Advanced Microeconomics I: Lecture 2

1 Lottery over monetary outcomes

1 The outcomes are amounts of money. It is convenient to treat money as a continuous variable, denoted by x.

2 We can describe a monetary lottery by a cumulative distribution function (cdf) $F\colon \mathbb{R} \to [0, 1]$. For each x, $F(x) = \text{Prob}(t \leq x)$.

We take the lottery space $\mathcal{L}$ to be the set of all distribution functions over nonnegative amounts of money.

3 We assume that the decision maker has a rational preference $\succsim$ over $\mathcal{L}$.

4 The expected utility theorem for general outcomes: There is an assignment of utility values $u(x)$ to nonnegative amounts of money x such that each monetary lottery F can be evaluated by a utility U of the form

$$U(F) = \int u(x)\,\mathrm{d}F(x).$$

5 Question: Is the result above consistent with the expected utility theorem for finite outcomes as discussed in Lecture 1?

6 The utility function U is defined on lotteries and the utility function u is defined on sure amounts of money.

To capture interesting economic attributes of choice behavior, we assume that u is increasing and continuous.

2 St. Petersburg paradox

7 A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The initial stake starts at 2 dollars and is doubled every time heads appears. The first time tails appears, the game ends and the player wins whatever is in the pot. Thus the player wins 2 dollars if tails appears on the first toss, 4 dollars if heads appears on the first toss and tails on the second, 8 dollars if heads appears on the first two tosses and tails on the third, and so on. Mathematically, the player wins $2k$ dollars, where k equals number of tosses (k must be a whole number and greater than zero). What would be a fair price to pay the casino for entering the game?

8 The paradox is the discrepancy between what people seem willing to pay to enter the game and the infinite expected value.

With probability $\frac{1}{2}$, the player wins 2 dollars; with probability $\frac{1}{4}$ the player wins 4 dollars; with probability $\frac{1}{8}$ the player wins 8 dollars, and so on. The expected value is thus

$$\frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \frac{1}{16} \cdot 16 + \cdots = 1 + 1 + 1 + 1 + \cdots = \infty.$$

Assuming the game can continue as long as the coin toss results in heads and in particular that the casino has unlimited resources, this sum grows without bound and so the expected win for repeated play is an infinite amount of money. Considering nothing but the expected value of the net change in one's monetary wealth, one should therefore play the game at any price if offered the opportunity.

9 Key point: The value of the game depends on the expectation of the value of the gain, rather than the expected gain.

Let w be the initial wealth and c be the cost charged to enter the game. For each possible event, the change in utility u(wealth after the event)$-u$(wealth before the event) will be weighted by the probability of that event occurring. The expected incremental utility of the lottery is

$$\Delta = \sum_{k=1}^{\infty} \frac{1}{2^k}\left[u(w+2^k-c)-u(w)\right]$$

So the fair price c for entering the game is such that $\Delta = 0$.

10 We consider the following cases:

(1a) Let $u(x) = \ln(x)$ and $w = 1000000$. Then the fair price for entering the game is 20.88.

(1b) Let $u(x) = \ln(x)$ and $w = 1000$. Then the fair price for entering the game is 10.95.

(1c) Let $u(x) = \ln(x)$ and $w = 2$. Then the fair price for entering the game is 3.35 (the player should borrow 1.35).

(2a) Let $u(x) = 2\sqrt{x}$ and $w = 1000000$. Then the fair price for entering the game is 22.89.

(2b) Let $u(x) = 2\sqrt{x}$ and $w = 1000$. Then the fair price for entering the game is 12.93.

11 Different utilities and different initial wealth lead to the distinct entering prices, which reflect the extent of risk (aversion).

3 Risk aversion

12 A decision maker is risk averter (or exhibits risk aversion) if for any lottery F, the degenerate lottery that yields the amount $\int x\,\mathrm{d}F(x)$ with certainty[1] is at least as good as the lottery F itself.

If the decision maker is always indifferent between these two lotteries, we say that he is risk neutral.

