ADVANCED MICROECONOMICS I: LECTURE 1

1 Review: Preference

1 Preference relation $\succsim$ is a binary relation on the set of alternatives X.

- Strict preference relation $\succ$: $x \succ y$ iff $x \succsim y$ and not $y \succsim x$.
- Indifference relation $\sim$: $x \sim y$ iff $x \succsim y$ and $y \succsim x$.

2 The preference relation $\succsim$ is rational if it possesses the following two properties:

- Completeness: for all $x, y \in X$, we have that $x \succsim y$ or $y \succsim x$ (or both).
- Transitivity: for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Reflexivity: for all $x \in X$, $x \succsim x$. It is implied by completeness.

3 A utility function $u(x)$ assigns a numerical value to each element in X.

4 A utility function $u\colon X \to \mathbb{R}$ represents a preference relation $\succsim$ if for all $x, y \in X$, $x \succsim y$ iff $u(x) \geq u(y)$.

5 Proposition: If a preference relation $\succsim$ is represented by a utility function u, then it is rational.

Proof. It suffices to show that $\succsim$ is complete and transitive.

- Since u is a real-valued function, for all $x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. Then $x \succsim y$ or $y \succsim x$. Thus, $\succsim$ is complete.
- Suppose that $x \succsim y$ and $y \succsim z$. Then we have that $u(x) \geq u(y)$ and $u(y) \geq u(z)$, and hence $u(x) \geq u(z)$. Thus, $x \succsim z$.

□

6 A rational preference relation need not be representable by a utility function.

Example. The lexicographic preference relation: Assume that $X = \mathbb{R}^2_+$. Define $x = (x_1, x_2) \succsim y = (y_1, y_2)$ if either "$x_1 > y_1$" or "$x_1 = y_1$ and $x_2 \geq y_2$".

It is easy to see that it is complete and transitive.

Suppose there is a utility function u that represents the lexicographic preference relation.

(1) For every $x_1 \in \mathbb{R}_+$, we have that $u(x_1, 2) > u(x_1, 1)$.

(2) We can a rational number $r(x_1) \in (u(x_1, 1), u(x_1, 2))$.

(3) If $x_1 > x'_1$, then $r(x_1) > u(x_1, 1) > u(x'_1, 2) > r(x'_1)$.

(4) r is a one-to-one function from $\mathbb{R}_+$ to $\mathbb{Q}$. It is impossible.

$\square$

7 If X is finite, then each rational preference relation can be represented by a utility function.

8 The preference relation $\succsim$ on X is continuous if whenever $x \succ y$, there are neighborhoods B_x of x and B_y of y such that for all $x' \in B_x$ and $y' \in B_y$, $x' \succ y'$.

Equivalent definitions:

- if for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for all n, $x = \lim_{n\to\infty} x^n$, and $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$.
- if for all x, the upper contour set $\{y \in X : y \succsim x\}$ and the lower contour set $\{y \in X : x \succsim y\}$ are both closed.
- if the graph of $\succsim$ (the set $\{(x,y) \in X \times X : x \succsim y\}$) is a closed set (with the product topology).

9 Lexicographic preference relations are not continuous:

(1) Consider $x^n = (\frac{1}{n}, 0)$ and $y^n = (0,1)$ for all n.

(2) Clearly, $x^n \succ y^n$ for each n.

(3) $\lim_{n\to\infty} y^n = (0,1) \succ (0,0) = \lim_{n\to\infty} x^n$.

10 Proposition: Suppose that the rational preference relation $\succsim$ on X is continuous. Then there is a continuous utility function u that represents $\succsim$.

Proof. We only prove the result in a simple case: $X = \mathbb{R}^L_+$, $\succsim$ is a monotone preference relation on X.[1]

(1) Let $e = (1, 1, \dots, 1)$.

(2) For each $x \in \mathbb{R}^L_+$, there is a unique value $\alpha(x)$ such that $\alpha(x)e \sim x$.

  - Since $\succsim$ is continuous, $A^+ = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$ and $A^- = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$ are nonempty and closed.
  - Since $\succsim$ is complete, $A^+ \cup A^- = \mathbb{R}_+$.
  - Since $\mathbb{R}_+$ is connected, $A^+ \cap A^- \neq \emptyset$, and hence there exists an $\alpha \in A^+ \cap A^-$.
  - $\alpha(x)$ is unique.

(3) For each x, let $u(x) = \alpha(x)$.

