# Advanced Microeconomics: Lecture 6

1 A seller has one indivisible object to sell and there are $N \geq 2$ risk-neutral potential buyers (or bidders) from the set $I = \{1, 2, \ldots, N\}$.

2 Buyer $i$'s utility if he purchases the good and pays a transfer $t_i$ to the seller is $\theta_i - t_i$. Buyer $i$'s utility if he does not purchase the good and pays a transfer of $t_i$ to the seller is $0 - t_i$.

The seller's utility if she obtains transfers $t_i$ from buyer $i$ ($i \in I$) is $\sum_{i \in I} t_i$.

3 Buyer $i$ knows $\theta_i$, but neither the seller nor any other buyer $j \neq i$ knows $\theta_i$.

We model $\theta_i$ as a random variable with cumulative distribution function $F_i$ with density $f_i$. The support of $\theta_i$ is $[\underline{\theta}, \bar{\theta}]$ where $0 \leq \underline{\theta} < \bar{\theta}$. The distributions $F_i$ are common knowledge among the buyers and the seller.

For technical convenience, we also assume that $f_i(\theta_i) > 0$ for all $i \in I$ and all $\theta_i \in [\underline{\theta}, \bar{\theta}]$.

We also assume that for $i \neq j$, $\theta_i$ and $\theta_j$ are independent.

4 Notations:

- $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ and $\Theta = [\underline{\theta}, \bar{\theta}]^N$.
- $\Theta_{-i} = [\underline{\theta}, \bar{\theta}]^{N-1}$ and $f_{-i}(\theta_{-i}) = \times_{j \neq i} f_j(\theta_j)$.
- $\Delta = \{(q_1, q_2, \ldots, q_N) \mid 0 \leq q_i \leq 1 \text{ for all } i \in I \text{ and } \sum_{i \in I} q_i \leq 1\}$.

## 1 Mechanism and the revelation principle

5 General criterions for the seller:

- The seller cannot force people to play—they have to be willing to play.
- The seller need to assume people will play an equilibrium within whatever game you define.

6 We assume that the seller has full commitment power—once he defines the rules of the game, the players have complete confidence that he will honor those rules. This is important—you had bid differently in an auction if you thought that, even if you won, the seller might demand a higher price or mess with you some other way.

7 Broadly speaking, mechanism design takes the environment as given—the players, their value distributions, and their preferences over the different possible outcomes—and designs a game for the players to play in order to select one of the outcomes. Outcomes can be different legislative proposals, different allocations of one or more objects, etc.

Here we will focus on the auction problem—designing a mechanism to sell a single object, and try to maximize the expected revenue or expected welfare. So the set of possible outcomes consists of who (if anyone) gets the object, and how much each person pays.

8 In general, a selling mechanism $(\mathcal{B}, \pi, \mu)$ has the following components:

- a set of possible messages $\mathcal{B}_i$ for each buyer; $\mathcal{B} = \times_{i \in I} \mathcal{B}_i$;
- an allocation rule $\pi\colon \mathcal{B} \to \Delta$;
- a payment rule $\mu\colon \mathcal{B} \to \mathbb{R}^N$.

An allocation rule determines, as a function of all $N$ messages, the probability $\mu_i(b)$ that $i$ will get the object.

A payment rule determines, as a function of all $N$ messages, for each buyer $i$, the expected payment $\mu_i(b)$ that $i$ must make.

9 A direct mechanism $(q, t)$ consists of functions $q$ and $t_i$ (for $i \in I$)

$$q\colon \Theta \to \Delta \text{ and } (t_1, t_2, \ldots, t_N)\colon \Theta \to \mathbb{R}^N.$$

The interpretation is that in a direct mechanism the buyers are asked to simultaneously and independently report their types.

