Advanced Microeconomics: Lecture 5

1 The theory of mechanism design can be thought of as the "engineering" side of economic theory. Much theoretical work focuses on existing economic institutions. The theorist wants to explain or forecast the economic or social outcomes that these institutions generate.

But in mechanism design theory the direction of inquiry is reversed. We begin by identifying our desired social goal. We then ask whether or not an appropriate institution (mechanism) could be designed to attain that goal.

Almost any kind of market institution or economic organization can be viewed, in principle, as a mechanism. Examples include: school choice, auction, kidney exchange, tax codes, contract design, etc.

2 Leonid Hurwicz defined a mechanism as a communication system in which participants send messages to each other and/or to a "message center", and where a pre-specified rule assigns an outcome (such as an allocation of goods and services) for every collection of received messages.

The difficulty in mechanism design is that the individuals have private information and different objectives, and so may not have the incentive to behave in a way that reveals what they know. The key point is how to design "incentive compatible" mechanisms that can generate the information needed as they are executed.

1 Envelope theorem

3 Consider a one-agent decision problem

$$V(\theta) = \max_{a \in A} h(a, \theta),$$

where a is the agent's chosen action and $\theta \in \Theta$ an exogenous parameter. In an auction, θ could be bidder's valuation, and a bidder's choice of bid.

4 A could be either discrete or continuous, but Θ is an interval.

5 Let $a^*(\theta)$ be the set of optimal choices, that is,

$$a^*(\theta) = \arg\max_{a \in A} h(a, \theta).$$

Let h_a and h_θ denote partial derivatives of h.

6 Theorem (Envelope theorem): Suppose for all $\theta \in \Theta$, $a^*(\theta)$ in non-empty, and for all a and θ, h_θ exists. Let $a(\theta)$ be any selection from $a^*(\theta)$, i.e., $a(\theta) \in a^*(\theta)$.

(i) If V is differentiable at θ, then

$$V'(\theta) = h_\theta\big(a(\theta), \theta\big).$$

(ii) If V is absolutely continuous, then for any $\theta' > \theta$,

$$V(\theta') - V(\theta) = \int_{\theta}^{\theta'} h_\theta\big(a(t), t\big)\,\mathrm{d}t.$$

Proof of (i). (1) If V is differentiable at θ, then

$$V'(\theta) = \lim_{\epsilon \downarrow 0} \frac{V(\theta+\epsilon) - V(\theta)}{\epsilon} = \lim_{\epsilon \downarrow 0} \frac{V(\theta) - V(\theta-\epsilon)}{\epsilon}.$$

(2) Take $a(\theta) \in a^*(\theta)$, then $V(\theta) = h\big(a(\theta), \theta\big)$, and

$$V(\theta+\epsilon) = \max_a h(a, \theta+\epsilon) \geq h\big(a(\theta), \theta+\epsilon\big).$$

(3) Then we have

$$V'(\theta) = \lim_{\epsilon \downarrow 0} \frac{V(\theta+\epsilon) - V(\theta)}{\epsilon} \geq \lim_{\epsilon \downarrow 0} \frac{h(a(\theta), \theta+\epsilon) - h(a(\theta), \theta)}{\epsilon} = h_\theta(a(\theta), \theta).$$

(4) For the same number $a(\theta)$, $V(\theta-\epsilon) = \max_a h(a, \theta-\epsilon) \geq h(a(\theta), \theta-\epsilon)$, and hence

$$V'(\theta) = \lim_{\epsilon \downarrow 0} \frac{V(\theta) - V(\theta-\epsilon)}{\epsilon} \leq \lim_{\epsilon \downarrow 0} \frac{h(a(\theta), \theta) - h(a(\theta), \theta-\epsilon)}{\epsilon} = h_\theta(a(\theta), \theta).$$

(5) So

$$h_\theta(a(\theta), \theta) \leq V'(\theta) \leq h_\theta(a(\theta), \theta).$$

□

Proof of (ii). (1) Absolute continuity: for all $\epsilon > 0$, there exists $\delta > 0$ such that for any finite, disjoint set of intervals $\{[x_k, y_k]\}_{k=1,2,\ldots,M}$ with $\sum_k |y_k - x_k| < \delta$,

$$\sum_k |V(y_k) - V(x_k)| < \epsilon.$$

(2) Absolute continuity is equivalent to V being differentiable almost everywhere and being the integral of its derivative, so the second part follows directly from the first part.