Finally, we say that he is strictly risk averse if indifference holds only when the two lotteries are the same, i.e., when F is degenerate.

13 The general definitions above do not presume an expected utility formulation.

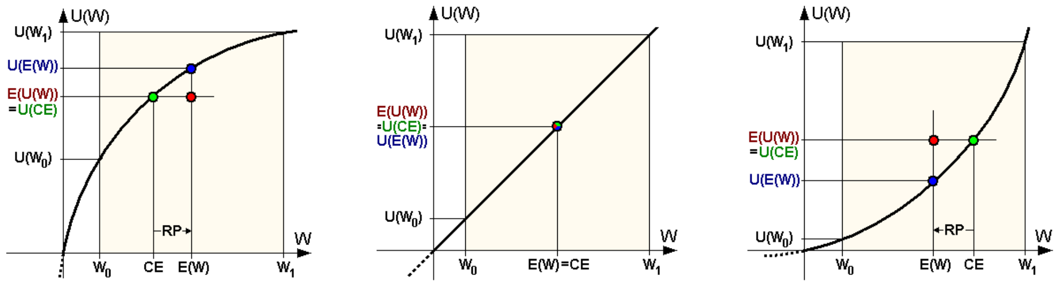
If preferences admit an expected utility representation with u, it follows directly from the definition of risk aversion that the decision maker is risk averse iff

$$\int u(x)\,\mathrm{d}F(x) \le u\left(\int x\,\mathrm{d}F(x)\right) \text{ for all } F.$$

The inequality is called Jensen's inequality, and it is the defining property of a concave function. Hence, in the context of expected utility theory, we see that risk aversion is equivalent to the concavity of u.

14 Graphs

[1]The degenerate lottery is $\tilde{F}$ such that $\tilde{F}(x) = 1$ for each $x \ge \int x\,\mathrm{d}F(x)$ and $\tilde{F}(x) = 0$ for each $x < \int x\,\mathrm{d}F(x)$.



(a) Risk averse (b) Risk neutral (c) Risk seeking

Figure 1

15 Given u, we define the following concepts:

- The certainty equivalent of F, denoted $\mathrm{CE}(F, u)$ or $\mathrm{CE}(F)$, is the amount of money for which the decision maker is indifferent between the gamble F and the certain amount $\mathrm{CE}(F)$, that is,

$$u\big(\mathrm{CE}(F)\big) = \int u(x)\,\mathrm{d}F(x) = U(F).$$

For a risk-averse decision maker, some expected return is traded for certainty. [为了得到“确定性”，付出了一定的期望收益]

- The risk premium of F is $R(F) = \int x\,\mathrm{d}F(x) - \mathrm{CE}(F)$. It implies that

$$u\left(\int x\,\mathrm{d}F(x) - R(F)\right) = u\big(\mathrm{CE}(F)\big) = U(F).$$

- For any fixed amount of money x and positive number ε, the probability premium denoted by $\pi(x, \varepsilon, u)$, is the excess in winning probability over fair odds that makes the decision maker indifferent between the certain outcome x and a gamble between the two outcomes $x + \varepsilon$ and $x - \varepsilon$. That is,

$$u(x) = \big(\tfrac{1}{2} + \pi(x, \varepsilon, u)\big)u(x + \varepsilon) + \big(\tfrac{1}{2} - \pi(x, \varepsilon, u)\big)u(x - \varepsilon).$$

Since $u(x) > \frac{1}{2}u(x + \varepsilon) + \frac{1}{2}u(x - \varepsilon)$, better than fair odds must bee given for the individual to accept the risk/lottery. [要使得人们愿意承担风险，就必须给予适当的回报]

16 Proposition: $\mathrm{CE}(F)$ exists.

17 The following properties are equivalent:

(a) The decision maker is risk averse.

(b) u is concave.

(c) $\mathrm{CE}(F) \le \int x\,\mathrm{d}F(x)$ for all F.

