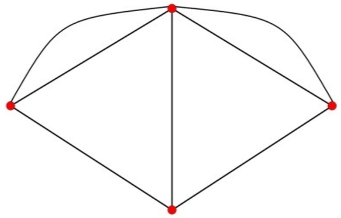
Medici = .430
Strozzi = .356
Guadagni = .289
Ridolfi=.341
Tornabuon=.326

Centrality



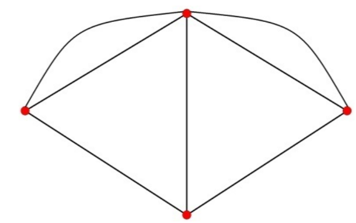
- Concepts related to eigenvector centrality:
- Google Page rank: score of a page is proportional to the sum of the scores of pages linked to it
- Random surfer model: start at some page on the web, randomly pick a link, follow it, repeat...





	Node 1	Node 3	Node 4
Degree	.33	.50	.33
Closeness	.40	.55	.60
N. Decay δ = .5	.50	.67	.67
N. Decay δ = .75	.69	.82	.84
N. Decay δ= .25	.39	.56	.50
Betweenness	.00	.53	.60
Eigenvector	.47	.63	.54

Bonacich Centrality



Builds on a measure by Katz

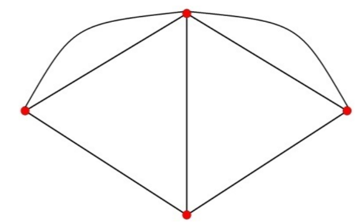
give each node a base value $ad_i(g)$ for some $a>0$

then add in all paths of length 1 from i to some j times b times j's base value

then add in all walks of length 2 from i to some j times b^2 times j's base value...

$$C^b(g) = ag\mathbf{1} + b\, g\, ag\mathbf{1} + b^2 g^2 ag\mathbf{1} \ldots$$

Bonacich Centrality



$$C^b(g) = ag\mathbf{1} + b\,g\,ag\mathbf{1} + b^2\,g^2\,ag\mathbf{1} \ldots$$

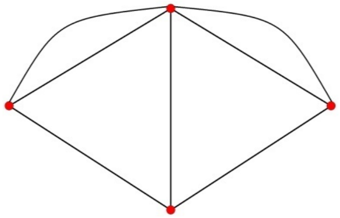
$$= a(\,g\mathbf{1} + b\,g^2\mathbf{1} + b^2\,g^3\mathbf{1} \ \ldots\,)$$

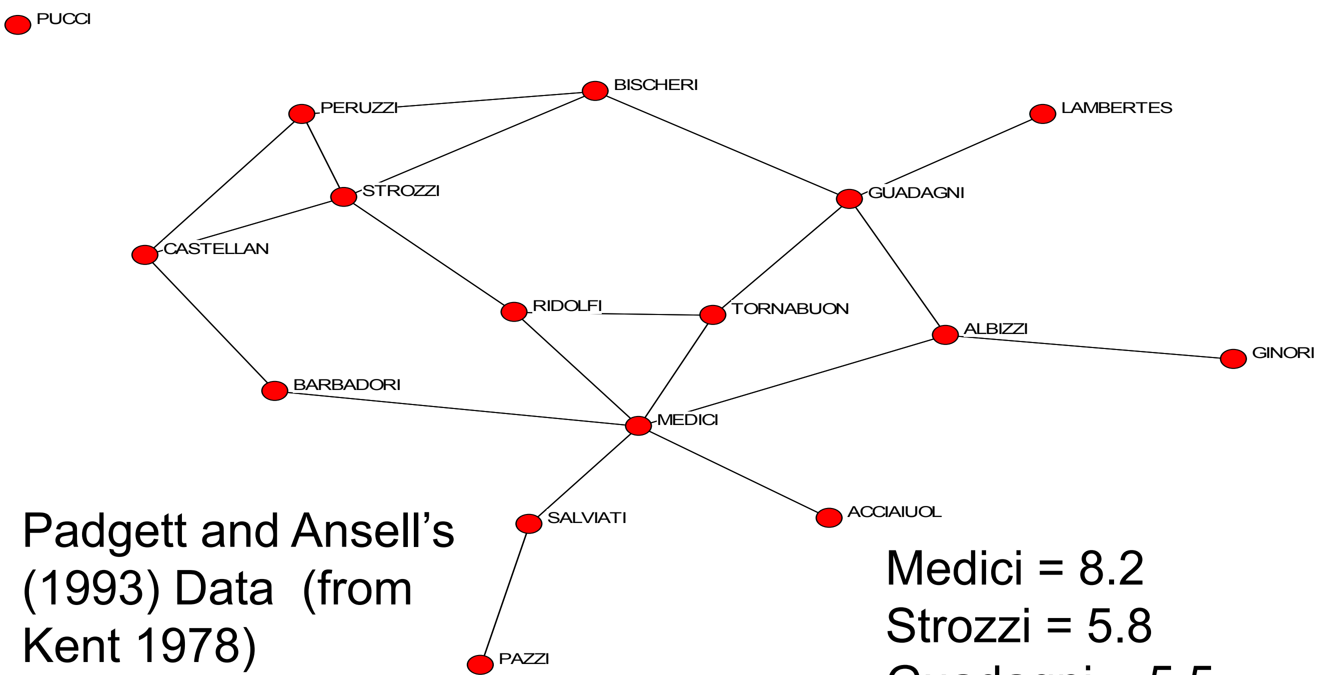
normalize a to 1, need *small b* to be finite