- The function $q(\theta)$ describes the rule by which the good is allocated if the reported type vector is $\theta$. We shall refer to $q$ as the "allocation rule." The probability $q_i(\theta)$ is the probability that agent $i$ obtains the good if the type vector is $\theta$. The probability $1 - \sum_{i \in I} q_i(\theta)$ is the probability with which the seller retains the good if the type vector is $\theta$.
- The functions $t_i$ describe the transfer payment that buyer $i$ makes to the seller. We shall also refer to it as the "payment rule" for buyer $i$. Note that we have assumed that this transfer payment is deterministic. This is without loss of generality.

10 For every mechanism $\Gamma = (\mathcal{B}, \pi, \mu)$ and Bayesian Nash equilibrium $\sigma$ of $\Gamma$, there exists a direct mechanism $\Gamma' = (q, t)$ and a Bayesian Nash equilibrium $\sigma'$ of $\Gamma'$ such that:

(i) For every $i$ and every $\theta_i$, the strategy vector $\sigma'$ satisfies $\sigma'_i(\theta_i) = \theta_i$, that is, $\sigma'$ prescribes telling the truth;

(ii) For every type vector $\theta$, the distribution over outcomes that result under $\Gamma$ if the agents play $\sigma$ is the same as the distribution over outcomes that result under $\Gamma'$ if the agents play $\sigma'$, and the expected value of the transfer payments that result under $\Gamma$ if the agents play $\sigma$ is the same as the transfer payments that result under $\Gamma'$ if the agents play $\sigma'$.

*Proof.* (1) Suppose that $\Gamma = (\mathcal{B}, \pi, \mu)$ is a mechanism and $\sigma$ is a Bayesian Nash equilibrium of this mechanism.

(2) Let $\Gamma' = (q, t)$ be defined as follows:

$$q(\theta) = \pi\big(\sigma(\theta)\big) \text{ and } t(\theta) = \mu\big(\sigma(\theta)\big).$$

$$\begin{tikzcd}
& \text{Messages } \mathcal{B} \arrow[dr, "(\pi,\mu)"] & \\
\text{Values} \arrow[ur, "\sigma"] \arrow[rr, "(\pi,\mu)\circ\sigma"'] & & \text{Outcomes}
\end{tikzcd}$$

Figure 1: Revelation principle for Bayesian Nash equilibrium

(3) It is clear that

$$\begin{aligned}
& \mathbf{E}_{\theta_{-i}}\big[q_i(\sigma_i'(\theta_i),\sigma_{-i}'(\theta_{-i}))\cdot\theta_i - t_i(\sigma_i'(\theta_i),\sigma_{-i}'(\theta_{-i}))\big] \\
={}& \mathbf{E}_{\theta_{-i}}\big[q_i(\theta_i,\theta_{-i})\cdot\theta_i - t_i(\theta_i,\theta_{-i})\big] \\
={}& \mathbf{E}_{\theta_{-i}}\big[\pi_i\big(\sigma_i(\theta_i),\sigma_{-i}(\theta_{-i})\big)\cdot\theta_i - \mu_i\big(\sigma_i(\theta_i),\sigma_{-i}(\theta_{-i})\big)\big] \\
\geq{}& \mathbf{E}_{\theta_{-i}}\big[\pi_i\big(\sigma_i(\theta_i'),\sigma_{-i}(\theta_{-i})\big)\cdot\theta_i - \mu_i\big(\sigma_i(\theta_i'),\sigma_{-i}(\theta_{-i})\big)\big] \\
={}& \mathbf{E}_{\theta_{-i}}\big[q_i(\theta_i',\theta_{-i})\cdot\theta_i - t_i(\theta_i',\theta_{-i})\big] \\
={}& \mathbf{E}_{\theta_{-i}}\big[q_i(\sigma_i'(\theta_i'),\sigma_{-i}'(\theta_{-i}))\cdot\theta_i - t_i(\sigma_i'(\theta_i'),\sigma_{-i}'(\theta_{-i}))\big]
\end{aligned}$$

for all $\theta_i' \in [\underline{\theta},\bar{\theta}]$, which implies that truthfully reporting is an equilibrium strategy for each buyer in the direct mechanism $(q,t)$, and its outcome is the same as in $\sigma$.