(d) $\pi(x, \varepsilon, u) \ge 0$ for all x and ε.

Proof: (b) is equivalent to (c).

$$\underbrace{\mathrm{CE}(F) \le \int x\,\mathrm{d}F(x) \Leftrightarrow u\big(\mathrm{CE}(F)\big) \le u\left(\int x\,\mathrm{d}F(x)\right)}_{u \text{ is increasing}} \Leftrightarrow \int u(x)\,\mathrm{d}F(x) \le u\left(\int x\,\mathrm{d}F(x)\right).$$

$\square$

Proof: (b) is equivalent to (d). (1) By the definition of $\pi(x, \varepsilon, u)$,

$$\pi(x, \varepsilon, u) \cdot [u(x+\varepsilon) - u(x-\varepsilon)] = u(x) - \tfrac{1}{2}[u(x+\varepsilon) - u(x-\varepsilon)].$$

(2) Since u is increasing, $u(x+\varepsilon) - u(x-\varepsilon) \geq 0$.

(3) $\pi(x, \varepsilon, u) \geq 0$ iff $u(x) - \frac{1}{2}[u(x+\varepsilon) - u(x-\varepsilon)] \geq 0$.

(4) $u(x) - \frac{1}{2}[u(x+\varepsilon) - u(x-\varepsilon)] \geq 0$ for all x and ε iff u is concave.

$\square$

18 Example: Insurance. Consider a strictly risk-averse decision maker who has an initial wealth of w but who runs a risk of a loss of D dollars. The probability of the loss is π. The decision maker wants to buy insurance: one unit of insurance costs q dollars and pays 1 dollar if the loss occurs. Thus, if α units of insurance are bought, the wealth of the individual will be $w - \alpha q$ if there is no loss and $w - \alpha q - D + \alpha$ if the loss occurs. The decision maker's problem is

$$\max_{\alpha \geq 0} (1-\pi) \cdot u(w - \alpha q) + \pi \cdot u(w - \alpha q - D + \alpha).$$

The insurance company's expected income is $\pi \cdot (q-1) + (1-\pi) \cdot q$. The fair price of one unit of insurance is such that the expected income is equal to zero, i.e., $q = \pi$.

If α^* is an optimum, it must satisfy the first-order condition:

$$-q(1-\pi)u'(w - \alpha^* q) + \pi(1-q)u'\big(w - D + \alpha^*(1-q)\big) \leq 0,$$

with equality if $\alpha^* > 0$. Since $q = \pi$, we have

$$-u'(w - \alpha^* q) + u'\big(w - D + \alpha^*(1-q)\big) \leq 0,$$

with equality if $\alpha^* > 0$. Since u' is strictly decreasing, we must have $\alpha^* > 0$, and hence

$$u'(w - \alpha^* q) = u'\big(w - D + \alpha^*(1-q)\big).$$

Therefore,

$$w - \alpha^* q = w - D + \alpha^*(1-q),$$

and $\alpha^* = D$.

Summary: If insurance is fair, the decision maker insures completely. The final wealth is then $w - \pi D$, regardless of the occurrence of the loss.

4 Arrow-Pratt coefficient of absolute risk aversion

19 The higher the curvature of u,[2] the higher the risk aversion. However, since expected utility functions are not uniquely defined (are defined only up to affine transformations), a measure that stays constant with respect to these transformations is needed.

[2] A plane curve is given as $y = f(x)$, then the curvature is $\frac{y''}{(1+(y')^2)^{\frac{3}{2}}}$.

20 Given a twice-differentiable utility function u, the Arrow-Pratt coefficient of absolute risk aversion at x is defined as

$$r_A(x,u) = r_A(x) = -\frac{u''(x)}{u'(x)}.$$

21 Example: Consider the utility function $u(x) = -e^{-ax}$ for $a > 0$. Then $r_A(u,x) = a$ for all x. This utility function is said to exhibit constant absolute risk aversion (CARA).