$$C^b(g) = g\mathbf{1} + b\,g^2\mathbf{1} + b^2\,g^3\mathbf{1} \ \ldots$$

$$= (\mathbf{I} - bg\,)^{-1}\,g\mathbf{1}$$

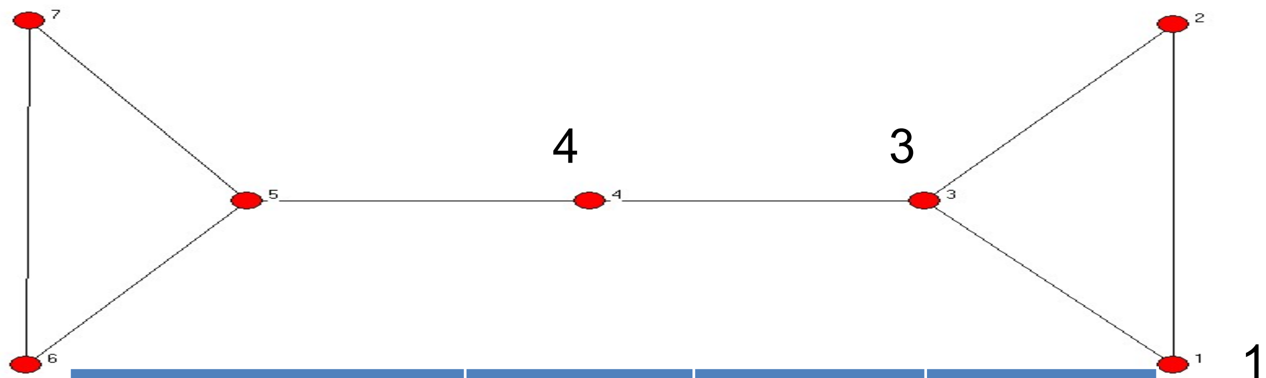
Bonacich Centrality

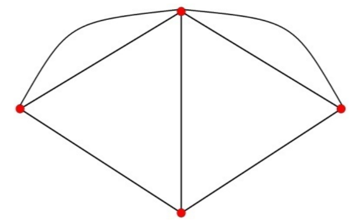




Padgett and Ansell's (1993) Data (from Kent 1978) Florentine Marriages, 1430's

Medici = 8.2
Strozzi = 5.8
Guadagni = 5.5
Ridolfi=4.9
Tornabuon=4.9





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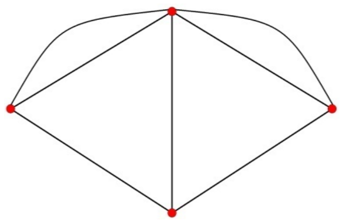
N. Decay δ = .25	.39	.56	.50
Betweenness	.00	.53	.60
Eigenvector	.47	.63	.54
Bonacich b=1/3	9.4	13	11
Bonacich b=1/4	4.9	6.8	5.4

Degree Distributions

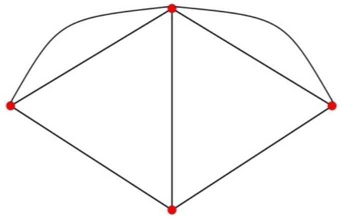
- The degree distribution, $P(d)$, of a network is a description of relative frequencies of nodes that have different degrees d.
    - For a given graph: $P(d)$ is a histogram, i.e., $P(d)$ is the fraction of nodes with degree d.
    - For a random graph model: $P(d)$ is a probability distribution.
- Two types of degree distributions:
    - $P(d) \leq c\, e^{-\alpha d}$, for some $\alpha > 0$ and $c > 0$: The tail of the distribution **falls off faster than an exponential**, i.e., large degrees are unlikely.
    - $P(d) = c\, d^{-\gamma}$, for some $\gamma > 0$ and $c > 0$: Power-law distribution: The tail of the distribution is **fat**, i.e., there tend to be many more nodes with very large degrees.
        - Appear in a wide variety of settings including networks describing incomes, city populations, WWW, and the Internet
        - Also known as a scale-free distribution: a distribution that is unchanged (within a multiplicative factor) under a rescaling of the variable
        - Appear linear on a log – log plot
- What is the degree distribution of the Erdös-Renyi model?