□

11 Remark: This result shows that the outcomes resulting from any equilibrium of any mechanism can be replicated by a truthful equilibrium of some direct mechanism. In this sense, there is no loss of generality in restricting attention to direct mechanisms.

## 2 Incentive compatibility and individual rationality

12 Given a direct mechanism $(q,t)$, define for each agent $i \in I$ a function $Q_i\colon [\underline{\theta},\bar{\theta}] \to [0,1]$ by setting:

$$Q_i(\theta_i) = \int_{\Theta_{-i}} q_i(\theta_i,\theta_{-i}) f_{-i}(\theta_{-i})\,\mathrm{d}\theta_{-i}.$$

Thus, $Q_i(\theta_i)$ is the conditional expected probability that agent $i$ obtains the good, conditioning on agent $i$'s type being $\theta_i$.

13 Similarly, define for each agent $i \in I$ a function $T_i\colon [\underline{\theta},\bar{\theta}] \to \mathbb{R}$ by setting:

$$T_i(\theta_i) = \int_{\Theta_{-i}} t_i(\theta_i,\theta_{-i}) f_{-i}(\theta_{-i})\,\mathrm{d}\theta_{-i}.$$

Thus, $T_i(\theta_i)$ is the conditional expected transfer that agent $i$ makes to the seller, conditioning on agent $i$'s type being $\theta_i$.

14 Note that both the probability of getting the object and the expected payment depend only on the reported value and not on the true value.

15 The expected payoff of buyer $i$ when his true value is $\theta_i$ and he reports $\theta_i'$, again assuming that all other buyers tell the truth, can then be written as

$$Q_i(\theta_i') \cdot \theta_i - T_i(\theta_i').$$

16 The direct mechanism $(q, t)$ is said to be Bayesian incentive compatible if truth telling is a Bayesian Nash equilibrium; that is, if for all $i$, for all $\theta_i$ and for all $\theta_i'$,

$$U_i(\theta_i) \triangleq Q_i(\theta_i) \cdot \theta_i - T_i(\theta_i) \geq Q_i(\theta_i') \cdot \theta_i - T_i(\theta_i').$$

We will refer to $U_i$ as the equilibrium payoff function.

17 Lemma: A direct mechanism is Bayesian incentive compatible, then for every $i$, the function $Q_i$ is increasing.

*Proof.* (1) Consider $\theta_i$ and $\theta_i'$ with $\theta_i > \theta_i'$.

(2) Bayesian incentive compatibility requires

$$Q_i(\theta_i) \cdot \theta_i - T_i(\theta_i) \geq Q_i(\theta_i') \cdot \theta_i - T_i(\theta_i') \text{ and } Q_i(\theta_i') \cdot \theta_i' - T_i(\theta_i') \geq Q_i(\theta_i) \cdot \theta_i' - T_i(\theta_i).$$

(3) Then we have

$$[Q_i(\theta_i) - Q_i(\theta_i')] \cdot (\theta_i - \theta_i') \geq 0,$$

and hence $Q_i(\theta_i) \geq Q_i(\theta_i')$.

□

18 Lemma: If a direct mechanism is Bayesian incentive compatible, then for every $i$, the function $U_i$ is increasing. It is also convex, and hence differentiable except in at most countably many points. For all $\theta_i$ for which it is differentiable, it satisfies

$$U_i'(\theta_i) = Q_i(\theta_i).$$

*Proof.* (1) Bayesian incentive compatibility implies that for all $\theta_i$,

$$U_i(\theta_i) = \max_{\theta_i' \in [\underline{\theta}, \overline{\theta}]} \big(Q_i(\theta_i') \cdot \theta_i - T_i(\theta_i')\big).$$

(2) Given any value of $\theta_i'$, $Q_i(\theta_i') \cdot \theta_i - T_i(\theta_i')$ is an increasing and affine (and hence convex) function.