22 Interpretation: Look at the preferences on the restricted domain of lotteries of the type (x_1, x_2) ($\frac{1}{2}$ for the outcome x_1 and $\frac{1}{2}$ for the outcome x_2). Let u be a continuously differentiable vNM utility function that represents a risk-averse preference.

Let $x_2 = \psi(x_1)$ be the function describing the indifference curve through (t,t) (the certain outcome t). Thus, $\psi(t) = t$.

It follows from risk aversion that all lotteries with expectation t (i.e., $\{(x_1,x_2) \mid \frac{1}{2}x_1 + \frac{1}{2}x_2 = t\}$) are below the indifference curve ψ. Thus, $\psi'(t) = -1$.

Then we have $u(t) = U(t,t) = \underbrace{U\big(x_1, \psi(x_1)\big) = \tfrac{1}{2}u(x_1) + \tfrac{1}{2}u(\psi(x_1))}_{\text{expected utility}}$. Differentiate twice with respect to x_1, we obtain

$$0 = \tfrac{1}{2}u''(x_1) + \tfrac{1}{2}u''(\psi(x_1))[\psi'(x_1)]^2 + \tfrac{1}{2}u'(\psi(x_1))\psi''(x_1).$$

At $x_1 = t$ we have

$$\tfrac{1}{2}u''(t) + \tfrac{1}{2}u''(t) + \tfrac{1}{2}u'(t)\psi''(t) = 0.$$

Therefore,

$$\psi''(t) = -2\frac{u''(t)}{u'(t)} = 2r_A(t,u).$$

Note that on this restricted space of lotteries, $\succsim_2$ is more risk averse than $\succsim_1$ iff the indifference curve of $\succsim_1$ through (t,t) (denoted by ψ_1) is below the indifference curve of $\succsim_2$ through (t,t) (denoted by ψ_2). Combined with $\psi_1'(t) = \psi_2'(t)$, we obtain that $\psi_1''(t) \le \psi_2''(t)$ and thus $r_A(t,u_2) \ge r_A(t,u_1)$.

23 Proposition: Given two utility function u_1 and u_2, the following statements are equivalent:

(i) $r_A(x,u_2) \ge r_A(x,u_1)$ for every x.

(ii) u_2 is more concave than u_1 (or a concave transformation of u_1): there exists an increasing concave function ψ such that $u_2 = \psi \circ u_1$.

(iii) $\mathrm{CE}(F,u_2) \le \mathrm{CE}(F,u_1)$ for any F.

(iv) $\pi(x,\varepsilon,u_2) \ge \pi(x,\varepsilon,u_1)$ for any x and ε.

(v) Any risk that u_2 would accept starting from a position of certainty would also be accepted by u_1: Whenever u_2 finds a lottery F at least as good as a riskless outcome $\bar{x}$, then u_1 also finds F at least as good as $\bar{x}$; that is, $\int u_2(x)\,\mathrm{d}F(x) \ge u_2(\bar{x})$ implies $\int u_1(x)\,\mathrm{d}F(x) \ge u_1(\bar{x})$ for any F and $\bar{x}$.

If they are satisfied, we say that u_2 is more risk averse than u_1.

Proof: (i) iff (ii). We always have $u_2 = \psi \circ u_1$ for some increasing function ψ. Assume that ψ is twice-differentiable. We have

$$u_2'(x) = \psi'(u_1(x)) \cdot u_1'(x) \text{ and } u_2''(x) = \psi'(u_1(x)) \cdot u_1''(x) + \psi''(u_1(x)) \cdot (u_1'(x))^2.$$

Then we have
$$r_A(x,u_2) = r_A(x,u_1) - \frac{\psi''(u_1(x))}{\psi'(u_1(x))}u_1'(x).$$
Thus, $r_A(x,u_2) \geq r_A(x,u_1)$ for all x iff $\psi'' \leq 0$. $\square$

Proof: (ii) iff (iii). Assume that $u_2 = \psi \circ u_1$. We have
$$\psi(u_1(\mathrm{CE}(F,u_2))) = u_2(\mathrm{CE}(F,u_2)) = \int u_2(x)\,\mathrm{d}F(x) = \int \psi(u_1(x))\,\mathrm{d}F(x) \leq \psi\left(\int u_1(x)\,\mathrm{d}F(x)\right).$$
Since ψ in increasing, $u_1(\mathrm{CE}(F,u_2)) \leq \int u_1(x)\,\mathrm{d}F(x) = u_1(\mathrm{CE}(F,u_1))$. Since u_1 is increasing, $\mathrm{CE}(F,u_2) \leq \mathrm{CE}(F,u_1)$.