Games on Networks



- Players on a network - explicitly modeled...
- Care about actions of neighbors
- Early literature: How complex is the computation of equilibrium in worse case games?
- Second branch: what can we say about behavior and how it relates to network structure

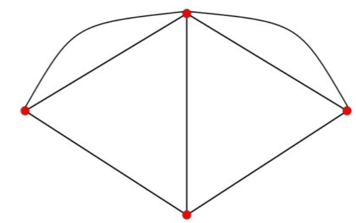
Start with a Canonical Special Case:



- Each player chooses action x_i in {0,1}

- payoff will depend on
  - how many neighbors choose each action
  - how many neighbors a player has

Definitions



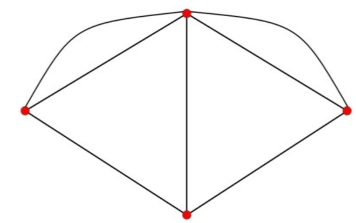
- Each player chooses action x_i in {0,1}

- Consider cases where i's payoff is

$$u_{d_i}(x_i, m_{N_i})$$

depends only on $d_i(g)$ and $m_{N_i(g)}$ - the number of neighbors of i choosing 1

Example: Simple Complement

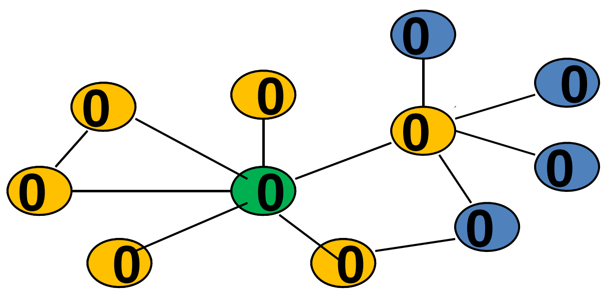


- agent i is willing to choose 1 if and only if at least t neighbors do:

- Payoff action 0: $u_{d_i}(0 ,m_{N_i}) = 0$

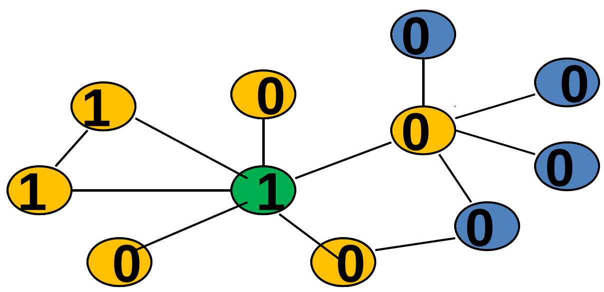
- Payoff action 1: $u_{d_i}(1 ,m_{N_i}) = -t + m_{N_i}$

Example:



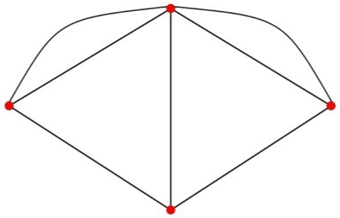
- An agent is willing to take action 1 if and only if at least two neighbors do

Example:



- An agent is willing to take action 1 if and only if at least two neighbors do

Example: Best Shot

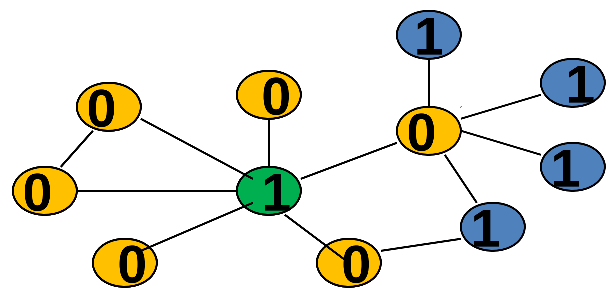


- agent i is willing to choose 1 if and only if no neighbors do:

- Payoff action 0: $u_{d_i}(0, m_{N_i}) = 1$ if $m_{N_i} > 0$
  $= 0$ if $m_{N_i} = 0$

- Payoff action 1: $u_{d_i}(1, m_{N_i}) = 1 - c$

Another Example:
Best Shot Public Goods



- An agent is willing to take action 1 if and only if no neighbors do

Complements/Substitutes

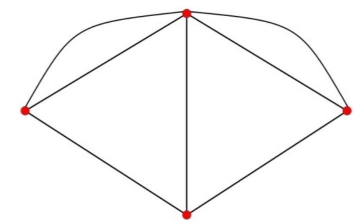
- strategic **complements** -- for all d, $m \geq m'$
  - Increasing differences:

  $u_d(1,m) - u_d(0,m) \geq u_d(1,m') - u_d(0,m')$

- strategic **substitutes** -- for all d, $m \geq m'$
  - Decreasing differences:

  $u_d(1,m) - u_d(0,m) \leq u_d(1,m') - u_d(0,m')$

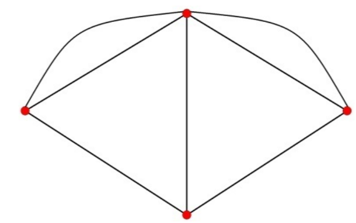
Externalities:



- Others’ behaviors affect my **utility/welfare**

- Others’ behaviors affect my ***decisions, actions, consumptions, opinions…***
  - others’ actions affect the ***relative*** payoffs to my behaviors

(Strategic) Complements/Substitutes



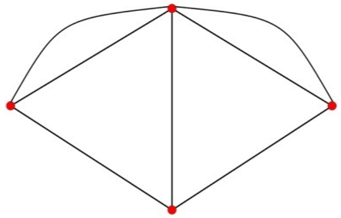
- **Complements:** Choice to take an action by my friends increases my relative payoff to taking that action (e.g., friend learns to play a video game)

- **Substitutes:** Choice to take an action by my friends decreases my relative payoff to taking that action (e.g., roommate buys a stereo/fridge)

Examples

- Complements:
  - education decisions
    - care about number of neighbors, access to jobs, etc. – invest if at least k neighbors do
  - smoking & other behavior among teens, peers, ...
  - technology adoption – how many others are compatible...
  - learn a language, ...
  - cheating, doping
- Substitutes
  - information gathering
    - e.g., payoff of 1 if anyone in neighborhood is informed, cost to being informed ($c<1$)
  - local public goods (shareable products...)
  - competing firms (oligopoly with local markets)
  - ...

Equilibrium

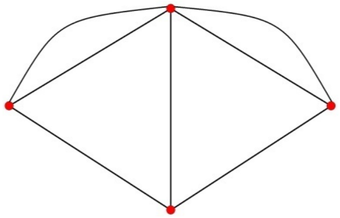


- Nash equilibrium: Every player's action is optimal for that player given the actions of others

- Often look for pure strategy equilibria

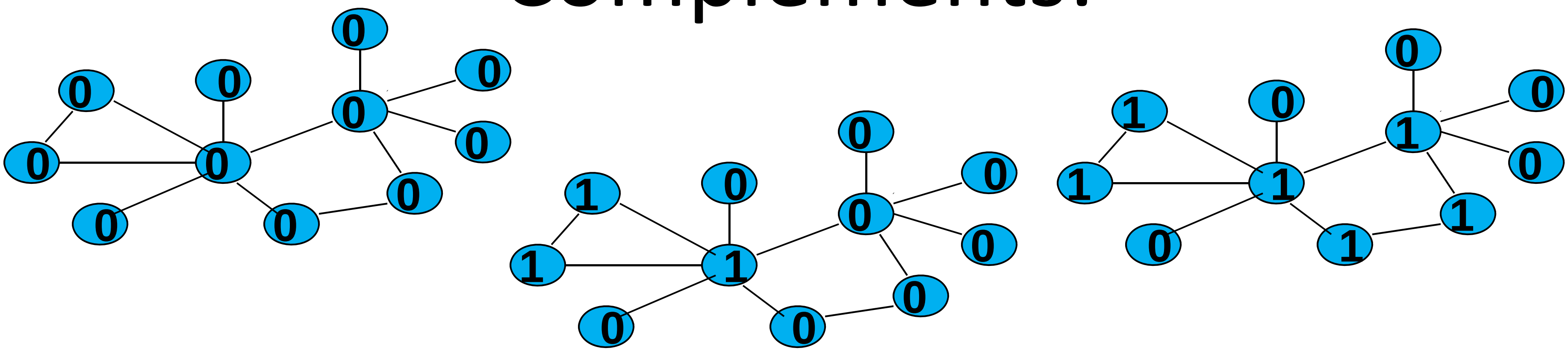
- May require some mixing

Useful Observation



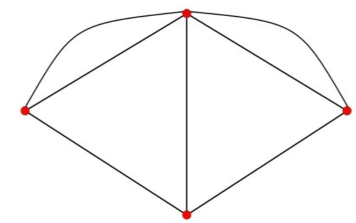
- Complements: there is a threshold t(d), such that i prefers 1 if $m_{N_i} > t(d)$ and 0 if $m_{N_i} < t(d)$
- Substitutes: there is a threshold t(d), such that i prefers 1 if $m_{N_i} < t(d)$ and 0 if $m_{N_i} > t(d)$
- Can be indifferent at the threshold

Complements:



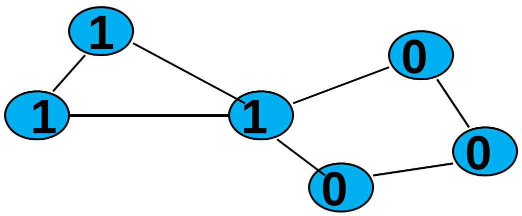
- threshold is two
- multiple equilibria
- lattice structure to set of equilibria

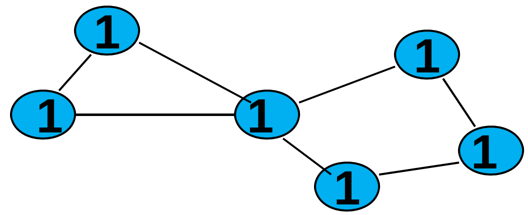
Complete lattice

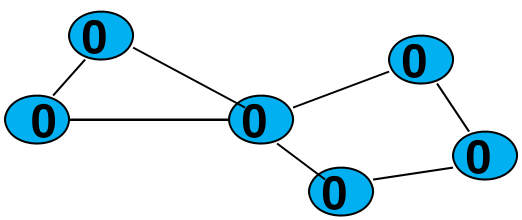


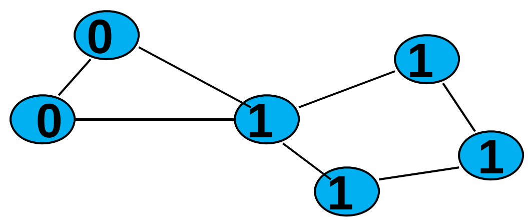
- Complete Lattice: for every set of equilibria X
  - there exists an equilibrium x’ such that x’≥x for all x in X, and
  - there exists an equilibrium x’’ such that x’’≤x for all x in X.

Lattice:

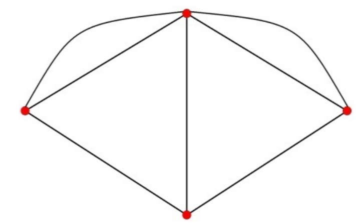








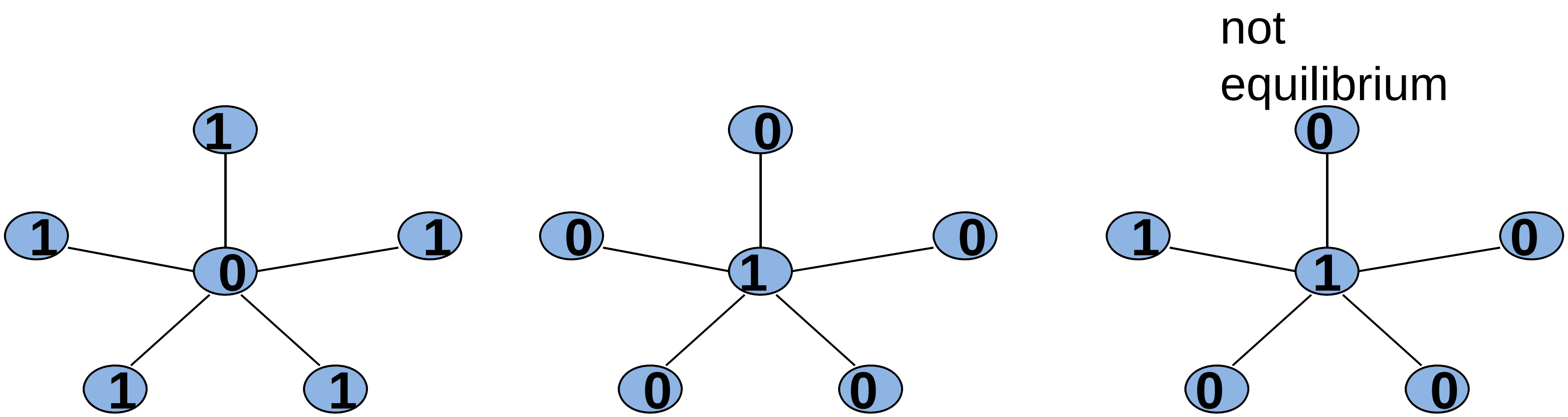
Proposition



In a game of strategic complements where the individual strategy sets are complete lattices:

the set of pure strategy equilibria are a (nonempty) complete lattice.

