Social and Economic Networks

Learning

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Outline

1. Bayesian learning
2. DeGroot model
    - Convergence
    - Consensus in beliefs
    - Social influence

Learning

Social networks play a central role in the sharing of information and the formation of opinions.

- Providing information about scientific research and results.
- Advising friends on which movies to see.
- Relaying information about the abilities and profit of a potential new employee in a firm.
- Debating the relative merits of politicians.

Learning (Cont.)

Given the role of social networks in the formation of opinions and beliefs, and the subsequent shaping of behaviors, it is critical that we have a thorough understanding of this how the structure of social networks affects learning.

- Whether individuals in a society come to hold a common belief or remain divided in opinions.
- Which individuals have the most influence over the beliefs in a society.
- How quickly individuals learn.
- Whether initially diverse information scattered throughout the society can be aggregated in an accurate manner.

Two learning models

- Bayesian learning
  - Individuals observe actions and results experienced by their neighbors and the information in a sophisticated manner.
  - It provides conditions under which individuals come to act similarly over time.
- DeGroot model
  - Individuals exchange information with their neighbors over time and then update by taking some weighted average of what they hear.
  - Tractable, and allows us to incorporate rich network structures.

Section 1

Bayesian learning

Bayesian learning

- Individuals observe actions and results experienced by their neighbors and the information in a sophisticated manner.
- Conclusion: If agents can observe each other's actions and outcomes over time, and all agents have the same preferences and face the same form of uncertainty, then they end up with similar payoffs over time.
- Idea: an agent who is doing significantly worse than a neighbor must come to realize this over time, and will eventually change actions and come to do as well as the neighbor.
- This then implies that all connected agents must end up with the same limiting payoffs.

Bala-Goyal model

- n players in an undirected connected network g.
- Choose action A or B in each period $t \in \{1, 2, \ldots\}$.
- In each period agent gets a payoff based on choice:
  - action A results in a payoff of 1.
  - action B results in a payoff of 2 with probability p and 0 with probability $1 - p$.
- p is unknown taking on finite set of values.

Bala-Goyal model (Cont.)

- Players also observe neighbors' choices.
- Each player maximizes discounted stream of payoffs

$$\mathbf{E}\left[\sum_{t=1}^{\infty} \delta^t \cdot \pi_{it}\right],$$

where $\delta \in (0, 1)$ is a discount parameter and π_{it} is the payoff that i receives at time t.

Bala-Goyal model: Challenges

- Seeing that a neighbor chooses an action B might indicate that the individual's neighbors have had good outcomes from B in the past.
- Beyond simply seeing actions and outcomes, an individual can make inferences about outcomes of indirect neighbors by observing the action choices of neighbors.

Bala-Goyal model: Result

Proposition:
If p is not exactly $\frac{1}{2}$, then with probability 1 there is a time such that all agents lay just one action (and all play the same action) from that time onward.

Proof

- Suppose contrary.
- Some agent plays B infinitely often.
- That agent will converge to true belief p by the law of large numbers.
- In order for agent to play B infinitely often, it must be that $p > \frac{1}{2}$, otherwise agent would stop playing B.

Proof (Cont.)

- With probability 1, all agents who see B played infinitely often converge to a belief that B pays 2 with probability $p > \frac{1}{2}$.
- Neighbors of agent must play B, after some time, and so forth.
- All agents must play B from some time on.

Play the right action?

- The fact that all agents end up choosing the same action does not imply that they end up with the same limiting beliefs, nor does it imply that they end up choosing the "right" action.
- If B is the right action then play the right action if converge to it, but might not.

* Each player starts with a low belief.

- If A is the right action, then must converge to right action.

Conclusions

- Consensus action chosen.
- Not necessarily consensus belief.
- Speed of convergence?

Limitations

- Homogeneity of actions and payoffs across players.
- What if heterogeneity?
- Repeated actions over time.
- Stationarity.
- Networks are not playing role here.