Let F be the lottery that assigns λ and $1-\lambda$ on x and y, respectively. Then we have
$$\lambda u_1(x) + (1-\lambda)u_1(y) = u_1(\mathrm{CE}(F,u_1)),$$
and hence
$$\psi\left(\lambda u_1(x) + (1-\lambda)u_1(y)\right) = u_2(\mathrm{CE}(F,u_1)).$$
On the other hand,
$$\lambda\psi(u_1(x)) + (1-\lambda)\psi(u_1(y)) = u_2(\mathrm{CE}(F,u_2)).$$
Thus,
$$\psi\left(\lambda u_1(x) + (1-\lambda)u_1(y)\right) \geq \lambda\psi(u_1(x)) + (1-\lambda)\psi(u_1(y)).$$
$\square$

Proof: (iii) iff (v). Assume that $\mathrm{CE}(F,u_2) \leq \mathrm{CE}(F,u_1)$. For any $\bar{x}$, if $\int u_2(x)\,\mathrm{d}F(x) \geq u_2(\bar{x})$, then $u_2(\mathrm{CE}(F,u_2)) \geq u_2(\bar{x})$. Thus, $\mathrm{CE}(F,u_1) \geq \mathrm{CE}(F,u_2) \geq \bar{x}$, and hence $\int u_1(x)\,\mathrm{d}F(x) = u_1(\mathrm{CE}(F,u_1)) \geq u_1(\bar{x})$.

Let $\bar{x} = \mathrm{CE}(F,u_2)$. Clearly, $\int u_2(x)\,\mathrm{d}F(x) \geq u_2(\bar{x})$. Then $u_1(\mathrm{CE}(F,u_1)) = \int u_1(x)\,\mathrm{d}F(x) \geq u_1(\bar{x})$. Thus, $\mathrm{CE}(F,u_1) \geq \bar{x} = \mathrm{CE}(F,u_2)$. $\square$

Proof: (iii) ⇒ (iv). (1) For any x and $\varepsilon > 0$, let F be the distribution that puts $\frac{1}{2} - \pi(u_2)$ on $x-\varepsilon$ and $\frac{1}{2} + \pi(u_2)$ on $x+\varepsilon$.

(2) Then $u_2(\mathrm{CE}(F,u_2)) = u_2(x+\varepsilon)\cdot(\frac{1}{2}+\pi(u_2)) + u_2(x-\varepsilon)\cdot(\frac{1}{2}-\pi(u_2)) = u_2(x)$. Thus, $\mathrm{CE}(F,u_2) = x$.

(3) Since $\mathrm{CE}(F,u_2) \leq \mathrm{CE}(F,u_1)$, we have $u_1(x) \leq u_1(\mathrm{CE}(F,u_1))$.

(4) By the definition of $\pi(u_1)$, $u_1(x) = (\frac{1}{2}+\pi(u_1))u_1(x+\varepsilon) + (\frac{1}{2}-\pi(u_1))u_1(x-\varepsilon)$.

(5) By the definition of $\mathrm{CE}(F,u_1)$, $u_1(\mathrm{CE}(F,u_1)) = (\frac{1}{2}+\pi(u_2))u_1(x+\varepsilon) + (\frac{1}{2}-\pi(u_2))u_1(x-\varepsilon)$.

(6) Then $\pi(u_2) \geq \pi(u_1)$.

