Advanced Microeconomics: Lecture 7

1 A seller has one indivisible object to sell and there are $N \geq 2$ risk-neutral potential buyers from the set $I = \{1, 2, \ldots, N\}$.

2 Buyer i's utility if he purchases the good and pays a transfer t_i to the seller is $\theta_i - t_i$. Buyer i's utility if he does not purchase the good and pays a transfer of t_i to the seller is $0 - t_i$.

The seller's utility if she obtains transfers t_i from buyer i ($i \in I$) is $\sum_{i \in I} t_i$.

3 Buyer i knows θ_i, but neither the seller nor any other buyer $j \neq i$ knows θ_i.

We model θ_i as a random variable with cumulative distribution function F_i with density f_i. The support of θ_i is $[\underline{\theta}, \bar{\theta}]$ where $0 \leq \underline{\theta} < \bar{\theta}$. The distributions F_i are common knowledge among the buyers and the seller.

For technical convenience, we also assume that $f_i(\theta_i) > 0$ for all $i \in I$ and all $\theta_i \in [\underline{\theta}, \bar{\theta}]$.

We also assume that for $i \neq j$, θ_i and θ_j are independent.

4 Notations:

- $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ and $\Theta = [\underline{\theta}, \bar{\theta}]^N$.
- $\Theta_{-i} = [\underline{\theta}, \bar{\theta}]^{N-1}$ and $f_{-i}(\theta_{-i}) = \Pi_{j \neq i} f_j(\theta_j)$.
- $\Delta = \{(q_1, q_2, \ldots, q_N) \mid 0 \leq q_i \leq 1 \text{ for all } i \in I \text{ and } \sum_{i \in I} q_i \leq 1\}$.

1 Mechanism and the revelation principle

5 In general, a selling mechanism $(\mathcal{B}, \pi, \mu)$ has the following components:

- a set of possible messages $\mathcal{B}_i$ for each buyer; $\mathcal{B} = \times_{i \in I} \mathcal{B}_i$;
- an allocation rule $\pi \colon \mathcal{B} \to \Delta$;
- a payment rule $\mu \colon \mathcal{B} \to \mathbb{R}^N$.

An allocation rule determines, as a function of all N messages, the probability $\mu_i(b)$ that i will get the object.

A payment rule determines, as a function of all N messages, for each buyer i, the expected payment $\mu_i(b)$ that i must make.

6 A direct mechanism (q, t) consists of functions q (allocation rule) and t_i (transfer) (for $i \in I$)

$$q \colon \Theta \to \Delta \text{ and } (t_1, t_2, \ldots, t_N) \colon \Theta \to \mathbb{R}^N.$$

The interpretation is that in a direct mechanism the buyers are asked to simultaneously and independently report their types.

7 Proposition (Revelation principle for dominant strategy mechanisms): Suppose a mechanism $\Gamma = (\mathcal{B}, \pi, \mu)$ and a strategy combination σ for Γ are such that for each type θ_i of each buyer i, the strategy $\sigma_i(\theta_i)$ is a dominant strategy in Γ. Then there exists a direct mechanism Γ' and a strategy combination σ' for Γ' such that for every type θ_i of each buyer i the strategy $\sigma'_i(\theta_i)$ is a dominant strategy in Γ', and:

(i) The strategy vector σ' satisfies for every i and every θ_i:

$$\sigma'_i(\theta_i) = \theta_i,$$

that is, σ' prescribes telling the truth;

(ii) For every vector θ of types, the distribution over allocations and the expected payments that result under Γ if the agents play σ is the same as the distribution over allocations and the expected payments that result under Γ' if the agents play σ'.

Proof. Let $q = \pi \circ \sigma$ and $t = \mu \circ \sigma$.

$$\begin{tikzcd} & \text{Messages } \mathcal{B} \arrow[dr, "(\pi, \mu)"] & \\ \text{Types/Values} \arrow[ur, "\sigma"] \arrow[rr, "(\pi, \mu) \circ \sigma"'] & & \text{Outcomes} \end{tikzcd}$$

Figure 1: Revelation principle for dominant strategy mechanisms

□

8 Remark: This result shows that the outcomes resulting from any equilibrium of any mechanism can be replicated by a truthful equilibrium of some direct mechanism. In this sense, there is no loss of generality in restricting attention to direct mechanisms.

2 Incentive compatibility and individual rationality

9 A direct mechanism (q, t) is dominant strategy incentive compatible if truth telling is a dominant strategy for each type of each buyer, that is, for all $i \in I$, all θ, θ' and all θ_{-i},

$$\theta_i q_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i}) \geq \theta_i q_i(\theta'_i, \theta_{-i}) - t_i(\theta'_i, \theta_{-i}).$$

10 A direct mechanism (q, t) is ex post individually rational if for each type of each buyer participation is a dominant strategy, that is, if for all $i \in I$, all θ_i and all θ_{-i},

$$\theta_i q_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i}) \geq 0.$$

11 Proposition: A direct mechanism (q, t) is dominant strategy incentive compatible if and only if for every i and every θ_{-i},

(i) $q_i(\theta_i, \theta_{-i})$ is increasing in θ_i.