Section 2

DeGroot model

DeGroot model

- Repeated communication.
- Information comes only once.
- See how information disseminates.
- Who has influence, convergence speed, network structure impact …

Bounded rationality

- Repeatedly average beliefs of self with neighbors.
- Non-Bayesian if weights do not adjust over time.
- Can under-weight neighbors (just as in experiments).

DeGroot model (Cont.)

- Individuals $\{1, 2, \ldots, n\}$.
- Individuals in a society start with initial opinions on a subject.
- Let these be represented by an n-dimensional vector of probabilities, $p(0) = \big(p_1(0), p_2(0), \cdots, p_n(0)\big)$.
- Each $p_i(0)$ lies in $[0, 1]$, and might be thought of as the probability that a given statement is true, or the quality of a given product, or the likelihood that the individual might engage in a given activity, etc.

DeGroot model: Updating

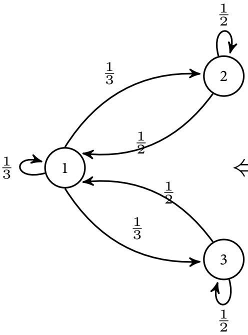
- The interaction patterns are captured through a possibly weighted and directed $n \times n$ nonnegative matrix T (social influence matrix).
- The interpretation of T_{ij} is that it represents the weight or trust that agent i places on the current belief of agent j in forming his or her belief for the next period.
- T: a (row) stochastic matrix, so that its entries across each row sum to one.
- Updating

$$p_i(t) = \sum_j T_{ij} \cdot p_j(t-1).$$

DeGroot model: Updating (Cont.)

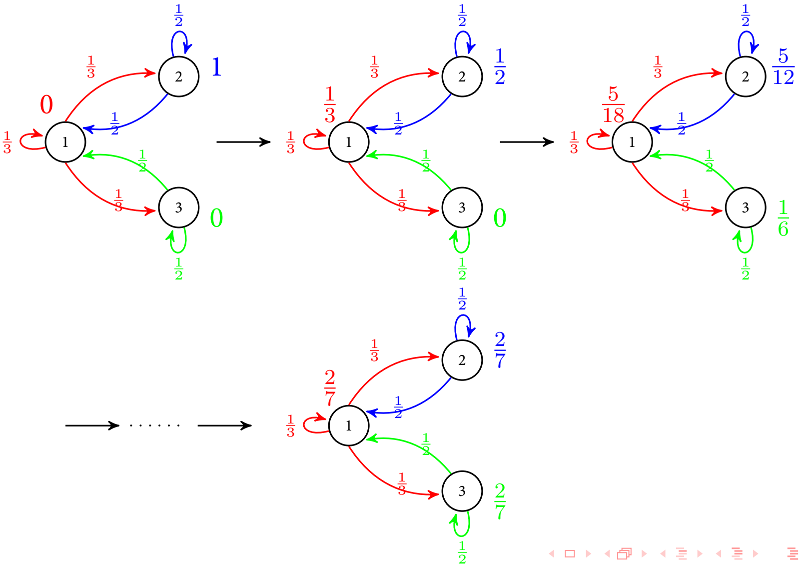
- Updating
$$p_i(t) = \sum_j T_{ij} \cdot p_j(t-1).$$
- $$p_i(t) = \sum_j T_{ij} \cdot p_j(t-1) = \big(T \cdot p(t-1)\big)_i.$$
- $$p(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_n(t) \end{pmatrix} = \begin{pmatrix} \big(T \cdot p(t-1)\big)_1 \\ \big(T \cdot p(t-1)\big)_2 \\ \vdots \\ \big(T \cdot p(t-1)\big)_n \end{pmatrix} = T \cdot p(t-1) = T^t \cdot p(0).$$

DeGroot model: Illustration 1

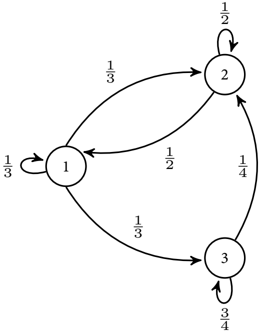


$$\iff T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

DeGroot model: Illustration 1 (Cont.)