(3) The maximum of increasing functions is increasing, and the maximum of convex functions is convex. Therefore, $U_i$ is increasing and convex.

(4) Convex functions are not differentiable in at most countably many points.

(5) Then, by envelope theorem, we have $U_i'(\theta_i) = Q_i(\theta_i)$ whenever $U_i$ is differentiable.

□

Remark: Bayesian incentive compatibility is equivalent to the requirement that for all $\theta_i$ and $\theta_i'$,

$$U_i(\theta_i') \geq U_i(\theta_i') + Q_i(\theta_i) \cdot (\theta_i' - \theta_i).$$

This implies that for all $\theta_i$, $Q_i(\theta_i)$ is the slope of a line that supports the function $U_i$ at the point $\theta_i$.

19 Proposition (Payoff equivalence): Consider a direct Bayesian incentive compatible mechanism. Then for all $i$ and all $\theta_i$, we have

$$U_i(\theta_i) = U_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i(x)\,\mathrm{d}x.$$

*Proof.* Since $U_i$ is convex, it is absolutely continuous. Since for all $\theta_i$ for which $U_i$ is differentiable, it satisfies $U_i'(\theta_i) = Q_i(\theta_i)$, we have

$$U_i(\theta_i) - U_i(\underline{\theta}) = \int_{\underline{\theta}}^{\theta_i} U_i'(\theta_i)\,\mathrm{d}x = \int_{\underline{\theta}}^{\theta_i} Q_i(x)\,\mathrm{d}x.$$

□

20 Remark: Proposition **??** shows that the interim expected payoffs of the different buyer values are pinned down by the functions $Q_i$ and the expected payoff of the lowest value. That is, Proposition **??** implies that up to an additive constant, the interim expected payoff to a buyer in a Bayesian incentive compatible direct mechanism $(q, t)$ depends only on the allocation rule $q$.

If $(q, t)$ and $(q, \bar{t})$ are two Bayesian incentive compatible mechanisms with the same allocation rule $q$ but different payment rules, then the expected payoff functions associated with the two mechanisms, $U_i$ and $\bar{U}_i$, respectively, differ by at most a constant; the two mechanisms are payoff equivalent. Put another way, the "shape" of the expected payoff function is completely determined by the allocation rule $q$ alone. The payment rule $t$ only serves to determine the constants $U_i(\underline{\theta})$.

21 Proposition: A direct mechanism $(q, t)$ is Bayesian incentive compatible if and only if for every $i$

(i) $Q_i$ is increasing;

(ii) For every $\theta_i$,

$$U_i(\theta_i) = U_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i(x)\,\mathrm{d}x.$$

*Proof.* (1) Let $\theta_i > \theta_i'$.

(2) Since $Q_i$ is increasing, we have

$$\int_{\theta_i'}^{\theta_i} Q_i(x)\,\mathrm{d}x \geq \int_{\theta_i'}^{\theta_i} Q_i(\theta_i')\,\mathrm{d}x = Q_i(\theta_i') \cdot (\theta_i - \theta_i').$$

(3) Since

$$\int_{\theta_i'}^{\theta_i} Q_i(x)\,\mathrm{d}x = \left[\int_{\underline{\theta}}^{\theta_i} - \int_{\underline{\theta}}^{\theta_i'}\right] Q_i(x)\,\mathrm{d}x = U_i(\theta_i) - U_i(\theta_i'),$$

we have

$$U_i(\theta_i) - U_i(\theta_i') \geq Q_i(\theta_i') \cdot (\theta_i - \theta_i').$$

(4) Then

$$U_i(\theta_i) \geq Q_i(\theta_i') \cdot (\theta_i - \theta_i') + U_i(\theta_i') = Q_i(\theta_i') \cdot (\theta_i - \theta_i') + Q_i(\theta_i') \cdot \theta_i' - T_i(\theta_i') = Q_i(\theta_i') \cdot \theta_i - T_i(\theta_i').$$

(5) If $\theta_i < \theta_i'$, the argument is analogous.