(4) Suppose that $x \succsim y$. Then $\alpha(x)e \sim x \succsim y \sim \alpha(y)e$. Monotonicity implies that $\alpha(x) \geq \alpha(y)$, i.e., $u(x) \geq u(y)$.

(5) Suppose that $u(x) \geq u(y)$. Then $\alpha(x) \geq \alpha(y)$. Monotonicity implies that $\alpha(x)e \succsim \alpha(y)e$. Since $x \sim \alpha(x)e$ and $y \sim \alpha(y)e$, we have $x \succsim y$.

(6) u is continuous.

$\square$

[1] A preference relation $\succsim$ is monotone if $x \gg y$ ($x_i > y_i$ for each i) implies $x \succ y$.

2 Lottery

11 The set of all possible outcomes C. It is assumed to be finite (N outcomes) for simplicity.

The probabilities of outcomes are objectively known.

12 A simple lottery L is a vector $L = (p_1, p_2, \ldots, p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

13 A simple lottery can be represented geometrically as a point in the $(N-1)$-dimensional simplex $\Delta = \{p \in \mathbb{R}^N_+ : p_1 + p_2 + \cdots + p_N = 1\}$.

[Graph]

14 Given K simple lotteries $L_k = (p^k_1, p^k_2, \ldots, p^k_N)$ and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the compound lottery $(L_1, L_2, \ldots, L_K; \alpha_1, \alpha_2, \ldots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k.

The corresponding reduced lottery $L = (p_1, p_2, \ldots, p_N)$, where each $p_n = \alpha_1 p^1_n + \alpha_2 p^2_n + \cdots + \alpha_K p^K_n$.

$$
\begin{array}{cccccc}
 & & c_1 & c_2 & \cdots & c_N \\
\alpha_1 & L_1 & p^1_1 & p^1_2 & \cdots & p^1_N \\
\alpha_2 & L_2 & p^2_1 & p^2_2 & \cdots & p^2_N \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_K & L_K & p^K_1 & p^K_2 & \cdots & p^K_N \\
 & L & p_1 & p_2 & \cdots & p_N
\end{array}
$$

Only the reduced lottery is of relevance to the decision maker.

3 Preferences over lotteries

15 The set of alternatives $\mathcal{L}$: the set of all simple lotteries over the set of outcomes C.

Assume that the decision maker has a rational preference relation $\succsim$ on $\mathcal{L}$.

16 The preference relation $\succsim$ on $\mathcal{L}$ is continuous if whenever $L_1 \succ L_2$, there are neighborhoods B_{L_1} of L_1 and B_{L_2} of L_2 such that for all $L'_1 \in B_{L_1}$ and $L'_2 \in B_{L_2}$, $L'_1 \succ L'_2$.

Alternative definitions:

- if for any $L, L', L'' \in \mathcal{L}$, the sets $\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succsim L''\}$ and $\{\alpha \in [0,1] : L'' \succsim \alpha L + (1-\alpha)L'\}$ are closed.
- if for any $L, L', L'' \in \mathcal{L}$ with $L \succ L' \succ L''$, there exists $\alpha \in (0,1)$ such that $L' \sim \alpha L + (1-\alpha)L''$.

The continuity axiom implies the existence of a continuous utility function representing $\succsim$.

17 The preference relation $\succsim$ on $\mathcal{L}$ satisfies the independence axiom if for all $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$ we have $L \succsim L'$ iff $\alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$.

18 The utility function $U : \mathcal{L} \to \mathbb{R}$ has an expected utility form if there is an assignment of numbers $(u_1, u_2, \ldots, u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, p_2, \ldots, p_N) \in \mathcal{L}$ we have

$$U(L) = p_1 u_1 + p_2 u_2 + \cdots + p_N u_N.$$

A utility function $U\colon \mathcal{L} \to \mathbb{R}$ with the expected utility form is called a von Neumann-Morgenstern (vNM) expected utility function.

19 Let L^n denote the lottery that yields outcome n with probability one, then $U(L^n) = u_n$.

The term "expected utility" is appropriate because with the vNM expected utility form, the utility of a lottery can be thought of as the expected value of the utilities u_n of the N outcomes.

20 Proposition: A utility function $U\colon \mathcal{L} \to \mathbb{R}$ has an expected utility form iff it is linear: $U(\sum_{k=1}^{K} \alpha_k L_k) = \sum_{k=1}^{K} \alpha_k U(L_k)$.

