Advanced Microeconomics I: Lecture 3

1 In lecture 2, we consider the differences between utility functions (risk aversion and the measurement of risk aversions). In this lecture, we would like to compare distributions/lotteries themselves.

2 There are two ways to compare distributions/lotteries:

- according to the level of returns: distribution F yields unambiguously higher returns than G;
- according to the dispersion of returns: F is unambiguously less risky than G.

3 For simplicity, we restrict our attention to the distributions F such that $F(0) = 0$ and $F(\bar{x}) = 1$ for some $\bar{x}$.

1 First-order stochastic dominance

4 Definition: A lottery F first-order stochastically dominates G if the decision maker prefers F to G regardless of what u is, as long as it is increasing.

That is, for every increasing function $u\colon \mathbb{R}_+ \to \mathbb{R}$ we have

$$\int u(x)\,\mathrm{d}F(x) \geq \int u(x)\,\mathrm{d}G(x).$$

Interpretation: Every individual with increasing utility function prefers F to G regardless of his risk preferences.

5 Theorem: F first-order stochastically dominates G iff $F(x) \leq G(x)$ for all x.

Interpretation: For every amount of money x, the probability of getting at least x is higher under F than under G, i.e., $1 - F(x) \geq 1 - G(x)$. F gives more wealth than G realization by realization.

6 *Proof.* $\Rightarrow$:

(1) Suppose that F first-order stochastically dominates G.

(2) Assume that $F(x^*) > G(x^*)$ for some x^*. We want to find some contradiction.

(3) Let $u(x) = \begin{cases} 1, & \text{if } x > x^*; \\ 0, & \text{if } x \leq x^*. \end{cases}$

(4) $\int u(x)\,\mathrm{d}F(x) = 1 \cdot \mathrm{Prob}_F(x > x^*) = 1 - F(x^*) < 1 - G(x^*) = 1 \cdot \mathrm{Prob}_G(x > x^*) = \int u(x)\,\mathrm{d}G(x)$. Contradiction.

$\Leftarrow$:

(1) Suppose that $F(x) \leq G(x)$ for all x.

(2) For simplicity, we assume that u is differentiable.

(3) Let $H(x) = F(x) - G(x) \leq 0$. Clearly, $H(0) = 0$ and $H(\bar{x}) = 0$ for some sufficiently large number $\bar{x}$.

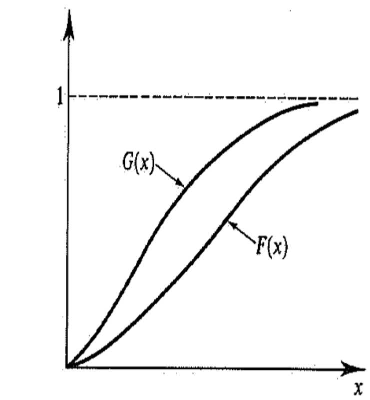
(4) Then we have

$$\int u(x)\,\mathrm{d}H(x) = \underbrace{u(x)H(x)\Big|_0^\infty}_{0} - \int u'(x)H(x)\,\mathrm{d}x.$$

(5) Since u is increasing, $\int u(x)\,\mathrm{d}F(x) - \int u(x)\,\mathrm{d}G(x) = \int u(x)\,\mathrm{d}H(x) = -\int u'(x)H(x)\,\mathrm{d}x \geq 0$.

□

7 Graphic illustration:



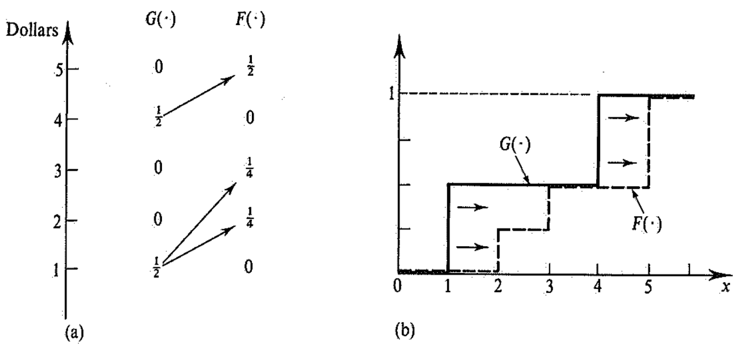
8 Example: Consider a lottery:

- In the first stage, we have a lottery over x distributed according to G;
- In the second stage, we have a "upward probabilistic shift" of x: the shift z is distributed according to H_x with $H_x(0) = 0$.
- The final amount of money is $x + z$. The resulting distribution is denoted by F.

For any increasing function $u\colon \mathbb{R}_+ \to \mathbb{R}$, we have

$$\int u(x)\,\mathrm{d}F(x) = \int \left[\int u(x+z)\,\mathrm{d}H_x(z)\right]\mathrm{d}G(x) \geq \int \underbrace{\left[\int u(x)\,\mathrm{d}H_x(z)\right]}_{\int \mathrm{d}H_x(z)=1}\mathrm{d}G(x) = \int u(x)\,\mathrm{d}G(x).$$

So F first-order stochastically dominates G.