□

7 Remark: The derivative of the value function is the derivative of the objective function, evaluated at the maximizer.

8 Corollary: Assume that

- for each $a \in A$, $h(a, \cdot)$ is differentiable,
- there exists $B > 0$, such that for all $a \in A$ and almost all $\theta \in \Theta$

$$|h_\theta(a, \theta)| \leq B,$$

- $a^*(\theta) = \arg\max_{a \in A} h(a, \theta) \neq \emptyset$.

Then V is Lipschitz continuous with Lipschitz constant 1, and hence absolutely continuous and almost everywhere differentiable. Therefore the two formulas in Theorem 6 still hold.

Proof. For any two distinct θ and θ', we have

$$|V(\theta) - V(\theta')| = |\max_{a \in A} h(a,\theta) - \max_{a \in A} h(a,\theta')| \leq \max_{a \in A} |h(a,\theta) - h(a,\theta')| \leq \max_{a \in A} B \cdot |\theta - \theta'| = B \cdot |\theta - \theta'|.$$

□

2 Pricing a single indivisible good

9 A seller seeks to sell a single indivisible good. The seller herself does not attach any value to the good. Her objective is to maximize the expected revenue from selling the good, that is, she is risk-neutral.

10 There is just one potential buyer. The buyer's von Neumann-Morgenstern utility if he purchases the good and pays a monetary transfer t to the seller is $\theta - t$.

Here θ is the number that we can interpret as the buyer's valuation of the good. We shall refer below to θ as the buyer's type.

We have assumed that the buyer's utility is additively separable—the sum of the utility derived from the good and the disutility resulting from the money payment.

We have also assumed that the buyer is risk-neutral with respect to money, that is, his utility is linear in money.

Utility functions that satisfy additive separability of utility and linearity of utility in money are referred to as "quasi-linear" utility functions.

11 We assume that the value of θ is known to the buyer, but it is not known to the seller. The seller has a subjective probability distribution over possible values of θ. To be more precise, θ is assumed to be a random variable with the cumulative distribution function F and the strictly positive density function f on $[\underline{\theta}, \bar{\theta}]$.

12 Our interest is in procedures for selling the good which the seller should adopt to maximize expected profits.

One obvious way would be to pick a price p and to say to the buyer that he can have the good if and only if he is willing to pay p. In this procedure, the probability that the buyer's value is above p, and hence he accepts a price offer p, is $1 - F(p)$. Seller's expected revenue is therefore $p \cdot (1 - F(p))$, and the optimal strategy for the seller is to pick some price that maximizes $p \cdot (1 - F(p))$.

13 Is picking a price p really the best the seller can do? What else could the seller do? The seller could, for example, negotiate with the buyer. The seller could offer the buyer a lottery where in return for higher or lower payments the buyer could be given a larger or smaller chance of getting the object.

Our objective is thus to study the optimality problem in which the seller's choices are a lottery (design) and a strategy in the game associated with the lottery and in which the seller's objective function is expected revenue, with the constraint that the buyer will choose an expected utility maximizing strategy and several other constraints introduced below.

3 Mechanism

14 A direct mechanism consists of functions q and t where

$$q \colon [\underline{\theta}, \bar{\theta}] \to [0,1], \quad t \colon [\underline{\theta}, \bar{\theta}] \to \mathbb{R}.$$

The interpretation is that in a direct mechanism the buyer is asked to report θ. The seller commits to transferring the good to the buyer with the probability $q(\theta)$ if the buyer reports that his type is θ, and the buyer has to pay the seller $t(\theta)$ if he reports that his type is θ.

15 Given a direct mechanism (q, t), we define the buyer's expected utility $u(\theta)$ conditional on his type being θ by $u(\theta) = \theta q(\theta) - t(\theta)$.