Proof. "$\Rightarrow$"

(1) Suppose U has the expected utility form.

(2) Consider K lotteries $L_k = (p_1^k, p_2^k, \ldots, p_N^k)$ and probabilities $(\alpha_1, \alpha_2, \ldots, \alpha_K)$ with $\sum \alpha_k = 1$.

(3) The reduced lottery $L = \sum_k \alpha_k L_k$ is $(\sum_k \alpha_k p_1^k, \sum_k \alpha_k p_2^k, \ldots, \sum_k \alpha_k p_N^k)$.

(4) $U(\sum_k \alpha_k L_k) = \sum_n u_n (\sum_k \alpha_k p_n^k) = \sum_k \alpha_k (\sum_n u_n p_n^k) = \sum_k \alpha_k U(L_k)$.

"$\Leftarrow$"

(1) Suppose that $U(\sum_{k=1}^{K} \alpha_k L_k) = \sum_{k=1}^{K} \alpha_k U(L_k)$.

(2) Let L^n be the degenerate lottery and $u_n = U(L^n)$.

(3) $L = (p_1, p_2, \ldots, p_N)$ can be written as $\sum_n p_n L^n$.

(4) $U(L) = U(\sum_n p_n L^n) = \sum_n p_n U(L^n) = \sum_n p_n u_n$.

$\square$

21 Proposition: Suppose that $U\colon \mathcal{L} \to \mathbb{R}$ is a vNM expected utility function for $\succsim$ on $\mathcal{L}$. Then $\tilde{U}\colon \mathcal{L} \to \mathbb{R}$ is another vNM utility function for $\succsim$ iff $\tilde{U}$ is a affine transformation of U, i.e., there are scalars $\beta > 0$ and γ such that $\tilde{U}(L) = \beta U(L) + \gamma$.

Proof. It suffices to show the "$\Rightarrow$" part.

(1) Since U is a vNM expected utility function, it is linear and continuous. Thus, $\arg\max U$ and $\arg\min U$ are nonempty.

(2) Let $\underline{L}$ and $\bar{L}$ be two lotteries such that $\bar{L} \succsim L \succsim \underline{L}$ for all $L \in \mathcal{L}$.

(3) Trivial case: if $\bar{L} \sim \underline{L}$, then every utility function is constant.

(4) Nontrivial case: $\bar{L} \succ \underline{L}$.

- Suppose $\tilde{U}$ is a vNM expected utility function for $\succsim$.
- For any $L \in \mathcal{L}$, define $\lambda_L \in [0,1]$ by $U(L) = \lambda_L U(\bar{L}) + (1-\lambda_L) U(\underline{L})$. $\lambda_L = \frac{U(L)-U(\underline{L})}{U(\bar{L})-U(\underline{L})}$.
- $U(\lambda_L \bar{L} + (1-\lambda_L)\underline{L}) = \lambda_L U(\bar{L}) + (1-\lambda_L) U(\underline{L})$.
- $U(L) = U(\lambda_L \bar{L} + (1-\lambda_L)\underline{L})$ and hence $L \sim \lambda_L \bar{L} + (1-\lambda_L)\underline{L}$.
- Since $\tilde{U}$ represents $\succsim$,

$$\tilde{U}(L) = \tilde{U}(\lambda_L \bar{L} + (1-\lambda_L)\underline{L}) = \lambda_L \tilde{U}(\bar{L}) + (1-\lambda_L)\tilde{U}(\underline{L}) = \beta U(L) + \gamma,$$

where $\beta = \frac{\tilde{U}(\bar{L})-\tilde{U}(\underline{L})}{U(\bar{L})-U(\underline{L})}$ and $\gamma = \tilde{U}(\underline{L}) - U(\underline{L}) \frac{\tilde{U}(\bar{L})-\tilde{U}(\underline{L})}{U(\bar{L})-U(\underline{L})}$.

∎

22 Proposition: If the preference relation $\succsim$ on $\mathcal{L}$ is represented by a vNM expected utility function U, then $\succsim$ satisfies the independence axiom.

Proof. Trivial. ∎

4 Expected utility theorem

23 Theorem: Suppose that the rational preference relation $\succsim$ on $\mathcal{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, there is an assignment of numbers $(u_1, u_2, \ldots, u_N)$ to the N outcomes such that for any two lotteries $L = (p_1, p_2, \ldots, p_N)$ and $L' = (p'_1, p'_2, \ldots, p'_N)$, we have

$$L \succsim L' \text{ iff } \sum_n u_n p_n \geq \sum_n u_n p'_n.$$

It is the most important result in the theory of choice under uncertainty.