9 Whenever F first-order stochastically dominates G, it is possible to generate F from G in the above manner.

2 Second-order stochastic dominance

10 First-order stochastic dominance involves the idea of "higher/better" vs. "lower/worse." The following comparison bases on relative riskiness or dispersion.

11 To avoid confusing the issue with the trade-off between returns and risk, we restrict ourselves to comparing distributions with the same mean.

12 Definition: A lottery F second-order stochastically dominates G (with the same mean) if the decision maker prefers F to G as long as he is risk averse and u is weakly increasing.[1]

That is, for any two distributions F and G with the same mean, F second-order stochastically dominates G if for every increasing concave function $u\colon \mathbb{R}_+ \to \mathbb{R}$, we have

$$\int u(x)\,\mathrm{d}F(x) \geq \int u(x)\,\mathrm{d}G(x).$$

13 Theorem: Consider two distributions F and G with the same mean. Then the following statements are equivalent.

(i) F second-order stochastically dominates G.

(ii) $\int_0^x G(t)\,\mathrm{d}t \geq \int_0^x F(t)\,\mathrm{d}t$ for all x.

Proof. (1) Let $I(x) = \int_0^x [F(t) - G(t)]\,\mathrm{d}t$. Then $I(0) = 0$ and $I(\bar{x}) = 0$ for some sufficiently large $\bar{x}$.

(2) We have

$$\begin{aligned}\int u(x)\,\mathrm{d}[F(x) - G(x)] &= \underbrace{u(x)[F(x) - G(x)]\Big|_0^\infty}_{0} - \int u'(x)[F(x) - G(x)]\,\mathrm{d}x \\ &= -\int u'(x)\,\mathrm{d}I(x) \\ &= \underbrace{-u'(x)I(x)\Big|_0^\infty}_{0} + \int u''(x)I(x)\,\mathrm{d}x\end{aligned}$$

(3) F second-order stochastically dominates G iff LHS is nonnegative for all u with $u'' \leq 0$ iff $I(x) \leq 0$.

□

14 Example: Consider a lottery:

- In the first stage, we have a lottery over x distributed according to F;
- In the second stage, we randomize each possible outcome x further so that the final payoff is $x + z$, where z has a distribution H_x with a mean of zero.
- The final amount of money is $x + z$. The resulting distribution is denoted by G.

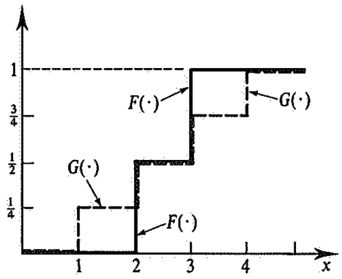
When lottery G can be obtained from lottery F in this manner for some distributions H_x, we say that G is a mean-preserving spread of F.

For any increasing concave function u, we have

$$\int u(x)\,\mathrm{d}G(x) = \int \left[\int u(x+z)\,\mathrm{d}H_x(z)\right]\mathrm{d}F(x) \leq \int u\left[\int (x+z)\,\mathrm{d}H_x(z)\right]\mathrm{d}F(x) = \int u(x)\,\mathrm{d}F(x).$$

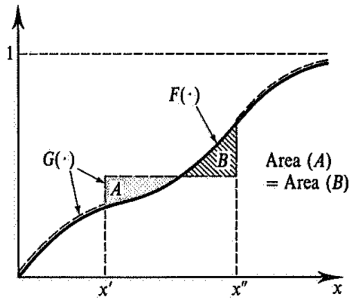
[1] One can generalize this definition without assuming the same mean.

Thus, F second-order stochastically dominates G.



Whenever F second-order stochastically dominates G, then G is a mean-preserving spread of F.

15 Example: An elementary increase in risk. We say that G constitutes an elementary increase in risk from F if G is generated from F by taking all the mass that F assigns to an interval $[x', x'']$ and transferring it to the endpoints x' and x'' in such a manner that the mean is preserved.



16 Suppose that F and G has the same mean and $F(\bar{x}) = G(\bar{x}) = 1$ for some $\bar{x}$. Then we have

$$\int_0^{\bar{x}} [F(x) - G(x)]\, \mathrm{d}x = \underbrace{[F(\bar{x}) - G(\bar{x})]}_{0}\, \bar{x} - \int_0^{\bar{x}} x\, \mathrm{d}\,[F(x) - G(x)] = 0.$$

That is, the areas below the two distributions are the same over the interval $[0, \bar{x}]$.

17 Theorem: Consider two distributions F and G with the same mean. Then the following statements are equivalent.

(i) F second-order stochastically dominates G.

(ii) G is a mean-preserving spread of F.

18 Proposition: If F second-order stochastically dominates G, then $\min_{x \in \text{support } F}(x) \geq \min_{x \in \text{support } G}(x)$. It implies that the left tail of G must be thicker than the left tail of F.

19 Proposition: For the general definition of SOSD: FOSD implies SOSD but not necessarily the reverse. In SOSD F and G can cross.

3 Homework

- Reading: 6.D

- Homework: 18 (assume that there are finite outcomes and support F is the subset of outcomes each of which has positive probability under F), 19

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