□

22 Proposition (Revenue equivalence): Consider a direct Bayesian incentive compatible mechanism. Then for all $i$ and all $\theta_i$, we have

$$T_i(\theta_i) = T_i(\underline{\theta}) + Q_i(\theta_i) \cdot \theta_i - Q_i(\underline{\theta}) \cdot \underline{\theta} - \int_{\underline{\theta}}^{\theta_i} Q_i(x)\,\mathrm{d}x.$$

*Proof.* Since $U_i(\theta_i) = Q_i(\theta_i) \cdot \theta_i - T_i(\theta_i)$ and $U_i(\underline{\theta}) = Q_i(\underline{\theta}) \cdot \underline{\theta} - T_i(\underline{\theta})$, by Proposition **??**, we have

$$Q_i(\theta_i) \cdot \theta_i - T_i(\theta_i) = Q_i(\underline{\theta}) \cdot \underline{\theta} - T_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i(x)\,\mathrm{d}x.$$

□

23 Proposition **??** shows similarly that the interim expected payments of the different buyer values are pinned down by the functions $Q_i$ and the expected payment of the lowest value. Note that this does not mean that the ex post payment functions $t_i$ are uniquely determined. Different functions $t_i$ might give rise to the same interim expected payments $T_i$.

24 Proposition: A direct mechanism $(q, t)$ is Bayesian incentive compatible if and only if for every $i$

(i) $Q_i$ is increasing;

(ii) For every $\theta_i$,

$$T_i(\theta_i) = T_i(\underline{\theta}) + Q_i(\theta_i) \cdot \theta_i - Q_i(\underline{\theta}) \cdot \underline{\theta} - \int_{\underline{\theta}}^{\theta_i} Q_i(x)\,\mathrm{d}x. \tag{1}$$

*Proof.* Similar with proof of Proposition **??**. □

25 Remark: We have now obtained a complete understanding of the implications of Bayesian incentive compatibility for the seller's choice. The seller can focus on two choice variables: firstly the allocation rule $q$, and secondly the interim expected payment by a buyer with the lowest type: $T_i(\underline{\theta})$.

As long as the seller picks an allocation rule $q$ such that the functions $\{Q_i\}_{i \in I}$ are increasing, she can pick the interim expected payments by the lowest values in any arbitrary way, and be assured that there will be some transfer scheme that makes the allocation rule Bayesian incentive compatible and that implies the given interim expected payments by the lowest values. Moreover, any such transfer scheme will give she the same expected revenue, and therefore the seller does not have to worry about the details of this transfer scheme.

26 A direct mechanism is individually rational if each agent, conditional on her type, is willing to participate, that is, if

$$U_i(\theta_i) \geq 0 \text{ for all } i \text{ and } \theta_i.$$

We are implicitly assuming here that by not participating, a buyer can guarantee herself a payoff of zero.

27 Proportion: A Bayesian incentive compatible direct mechanism is individually rational if and only if for every $i$, we have

$$U_i(\underline{\theta}) \geq 0.$$

*Proof.* $U_i$ is increasing for Bayesian incentive compatible direct mechanisms. Therefore, $U_i(\theta_i)$ is non-negative for all $\theta_i$ if and only if it is non-negative for the lowest $\theta_i$, which is zero. □

Since $U_i(\underline{\theta}) = Q_i(\underline{\theta}) \cdot \underline{\theta} - T_i(\underline{\theta})$, this is equivalent to the requirement that $T_i(\underline{\theta}) \leq Q_i(\underline{\theta}) \cdot \underline{\theta}$.