24 *Proof.* (1) Since C is finite, there are best and worst outcomes in C. Let $\bar{L}$ be the lottery that yields a particular best outcome with probability one and $\underline{L}$ be the lottery that yields a particular worst outcome with probability one. It is easy to see that $\bar{L} \succsim L \succsim \underline{L}$ for all $L \in \mathcal{L}$.

(2) We assume that $\bar{L} \succ \underline{L}$.

(3) If $L \succ L'$ and $\alpha \in (0,1)$, then $L \sim \alpha L + (1-\alpha)L \succ \alpha L + (1-\alpha)L' \succ \alpha L' + (1-\alpha)L' \sim L'$.

(4) Let $\alpha, \beta \in [0,1]$. Then $\beta\bar{L} + (1-\beta)\underline{L} = \gamma\bar{L} + (1-\gamma)[\alpha\bar{L} + (1-\alpha)\underline{L}] \succ \alpha\bar{L} + (1-\alpha)\underline{L}$ iff $\beta > \alpha$, where $\gamma = \frac{\beta-\alpha}{1-\alpha} \in (0,1]$.

(5) For any $L \in \mathcal{L}$, there is a unique α_L such that $\alpha_L\bar{L} + (1-\alpha_L)\underline{L} \sim L$.
- The existence is implied by the continuity and completeness.
- The uniqueness is implied by the previous step.

(6) Let $U(L) = \alpha_L$ for all $L \in \mathcal{L}$. It represents $\succsim$.

(7) U is linear and therefore has the expected utility form.
- For any $L, L' \in \mathcal{L}$ and $\beta \in [0,1]$, we have $L \sim U(L)\bar{L} + (1-U(L))\underline{L}$ and $L' \sim U(L')\bar{L} + (1-U(L'))\underline{L}$.
- By the independence axiom,

$$\begin{aligned}\beta L + (1-\beta)L' &\sim \beta[U(L)\bar{L} + (1-U(L))\underline{L}] + (1-\beta)L' \\ &\sim \beta[U(L)\bar{L} + (1-U(L))\underline{L}] + (1-\beta)[U(L')\bar{L} + (1-U(L'))\underline{L}] \\ &\sim [\beta U(L) + (1-\beta)U(L')]\bar{L} + [1-\beta U(L) - (1-\beta)U(L')]\underline{L}.\end{aligned}$$

- By the definition of U, $U(\beta L + (1-\beta)L') = \beta U(L) + (1-\beta)U(L')$.

∎

25 If $C = \mathbb{R}$, we will describe a lottery by a cumulative distribution function $F\colon \mathbb{R} \to [0,1]$. Then $\mathcal{L}$ is the set of all distribution functions. Given a rational preference $\succsim$ on $\mathcal{L}$ that satisfies the continuity and independence axioms, then there is an assignment of utility values $u(x)$ such that any F can be evaluated by a utility function U of the form

$$U(F) = \int_{\mathbb{R}} u(x)\,\mathrm{d}F(x).$$

5 Homework

- Reading: Section 3.B, 3.C, 6.B
- Homework: 6.B.3,
  - Consider the sequence of preference relations $\{\succsim^n\}_{n=1}^{\infty}$ defined on $\mathbb{R}_+^2$ where $\succsim^n$ is represented by the utility function $u^n(x_1, x_2) = x_1^n + x_2^n$. We will say that the sequence $\succsim^n$ converges to the preferences $\succsim^*$ if for every x and y, such that $x \succ^* y$, there is an N such that for every $n > N$ we have $x \succ^n y$. Show that the sequence of preference relations $\succsim^n$ converges to the preference $\succsim^*$, which is represented by the function max$\{x_1, x_2\}$.
  - One way to construct preferences over lotteries with monetary prizes is by evaluating each lottery L on the basis of two numbers: $\mathbf{E}(L)$, the expectation of L, and var(L), L's variance. Such a construction may or may not be consistent with vNM assumptions.
    * Show that the function $U(L) = \mathbf{E}(L) - \frac{1}{4}\,\text{var}(L)$ induces a preference relation that is not consistent with the vNM assumptions.
    * Show that the utility function $U(L) = \mathbf{E}(L) - (\mathbf{E}(L))^2 - \text{var}(L)$ is consistent with vNM assumptions.

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