$\square$

Proof: (iv) ⇒ (i). (1) Assume that $\pi(x,\varepsilon,u_2) \geq \pi(x,\varepsilon,u_1)$ for any x and ε.

(2) By the definition of $\pi(x,\varepsilon,u)$, we have $\pi(x,0,u_1) = \pi(x,0,u_2) = 0$.

(3) Then $\frac{\partial\pi(x,0,u_2)}{\partial\varepsilon} \geq \frac{\partial\pi(x,0,u_1)}{\partial\varepsilon}$.

(4) Differentiate $u(x) = (\frac{1}{2}+\pi(x,\varepsilon,u))u(x+\varepsilon) + (\frac{1}{2}-\pi(x,\varepsilon,u))u(x-\varepsilon)$ twice with respect to ε, we have
$$4\pi'(x,0,\varepsilon)\cdot u'(x) + u''(x) = 0.$$

(5) $r_A(x,u) = -\frac{u''(x)}{u'(x)} = 4\pi'(x,0,\varepsilon)$.

(6) $r_A(x,u_2) \geq r_A(x,u_1)$.

$\square$

24 The utility function u for money exhibits decreasing absolute risk aversion if $r_A(x,u)$ is a decreasing function of x. Individuals whose preference satisfies the decreasing absolute risk aversion property take more risk as they become wealthier.

25 The following properties are equivalent.

(i) u exhibits decreasing absolute risk aversion.

(ii) Whenever $x_2 < x_1$, $u_2(z) = u(x_2+z)$ is a concave transformation of $u_1(z) = u(x_1+z)$.

(iii) For any risk F, the certainty equivalent of the lottery formed by adding risk z to wealth level x, given by the amount c_x at which $u(c_x) = \int u(x+z)\,\mathrm{d}F(z)$, is such that $x - c_x$ is decreasing in x. That is, the higher x is, the less is the individual willing to pay to get rid of the risk.

(iv) $\pi(x,\varepsilon,u)$ is decreasing in x.

(v) For any F, if $\int u(x_2+z)\,\mathrm{d}F(z) \geq u(x_2)$ and $x_2 < x_1$, then $\int u(x_1+z)\,\mathrm{d}F(z) \geq u(x_1)$.

Proof. Consider $u_1(z) = u(x_1+z)$ and $u_2(z) = u(x_2+z)$. We have $r_A(z,u_1) = -\frac{u_1''(z)}{u_1'(z)} = -\frac{u''(x_1+z)}{u'(x_1+z)} = r_A(x_1+z,u)$ and $r_A(z,u_2) = -\frac{u_2''(z)}{u_2'(z)} = -\frac{u''(x_2+z)}{u'(x_2+z)} = r_A(x_2+z,u)$. $\square$

5 Relative risk aversion

26 The concept of absolute risk aversion is suited to the comparison of attitudes toward risky projects whose outcomes are absolute gains or losses from the current wealth. But it is also of interest to evaluate risky projects whose outcomes are percentage gains or losses of current wealth.

27 Given a utility function u, the coefficient of relative risk aversion at x is $r_R(x,u) = -x\frac{u''(x)}{u'(x)}$.

28 Example: Consider the utility function $u(x) = -\frac{x^{1-\rho}}{1-\rho}$. It is clear that $r_R(x,u) = \rho$ for all x. This function is said to exhibit constant relative risk aversion (CRRA).

29 Proposition: The following statements are equivalent:

(i) $r_R(x,u)$ is decreasing in x.

(ii) Whenever $x_2 < x_1$, $\tilde{u}_2(t) = u(tx_2)$ is a concave transformation of $\tilde{u}_1(t) = u(tx_1)$.

(iii) Given any F, the certainty equivalent $\bar{c}_x$ defined by $u(\bar{c}_x) = \int u(tx)\,\mathrm{d}F(t)$ is such that $\frac{x}{\bar{c}_x}$ is decreasing in x.

6 Homework

- Reading: Section 6.C
- Homework: 6.C.1, 6.C.11, 6.C.12

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