Best shot



- Maximal independent set: each 1 has no 1's in its neighborhood, each 0 has at least one 1
- Different distributions of utilities, and different total costs

Maximal Independent Set

- Independent Set: a set S of nodes such that no two nodes in S are linked,
- Maximal: every node in N is either in S or linked to a node in S

Basic model

- n agents in a network; each exerts $e_i \in [0, +\infty)$ effort; $mc = c$;
- An agent i's payoff from profile $\mathbf{e} = (e_1, \cdots, e_n)$ in a network $\mathbf{g}$ is

$$U_i(\mathbf{e}; \mathbf{g}) = b\left(e_i + \sum_{j \in N_i} e_j\right) - ce_i$$

  strategic substitutes
- $b(\cdot)$ strictly increasing and concave benefit. let e^* solves $b'(e) - c = 0$.
- Let $\bar{e}_i = \sum_{j \in N_i} e_j$. Then every agent i either (1) $\bar{e}_i \geq e^*$ and $e_i = 0$ or (2) $\bar{e}_i \leq e^*$ and $e_i = e_i^* - \bar{e}_i$.
- Equilibrium always exists by Brouwer's Fixed Point Theorem. multiple equilibria;

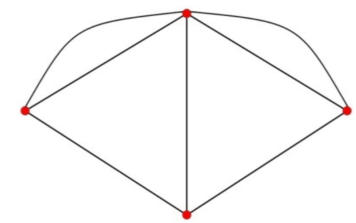
main result

- maximal independent sets. An *independent set* I of a graph $\mathbf{g}$ is a set of agents such that no two agents who belong to I are linked; i.e., $\forall i, j \in I$ such that $i \neq j, g_{ij} = 0$. An independent set is maximal when it is not a proper subset of any other independent set.
- We say a profile e is specialized when every agent either exerts the maximum amount of effort e^* or exerts no effort; for all agents i either $e_i = 0$ or $e_i = e^*$. We call an agent who exerts e^* a specialist.

Theorem

A specialized profile is a Nash equilibrium if and only if its set of specialists is a maximal independent set of the structure $\mathbf{g}$. Since for every $\mathbf{g}$ there exists a maximal independent set, there always exists a specialized Nash equilibrium.

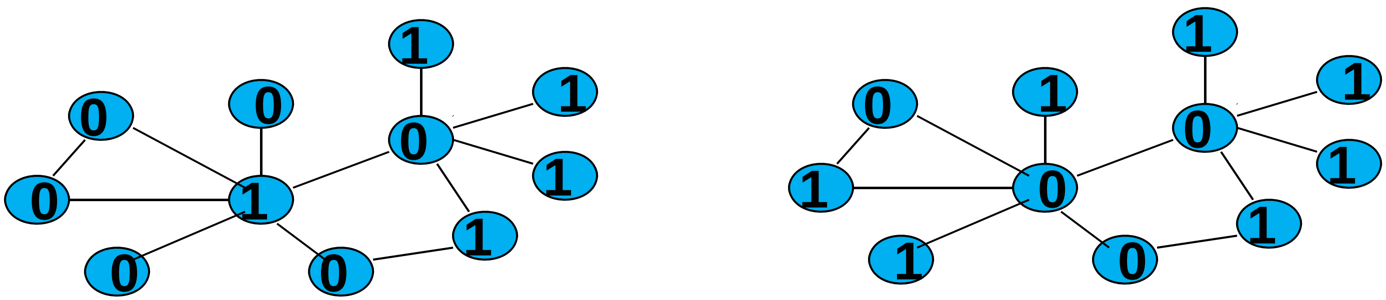
Contrast: Complements and Substitutes



- In a game of complements: pure strategy equilibria are a nonempty complete lattice

- In a game of strategic substitutes:
  - Best shot game: pure strategy equilibria exist and are related to maximal independent sets
  - Others: pure strategy may not exist, but mixed will (with finite action spaces)
  - Equilibria usually do not form a lattice

Best Shot Public Goods



- invest if and only if no neighbors do (threshold is 1)
- again, multiple equilibria
- but, no lattice structure...

Model

- A set of players $\mathcal{N}$ in a social network G.
- Each i choose x_i simultaneously. The payoff for player i,

$$\pi_i(x_1, x_2, \cdots, x_n) = \alpha_i x_i - \frac{1}{2}x_i^2 + \delta \sum_{j=1}^{N} g_{ij} x_i x_j,$$

- x_i: contribution (time, effort)
- α_i : intrinsic marginal utility
- $G = (g_{ij})$ network matrix; $g_{ii} = 0$, $g_{ij} \geq 0$
  - local network effect
  - e.g., G: adjacent matrix of undirected graph, $g_{ij} \in \{0, 1\}$;
- $\delta > 0$ sufficiently small (stability)

Equilibrium

- Best Responses:

$$x_i^N = BR_i(x_{-i}^N) = \alpha_i + \delta \sum_{j \neq i} g_{ij} x_j^N.$$

- Matrix representation $[\alpha = (\alpha_1, \cdots, \alpha_n)']$:

$$\mathbf{x}^N = \alpha + \delta G \cdot \mathbf{x}^N \Leftrightarrow \mathbf{x}^N = [\mathbf{I} - \delta G]^{-1} \alpha = \mathbf{b}(\mathbf{G}, \delta, \mathbf{a}).$$