# 3 Optimal auction

28 In this section we view the seller as the designer of the mechanism and examine mechanisms that maximize the expected revenue—the sum of the expected payments of the buyers—among all mechanisms that are Bayesian incentive compatible and individually rational. We reiterate that when carrying out this exercise, the revelation principle guarantees that there is no loss of generality in restricting attention to direct mechanisms. Suppose that the seller uses the direct mechanism $(q, t)$.

We will refer to a mechanism that maximizes expected revenue, subject to the Bayesian incentive compatibility and individual rationality constraints, as an optimal mechanism.

29 If a Bayesian incentive compatible and individually rational direct mechanism maximizes the seller's expected revenue, then for every $i \in I$:

$$T_i(\underline{\theta}) = Q_i(\underline{\theta})\underline{\theta}.$$

Otherwise, the seller could increase expected revenue by choosing a direct mechanism with the same $q$, but a higher $T_i(\underline{\theta})$. Then all values' payments would increase.

30 We can now simplify the seller's problem further by :

$$T_i(\theta_i) = \theta_i Q_i(\theta_i) - \int_{\underline{\theta}}^{\theta_i} Q_i(x)\,\mathrm{d}x.$$

31 Seller's expected revenue from buyer $i$ is

$$\begin{aligned}
\int_{\underline{\theta}}^{\bar{\theta}} T_i(\theta_i) f_i(\theta_i)\,\mathrm{d}\theta_i &= \int_{\underline{\theta}}^{\bar{\theta}} \theta_i Q_i(\theta_i) f_i(\theta_i)\,\mathrm{d}\theta_i - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta_i} Q_i(x)\,\mathrm{d}x f_i(\theta_i)\,\mathrm{d}\theta_i \\
&= \int_{\underline{\theta}}^{\bar{\theta}} \theta_i Q_i(\theta_i) f_i(\theta_i)\,\mathrm{d}\theta_i - \int_{\underline{\theta}}^{\bar{\theta}} \int_{x}^{\bar{\theta}} f_i(\theta_i)\,\mathrm{d}\theta_i Q_i(x)\,\mathrm{d}x \\
&= \int_{\underline{\theta}}^{\bar{\theta}} \theta_i Q_i(\theta_i) f_i(\theta_i)\,\mathrm{d}\theta_i - \int_{\underline{\theta}}^{\bar{\theta}} [F_i(\bar{\theta}) - F_i(x)] Q_i(x)\,\mathrm{d}x \\
&= \int_{\underline{\theta}}^{\bar{\theta}} \theta_i Q_i(\theta_i) f_i(\theta_i)\,\mathrm{d}\theta_i - \int_{\underline{\theta}}^{\bar{\theta}} [1 - F_i(\theta_i)] Q_i(\theta_i)\,\mathrm{d}\theta_i \\
&= \int_{\underline{\theta}}^{\bar{\theta}} Q_i(\theta_i) \left[\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}\right] f_i(\theta_i)\,\mathrm{d}\theta_i.
\end{aligned}$$

32 The total expected transfer by all buyers is

$$\begin{aligned}
&\sum_{i \in I} \int_{\underline{\theta}}^{\bar{\theta}} Q_i(\theta_i) \left[\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}\right] f_i(\theta_i)\,\mathrm{d}\theta_i \\
= &\sum_{i \in I} \int_{\underline{\theta}}^{\bar{\theta}} \int_{\Theta_{-i}} q_i(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i})\,\mathrm{d}\theta_{-i} \left[\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}\right] f_i(\theta_i)\,\mathrm{d}\theta_i \\
= &\sum_{i \in I} \int_{\Theta} q_i(\theta) \left[\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}\right] f(\theta)\,\mathrm{d}\theta.
\end{aligned}$$

33 The seller's objective therefore is to find a mechanism that maximizes

$$\sum_{i \in I} \int_{\Theta} q_i(\theta) \left[ \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right] f(\theta) \, \mathrm{d}\theta,$$

subject to the constraint that the mechanism is Bayesian incentive compatible and individually rational.