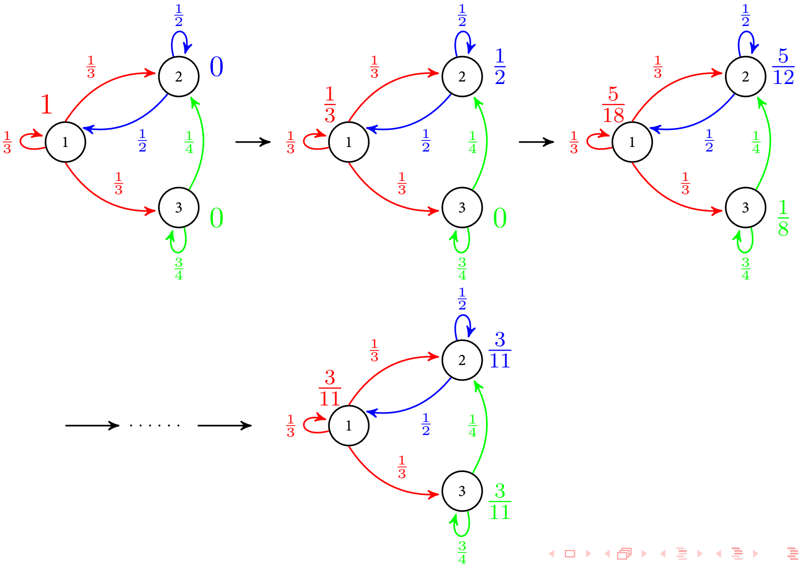


DeGroot model: Illustration 2

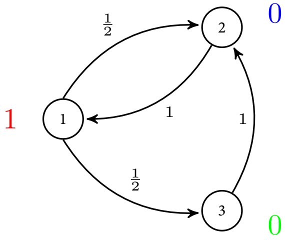


$$\Longleftrightarrow T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

DeGroot model: Illustration 2 (Cont.)



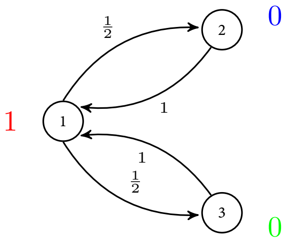
DeGroot model: Illustration 3



$$\Longleftrightarrow T = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 0 \end{pmatrix} \to \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{1}{2} \end{pmatrix} \to \cdots \to \begin{pmatrix} \frac{2}{5} \\ \frac{2}{5} \\ \frac{2}{5} \end{pmatrix}$$

DeGroot model: Illustration 4



$$\Longleftrightarrow T = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

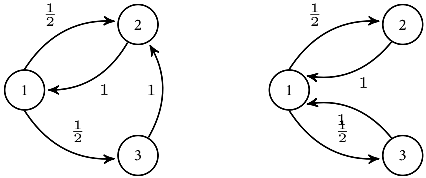
$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to p(1) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \cdots$$

Subsection 1

Convergence

Convergence

- T converges if $\lim_{t \to \infty} T^t \cdot p(0)$ exists for all $p(0)$.
- T is aperiodic (非周期) if the greatest common divisor of its cycle lengths is one.
- Left: aperiodic; Right: periodic.



Convergence result

- Suppose T is strongly connected (so that there is a directed path from any node to any other node, also referred to as being irreducible (不可约)).
- Result: T is convergent if and only if it is aperiodic.
- Result: T is convergent if and only if $\lim_{t\to\infty} T^t = (1, 1, \cdots, 1)^T \cdot s$, where s is the unique left eigenvector of T associated with the eigenvalue 1.

Proof: Sufficiency

- Definition: T is primitive (素矩阵) if $T^t_{ij} > 0$ for all i and j after some t.
- Perkins (1961): If T is strongly connected and (row) stochastic, then it is aperiodic if and only if it is primitive.
- Meyer (2000): If T is strongly connected and primitive, then $\lim_{t\to\infty} T^t = (1, 1, \cdots, 1)^T \cdot s$, where s is the unique left eigenvector of T associated with the eigenvalue 1.
- So strongly connected and aperiodic implies convergence.