16 A direct mechanism (q, t) is incentive compatible (IC) if truth telling is optimal for every type $\theta \in [\underline{\theta}, \bar{\theta}]$, that is, if

$$u(\theta) \geq \theta q(\theta') - t(\theta') \text{ for all } \theta, \theta' \in [\underline{\theta}, \bar{\theta}].$$

17 Lemma: If a direct mechanism (q, t) is IC, then q is increasing in θ.

Proof. (1) Consider θ and θ' with $\theta > \theta'$.

(2) Bayesian incentive compatibility requires

$$\theta q(\theta) - t(\theta) \geq \theta q(\theta') - t(\theta') \text{ and } \theta' q(\theta') - t(\theta') \geq \theta' q(\theta) - t(\theta).$$

(3) Then we have

$$[q(\theta) - q(\theta')] \cdot (\theta - \theta') \geq 0,$$

and hence $q(\theta) \geq q(\theta')$.

□

18 Lemma: If a direct mechanism (q, t) is IC, then

- u is increasing,
- u is convex, and hence differentiable except in at most countably many points,
- for all θ for which it is differentiable, it satisfies $u'(\theta) = q(\theta)$.

Proof. (1) IC means that for all θ we have

$$u(\theta) = \max_{\theta' \in [\underline{\theta}, \bar{\theta}]} \theta q(\theta') - t(\theta').$$

(2) Given any value of θ', $\theta q(\theta') - t(\theta')$ is an increasing and affine (and hence convex) function of θ.

(3) The maximum of increasing functions is increasing, and the maximum of convex functions is convex. Therefore, u is increasing and convex.

(4) Convex functions are not differentiable in at most countably many points.

(5) Consider any θ for which u is differentiable. By IC we have

$$\lim_{\epsilon \downarrow 0} \frac{u(\theta + \epsilon) - u(\theta)}{\epsilon} \geq \lim_{\epsilon \downarrow 0} \frac{[(\theta + \epsilon) q(\theta) - t(\theta)] - [\theta q(\theta) - t(\theta)]}{\epsilon} = q(\theta).$$

Similarly,

$$\lim_{\epsilon \downarrow 0} \frac{u(\theta) - u(\theta - \epsilon)}{\epsilon} \leq \lim_{\epsilon \downarrow 0} \frac{[\theta q(\theta) - t(\theta)] - [(\theta - \epsilon) q(\theta) - t(\theta)]}{\epsilon} = q(\theta).$$

(6) Putting the two inequalities together, we obtain $u'(\theta) = q(\theta)$ whenever u is differentiable.

$\square$

19 Proposition (Payoff equivalence): Consider an IC direct mechanism (q, t). Then for all $\theta \in [\underline{\theta}, \bar{\theta}]$ we have

$$u(\theta) = u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(x)\,\mathrm{d}x.$$

Proof. (1) u is convex, then u is absolutely continuous; see Corollary 17 in **?**, Chapter 5.

(2) By Theorem 10 in **?**, Chapter 5, it is the integral of its derivative.

$\square$

20 Proposition (Revenue equivalence): Consider an IC direct mechanism (q, t). Then for all $\theta \in [\underline{\theta}, \bar{\theta}]$ we have

$$t(\theta) = t(\underline{\theta}) + [\theta q(\theta) - \underline{\theta} q(\underline{\theta})] - \int_{\underline{\theta}}^{\theta} q(x)\,\mathrm{d}x.$$

21 Proposition: A direct mechanism (q, t) is IC if and only if

- q is increasing.
- For every $\theta \in [\underline{\theta}, \bar{\theta}]$ we have

$$t(\theta) = t(\underline{\theta}) + [\theta q(\theta) - \underline{\theta} q(\underline{\theta})] - \int_{\underline{\theta}}^{\theta} q(x)\,\mathrm{d}x.$$

Proof. (1) The "only if" part is due to Lemma 17 and Proposition 20.

(2) For any θ and θ', we have

$$[\theta q(\theta) - t(\theta)] - [\theta q(\theta') - t(\theta')] = \int_{\theta'}^{\theta} q(x)\,\mathrm{d}x - q(\theta')(\theta - \theta') \geq 0.$$

$\square$

22 A direct mechanism is individually rational (IR) if the buyer, conditional on his type, is willing to participate, that is, if

$$u(\theta) \geq 0 \text{ for all } \theta \in [\underline{\theta}, \bar{\theta}].$$

23 Proposition: An IC direct mechanism (q, t) is IR if and only if $u(\underline{\theta}) \geq 0$, or equivalently $\underline{\theta} q(\underline{\theta}) \geq t(\underline{\theta})$.