34 We first ask which function $q$ the seller would choose if she did not have to make sure that the functions $Q_i$ are increasing. In a second step, we introduce an assumption that makes sure that the optimal $q$ from the first step implies increasing functions $Q_i$.

(1) Let

$$\psi_i(\theta_i) = \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \text{ for all } i \text{ and } \theta_i,$$

which is referred to as virtual valuation of a buyer with value $\theta_i$.

(2) Focus on

$$\max_q \int_{\Theta} \Big[ \sum_{i \in I} \psi_i(\theta_i) q_i(\theta) f(\theta) \, \mathrm{d}\theta \Big],$$

then the optimal choice of $q$ without constraints is: for all $i$ and $\theta$,

$$q_i(\theta) \begin{cases} > 0, & \text{if } \psi_i(\theta_i) = \max_{j \in I} \psi_j(\theta_j) \geq 0, \\ = 0, & \text{otherwise.} \end{cases}$$

35 Condition of regularity: for every $i$, the function $\psi_i(\theta_i)$ is strictly increasing.

Since

$$\psi_i(\theta_i) = \theta_i - \frac{1}{\lambda_i(\theta_i)},$$

where $\lambda_i = \frac{f_i}{1 - F_i}$ is the hazard rate function associated with $F_i$, a sufficient condition for regularity is that for all $i$, $\lambda_i$ is increasing.

36 Lemma: If $\psi_i(\theta_i)$ is strictly increasing, then $Q_i$ is increasing.

*Proof.* (1) Suppose $\theta_i' < \theta_i$. Then by the regularity condition, $\psi_i(\theta_i') < \psi_i(\theta_i)$.

(2) For any $\theta_{-i}$, if $q_i(\theta_i', \theta_{-i}) > 0$, then

$$\psi_i(\theta_i') = \max_{j \in I} \psi_j(\theta_j) \geq 0,$$

and hence

$$\psi_i(\theta_i) > \psi_i(\theta_i') \geq \max_{j \neq i} \psi_j(\theta_j) \geq 0.$$

Therefore, $q_i(\theta_i, \theta_{-i}) = 1 \geq q_i(\theta_i', \theta_{-i})$.

(3) If $q_i(\theta_i', \theta_{-i}) = 0$, it means the virtual value of bidder $i$ with value $\theta_i'$ is not the highest. Now when her value is $\theta_i$, the virtual value is either still not the highest, which gives zero, or the virtual value becomes the highest among all bidders which give strictly positive number. Thus $q_i(\theta_i, \theta_{-i}) \geq q_i(\theta_i', \theta_{-i})$.

(4) Therefore, $Q_i$ is an increasing function.

□

37 Theorem: Suppose the design problem is regular. Then the following is an optimal mechanism:

$$q_i(\theta) \begin{cases} \geq 0, & \text{if } \psi_i(\theta_i) = \max_{j \in I} \psi_j(\theta_j) \geq 0, \\ = 0, & \text{otherwise,} \end{cases}$$

and

$$T_i(\theta_i) = Q_i(\theta_i) \cdot \theta_i - \int_{\underline{\theta}}^{\theta_i} Q_i(x)\,\mathrm{d}x.$$

*Proof.* It is clear that $(q,t)$ is Bayesian incentive compatible and individually rational. It is optimal, since it separately maximizes

$$\sum_{i \in I} T_i(\underline{\theta}) \text{ and } \sum_{i \in I} \int_{\Theta} \psi_i(\theta_i) q_i(\theta) f(\theta)\,\mathrm{d}\theta$$

over all $q\colon \Theta \to \Delta$. In particular, it gives positive weight only to non-negative and maximal terms in

$$\sum_{i \in I} \psi_i(\theta_i) q_i(\theta).$$

□

38 Remark: We have characterized the optimal choice of the allocation rule $q$ and of the interim expected payments. We have not described the actual transfer schemes that make these choices Bayesian incentive compatible and individually rational.