  - Weighted Katz-Bonacich Centrality $b_i(\mathbf{G}, \delta, \mathbf{a})$
- When $\delta < 1/\rho(G)$, $\mathbf{M} := [\mathbf{I} - \delta G]^{-1}$ is well defined with

$$m_{ij} = \sum_{k=0}^{+\infty} \delta^k g_{ij}^{[k]} = ((\mathbf{I} - \delta G)^{-1})_{ij} = 1\{i = j\} + \delta g_{ij} + \delta^2 g_{ij}^{[2]} + \cdots,$$

  - counts paths from node i to j, weighted by δ^k.
- therefore,

$$\mathbf{x}^N = [\mathbf{I} - \delta G]^{-1} \alpha = \alpha + \delta G \alpha + \delta^2 G^2 \alpha + \cdots,$$

Examples: K_2

Figure: The graph for $\mathbf{K}_2$.

- $\mathbf{K}_2$, the complete graph with 2 nodes.
- the adjacency matrix is

$$\mathbf{G} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- by induction,

$$\mathbf{G}^{2k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{G}^{2k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad k = 1, 2, 3, \cdots$$

- The M matrix, well defined when $\delta < 1$, is
  $\mathbf{M} = [\mathbf{I}_n - \delta \mathbf{G}]^{-1} = \frac{1}{1-\delta^2} \begin{bmatrix} 1 & \delta \\ \delta & 1 \end{bmatrix}$
- unique Nash Equilibrium

$$\mathbf{x}^N = (x_1^N, x_2^N)' = \left(\frac{\alpha_1 + \delta \alpha_2}{1 - \delta^2}, \frac{\alpha_2 + \delta \alpha_1}{1 - \delta^2} \right)'.$$

Examples: K_n the complete graph with n nodes

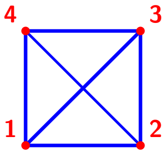


Figure: A graph for $\mathbf{K}_4$.

- The adjacency matrix of $\mathbf{K}_n$ is $\mathbf{G} = \mathbf{J}_{nn} - \mathbf{I}_n$.
- For $\mathbf{K}_n$, we can verify that

$$\mathbf{M} \quad = \quad [\mathbf{I}_n - \delta \mathbf{G}]^{-1} = \frac{1}{(1+\delta)}[\mathbf{I}_n + \frac{\delta}{1-(n-1)\delta}\mathbf{J}_{nn}],$$

well defined when $\delta < 1/(n-1)$.

- In equilibrium,

$$x_i^N = \frac{1}{1+\delta}\left(a_i + \frac{\delta \sum_k a_k}{1-(n-1)\delta}\right), i = 1, \cdots, n.$$

- Clearly, $x_i^N \geq x_j^N$ if and only if $a_i \geq a_j$.

Regular graph with degree d

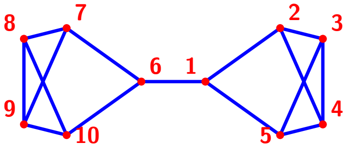


Figure: A regular graph with degree three.

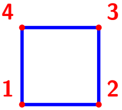


Figure: A circle of four nodes $\mathbf{O}_4$, which is also a regular graph with degree 2.

- $\mathbf{G}$ is regular with degree d, if each node has exactly d neighbors, i.e., $\mathbf{G}\mathbf{1}_n = d\mathbf{1}_n$.
- Assume $a_i = a$ for all i, then

$$x_i^N = \frac{a}{1 - d\delta}, \forall i.$$

Examples: $K_{1,2}$

- star network. center: 1; spokes: 2 and 3;
- the adjacency matrix is

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- by induction,

$$\mathbf{G}^{2k} = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 2^{k-1} & 2^{k-1} \\ 0 & 2^{k-1} & 2^{k-1} \end{bmatrix}, \mathbf{G}^{2k+1} = \begin{bmatrix} 0 & 2^k & 2^k \\ 2^k & 0 & 0 \\ 2^k & 0 & 0 \end{bmatrix}, \quad k \geq 1$$

- M matrix, well-defined when $\delta < 1/\sqrt{2}$, is

$$\mathbf{M} = [\mathbf{I}_n - \delta \mathbf{G}]^{-1} = \frac{1}{1-2\delta^2} \begin{bmatrix} 1 & \delta & \delta \\ \delta & 1-\delta^2 & \delta^2 \\ \delta & \delta^2 & 1-\delta^2 \end{bmatrix}$$

K_{pq}

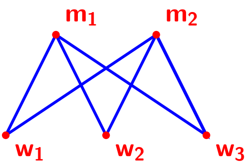


Figure: A graph for $\mathbf{K}_{2,3}$.