Proof. Due to Lemma 18. $\square$

24 Proposition: If an IC and IR direct mechanism (q, t) maximizes the seller's expected revenue, then

$$\underline{\theta} q(\underline{\theta}) = t(\underline{\theta}).$$

Proof. If $\underline{\theta} q(\underline{\theta}) > t(\underline{\theta})$, then the seller could increase expected revenue by choosing a direct mechanism with the same q, but a higher $t(\underline{\theta})$. $\square$

25 Remark: If the buyer has only two possible types, the revenue equivalence principle does not necessarily hold.

Let θ_H and θ_L be buyer's types. Firstly, IR for the low type should be binding, that is, $\theta_L q(\theta_L) = t(\theta_L)$.

Next, IC for the high type should be binding. Otherwise, we have

$$\theta_H q(\theta_H) - t(\theta_H) > \theta_H q(\theta_L) - t(\theta_L) \geq \theta_L q(\theta_L) - t(\theta_L) = 0.$$

Then the seller can increase $t(\theta_H)$ without breaking IC and IR.

Thus, we have

$$t(\theta_H) = t(\theta_L) + [\theta_H q(\theta_H) - \theta_H q(\theta_L)].$$

Since $\int_{\theta_L}^{\theta_H} q(x)\,\mathrm{d}x$ may not be zero, the revenue equivalence principle does not necessarily hold.

In the discrete case, the distance between two types allows the variety of the transfers.

For example, let $\theta_L = 1$, $\theta_H = 2$, $\text{Prob}(\theta_H) = \text{Prob}(\theta_L) = \frac{1}{2}$, $q(\theta_L) = 0$, $q(\theta_H) = 1$, $t(\theta_L) = 0$, $t(\theta_H) = 2$.

26 Let $\mathcal{M}$ be the set of all increasing functions $q\colon [\underline{\theta}, \bar{\theta}] \to [0, 1]$.

Lemma:

- $\mathcal{M}$ is compact and convex.
- Let X be a compact, convex subset of a vector space, $f\colon X \to \mathbb{R}$ be a continuous linear function. Then the set E of extreme points of X is nonempty, and there exists an $e \in E$ such that $f(e) \geq f(x)$.
- A function $q \in \mathcal{M}$ is an extreme point of $\mathcal{M}$ if and only if $q(\theta) \in \{0, 1\}$ for almost all θ.

27 The lemma above implies that the seller can restrict her attention to nonstochastic mechanisms. But a nonstochastic mechanism is monotonic if and only if there is some $p^* \in [\underline{\theta}, \bar{\theta}]$ such that $q(\theta) = 0$ if $\theta < p^*$ and $q(\theta) = 1$ if $\theta > p^*$.

Proposition: The following direct mechanism maximizes the seller's expected revenues among all IC and IR direct mechanisms:

$$p^* \in \operatorname*{arg\,max}_{p \in [\underline{\theta}, \bar{\theta}]} p(1 - F(p)),\ q(\theta) = \begin{cases} 0, & \text{if } \theta < p^* \\ 1, & \text{if } \theta > p^* \end{cases},\ t(\theta) = \begin{cases} 0, & \text{if } \theta < p^* \\ p^*, & \text{if } \theta > p^* \end{cases}.$$

28 It may seem that we have gone to considerable lengths to derive a disappointing result, namely, a result that does not offer the seller any more sophisticated selling mechanisms than we are familiar with from elementary microeconomics.

4 Homework

- Reading: Börgers 2.2
- Homework: Consider the following two-stage mechanism: In stage 1 the seller posts a price 0.5. Then the buyer can choose to buy or not buy. If the buyer buys, the game is over. If he does not buy, then a third party draws a price randomly from the interval $[\underline{\theta}, \bar{\theta}]$, using the uniform distribution. The buyer can then either buy or not buy at the random price. Find the buyer's optimal strategy for this mechanism. Then find an equivalent direct mechanism in which truth telling is an optimal strategy for the buyer.