# 4 Maximizing welfare

39 Suppose that the seller is not maximizing expected revenue but expected welfare. So the seller uses the following utilitarian welfare function, where each agent has equal weight:

$$\sum_{i \in I} q_i(\theta) \cdot \theta_i.$$

Note that this seller is no longer concerned with transfer payments, and expected welfare depends only on the allocation rule $q$.

40 By Lemma **??**, the seller can choose any rule $q$ that is such that the functions $Q_i$ are increasing. By Proposition **??**, she can choose any transfer payments such that $T_i(\underline{\theta}) \leq \underline{\theta} Q_i(\underline{\theta})$ for all $i$.

41 If values were known, maximization of the welfare function would require that the object be allocated to the potential buyer for whom $\theta_i$ is largest.

Because transferring to the buyer for whom $\theta_i$ is largest maximizes welfare for every type vector, it also maximizes expected welfare.

In this case, it is clear that $Q_i$ is increasing.

42 Proposition: Among all Bayesian incentive compatible, individually rational direct mechanisms, a mechanism maximizes expected welfare if and only if for all $i$ and all $\theta$:

(i)

$$q_i(\theta) = \begin{cases} 1, & \text{if } \theta_i > \theta_j \text{ for all } j \neq i, \\ 0, & \text{otherwise.} \end{cases}$$

(ii)

$$T_i(\theta_i) \leq Q_i(\theta_i) \cdot \theta_i - \int_{\underline{\theta}}^{\theta_i} Q_i(x)\,\mathrm{d}x.$$

43 Remark: Note that this result does not rely on regularity condition.

44 Differences between welfare maximizing and revenue maximizing mechanisms in the case that regularity condition holds.

- Revenue maximizing mechanism allocates the object to the highest virtual type whereas the welfare maximizing mechanism allocates the object to the highest actual type. In the symmetric case, the functions $\psi_i$ are the same for all $i$ and there is no difference between these two rules. But in the asymmetric case the revenue maximizing mechanism might allocate the object inefficiently.
- Revenue maximizing mechanism sometimes does not sell the object at all, whereas the welfare maximizing mechanism always sells the object. This is an instance of the well-known inefficiency that monopoly sellers make goods artificially scarce.

45 Example: Suppose that $[\underline{\theta}, \bar{\theta}] = [0, 1]$, and that $\theta_1$ and $\theta_2$ are independently and uniformly distributed on $[0, 1]$. Then $F_i(\theta_i) = \theta_i$ and $f_i(\theta_i) = 1$, and

$$\psi_i(\theta_i) = \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} = 2\theta_i - 1.$$

Note that the regularity condition is satisfied.

In an expected revenue maximizing auction, the good is sold to neither bidder if

$$\psi_1(\theta_1), \psi_2(\theta_2) < 0,$$

that is,

$$\theta_1, \theta_2 < \tfrac{1}{2}.$$

If the good is sold, it is sold to bidder 1 if and only if

$$\psi_1(\theta_1) > \psi_2(\theta_2) \Leftrightarrow \theta_1 > \theta_2.$$

The expected revenue maximizing auction will allocate the object to the buyer with the highest value provided that this value is larger than $\frac{1}{2}$. A first- or second-price auction with reserve price $\frac{1}{2}$ will implement this mechanism.

# 5 Homework

- Reading: Börgers 3.2
- Homework: Suppose that $[\underline{\theta}, \bar{\theta}] = [0, 1]$, and that $F_1(\theta_1) = \theta_1^2$ and $F_2(\theta_2) = 2\theta_2 - \theta_2^2$. Find the optimal auction mechanism. Does this mechanism maximize the welfare?