- In a complete bipartite graph $\mathbf{K}_{pq}$, there are two disjoint groups P and Q in $\mathbf{K}_{pq}$ such that any node in P is connected to any node in Q. Let $p = |P|, q = |Q|$. Thus, the network size satisfies $n = p + q$.
- The adjacency matrix is $\mathbf{G} = \begin{bmatrix} 0 & \mathbf{J}_{pq} \\ \mathbf{J}_{qp} & 0 \end{bmatrix}$.

$$\mathbf{M} = [\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \begin{bmatrix} [\mathbf{I}_p + \frac{\delta^2 q}{1-\delta^2 qp}\mathbf{J}_{pp}] & \frac{\delta}{1-\delta^2 pq}\mathbf{J}_{pq} \\ \frac{\delta}{1-\delta^2 pq}\mathbf{J}_{qp} & [\mathbf{I}_q + \frac{\delta^2 p}{1-\delta^2 qp}\mathbf{J}_{qq}] \end{bmatrix}$$

- for any $i \in P$, $x_i^N = [a_i + \frac{\delta^2 q}{1-\delta^2 qp}\sum_{s\in P} a_s] + \frac{\delta}{1-\delta^2 pq}\sum_{t\in Q} a_t$.

Discussions

- Linear-quadratic payoff structure
- explicit equilibrium characterization; related to sociology literature
- applicable for various scenarios:
  - monopoly pricing;
  - crime;
  - team production;
  - education and peer effects;

Key player problem

- Question: Within a crime organization the police/government has the ability to remove one player, who should it be?
- Mathematically,the key player program is formulated as follows:

$$\max_i \left\{ \sum_{k=1}^{n} b_i(\mathbf{G}, \delta, \mathbf{a}) - \sum_{k \neq i} b_k(\mathbf{G}_{-i}, \delta, \mathbf{a}_{-i}) \right\}.$$

Here $\mathbf{G}_{-i}$ is the resulting network when player i is removed. The first term $\sum_{k=1}^{n} b_i(\mathbf{G}, \delta, \mathbf{a})$ is the sum of total activities in the original network $\mathbf{G}$, while the second term $\sum_{k \neq i} b_k(\mathbf{G}_{-i}, \delta, \mathbf{a}_{-i})$ is the resulting equilibrium total activity when i is removed.

Lemma

The following identity holds:

$$\left\{ \sum_{k=1}^{n} b_k(\mathbf{G}, \delta, \mathbf{a}) - \sum_{k \neq i} b_k(\mathbf{G}_{-i}, \delta, \mathbf{a}_{-i}) \right\} = \frac{b_i(\mathbf{G}, \delta, \mathbf{1}_n) b_i(\mathbf{G}, \delta, \mathbf{a})}{m_{ii}(\mathbf{G}, \delta)}.$$

- Example

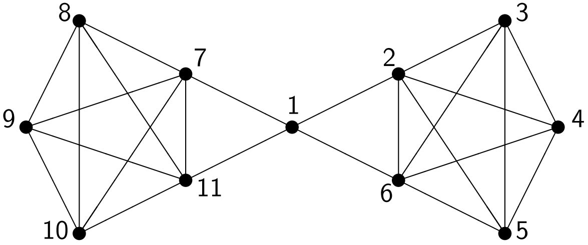


Figure: An example to illustrate the key player policy.

- key player for different δ:

δ	0.18		0.2	
Player Type	b_i	c_i	b_i	c_i
1	4.77	17.03	8.33	41.67*
2	5.23*	17.62*	9.17*	40.33
3	4.51	14.07	7.78	32.67

Pricing stage(Candogan et al.(OR '12), Bloch and Querou(GEB '13))

- Monopoly seller sets up price vector $p = (p_i)_{i \in N}$ (full discrimination)
- Player i's net utility:

$$u_i(x_1, x_2, \cdots, x_n) = \alpha_i x_i - \frac{1}{2}x_i^2 + \delta \sum_{j \neq i} g_{ij} x_i x_j - p_i x_i.$$

- two-stage game, using Backward induction:
  - Players' optimal consumption decisions: $x = \mathbf{M}(\alpha - p)$.
  - Seller's problem:

$$\max_{\mathbf{p} \in \mathbf{R}^n} \overbrace{(\mathbf{p} - \mathbf{c})' \underbrace{\mathbf{M}(\mathbf{a} - \mathbf{p})}_{=\mathbf{x}^*}}^{=\Pi(\mathbf{p})}, \quad \Longrightarrow \mathbf{p}^* = \frac{\mathbf{a} + \mathbf{c}}{2}, \quad \textbf{independent of } G, \delta$$

- Seller's equilibrium profit:

$$\Pi^* = \langle \frac{\mathbf{a} - \mathbf{c}}{2}, \mathbf{M}\frac{\mathbf{a} - \mathbf{c}}{2} \rangle, \text{ which is } \uparrow G, \delta$$