Proof: Necessity

- Claim: If T is strongly connected, stochastic and convergent, then it is primitive.
- Let $S = \lim_{t\to\infty} T^t$.
- Then $ST = \lim_{t\to\infty} T^t T = S$.
- So each row is a left eigenvector of T with eigenvalue 1.
- It is a positive vector by Perron-Frobenius theorem.
- Since S is all positive, T is primitive.
- T is primitive then Perron-Frobenius theorem implies the eigenvector is unique, and all rows of S are the same s.

Convergence

- Aperiodicity is easy to satisfy.
- Have some agent weight him or herself.
  - * If T is strongly connected and $T_{ii} > 0$ for some i, then T is aperiodic, and hence T is convergent.
- Have at least one communicating dyad and a transitive triple.
  - * T is aperiodic if the greatest common divisor of its cycle lengths is one.

Subsection 2

Consensus in beliefs

Consensus

- Beyond knowing whether or not beliefs converge, we are also interested in characterizing:
  - what beliefs converge to when they converge,
  - which agents have substantial influence in the society,
  - when it is that a consensus is reached.
- Agents reaches a consensus (共识) under T for an initial vector of beliefs $p(0)$ if $\lim_{t\to\infty} p_i(t) = \lim_{t\to\infty} p_j(t)$ for each i and j.

Consensus/convergence and aperiodicity

- Theorem: Agents reaches a consensus for every initial vector of beliefs under T if and only if T is aperiodic.
- Necessity: Consensus $\Rightarrow$ convergence $\Rightarrow$ aperiodicity.
- Sufficiency:

$$p(\infty) = \lim_{t\to\infty} T^t p(0) = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} s_1 & s_2 & \cdots & s_n \end{pmatrix} \begin{pmatrix} p_1(0) \\ p_2(0) \\ \vdots \\ p_n(0) \end{pmatrix}$$

$$= \begin{pmatrix} s_1 p_1(0) + \cdots + s_n p_n(0) \\ s_1 p_1(0) + \cdots + s_n p_n(0) \\ \vdots \\ s_1 p_1(0) + \cdots + s_n p_n(0) \end{pmatrix} = \begin{pmatrix} s \cdot p(0) \\ s \cdot p(0) \\ \vdots \\ s \cdot p(0) \end{pmatrix}.$$

Consensus in beliefs

- The agents reach a consensus whenever T converges.
- The limit belief $p_i(\infty)$ is $s \cdot p(0)$, where s is the left eigenvector of T associated with eigenvalue 1.
- The belief converges to (normalized) eigenvector weighted (s) sum of original beliefs $p(0)$.

Subsection 3

Social influence

Limiting beliefs

- Limiting beliefs would be weighted averages of the initial beliefs.
- The relative weights would be the influences that the various agents have on the final consensus beliefs.
- $p_i(\infty) = s \cdot p(0)$.
- $s_i = \sum_j s_j T_{ji}$.

$\Rightarrow$ High influence from being paid attention to by people with high influence.

- Related to eigenvector centrality.

Stubborn agents

- An agent who places high weight on self will maintain belief while others converge to that agent's belief.
- Groups that are highly introspective will have substantial influence.

Equal weights

- Suppose equally weight connections.
- Suppose also that $T_{ij} > 0$ if and only if $T_{ji} > 0$.
- d_i is i's degree.
- So, $T_{ij} = \frac{1}{d_i}$ for each i and j that i has a (directed) link to.

$\Rightarrow$ Weight friends equally.

Equal weights (Cont.)

- Let $D = \sum_k d_k$.
- Claim: $s_i = \frac{d_i}{D}$ for each i.
- Verify: $s_i = \sum_j s_j T_{ji} = \sum_{j:\ ji \in g} \frac{1}{d_j} \frac{d_j}{D} = \frac{d_i}{D}$ (degree centrality).