(ii) for every θ_i,

$$t_i(\theta_i, \theta_{-i}) = t_i(\underline{\theta}, \theta_{-i}) + \theta_i q_i(\theta_i, \theta_{-i}) - \underline{\theta} q_i(\underline{\theta}, \theta_{-i}) + \int_{\underline{\theta}}^{\theta_i} q_i(\theta_i', \theta_{-i})\, \mathrm{d}\theta_i'.$$

12 Proposition: A dominant strategy incentive compatible direct mechanism (q, t) is ex post individually rational if and only if for every i and every θ_{-i} we have

$$t_i(\underline{\theta}, \theta_{-i}) \leq \underline{\theta} q_i(\underline{\theta}, \theta_{-i}).$$

13 A direct mechanism (q, t) is called a canonical auction if there are strictly increasing and continuous functions $\psi_i \colon \Theta_i \to \mathbb{R}$ for i such that for all θ and i,

$$q_i(\theta) = \begin{cases} \frac{1}{n}, & \text{if } \psi_i(\theta_i) \geq 0 \text{ and } \psi_i(\theta_i) \geq \psi_j(\theta_j) \text{ for all } j \neq i, \\ 0, & \text{otherwise,} \end{cases}$$

where n is the number of agents k such that $\psi_k(\theta_k) = \psi_i(\theta_i)$, and

$$t_i(\theta) = \begin{cases} \frac{1}{n} \min\{\theta_i' \mid q_i(\theta_i', \theta_{-i}) > 0\}, & \text{if } q_i(\theta) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

for all θ.

14 It is worth considering the transfer rule in detail. If bidder i does not win the auction, either because her bid is too low, or because she won, but there was a tie and she was not selected, then bidder i does not have to pay anything. If bidder i does win the auction, then bidder i's payment equals the lowest type that she might have had that would have allowed her to win the auction with positive probability.

The assumed continuity of ψ_i guarantees that this minimum exists. If bidder i wins with probability 1, then the minimum will either be the value of θ_i at which i would have tied, or, if no such value exists, $\underline{\theta}$. If bidder i ties, then the minimum equals her θ_i.

15 Proposition: Every canonical auction (q, t) is DSIC and ex post IR. Moreover, for every i, $u_i(\underline{\theta}, \theta_{-i}) = 0$ for all θ_{-i}.

16 *Proof.* Case 1: Suppose that buyer i is of type θ_i, that the other buyers have types θ_{-i}, and that $q_i(\theta_i, \theta_{-i}) = 0$. Does buyer i have an incentive to report a different type θ_i'?

- If $q_i(\theta_i', \theta_{-i}) = 0$, then her utility doesn't change.
- If $q_i(\theta_i', \theta_{-i}) = \frac{1}{n} > 0$, it will have to be the case that $\theta_i' > \theta_i$. Moreover, buyer i's payment will be larger than $\frac{1}{n}\theta_i$, as her payment will be the lowest type of buyer i that wins against θ_{-i}, and by assumption θ_i is not large enough. Thus, buyer i can win the auction, but only by paying more than the object is worth to her.

Thus, she has no incentive to change her strategy.

Case 2: Consider next the case that $q_i(\theta_i, \theta_{-i}) = 1$. Note first that reporting θ_i truthfully yields strictly positive utility, because buyer i obtains the object with probability 1 and pays less than θ_i.

- If buyer i changes her report to another type θ_i' for which $q_i(\theta_i', \theta_{-i}) = 1$, then her utility doesn't change, as her payment does not depend on her report.

- If she changes her report to a type θ_i' for which $0 < q_i(\theta_i', \theta_{-i}) < 1$, i.e. there is a tie, her utility decreases because she obtains the object with probability less than 1, and if she obtains it, pays the same as she would if she reported truthfully. In other words: she obtains the positive surplus that results from truthful reporting with probability less than 1.
- Lastly, if buyer i reports a type θ_i' for which $q_i(\theta_i', \theta_{-i}) = 0$, then she loses the positive surplus.

Case 3: Consider finally the case that $q_i(\theta_i, \theta_{-i}) = \frac{1}{n}$ where $n \geq 2$. Then buyer i's transfer payment is $\frac{1}{n}\theta_i$, and her expected utility will be zero.

- If buyer i changes her report to another type $\theta_i' > \theta_i$ so that $q_i(\theta_i', \theta_{-i}) = 1$, then her expected utility doesn't change, as her payment will be θ_i.
- If she changes her report to a type θ_i' for which $q_i(\theta_i', \theta_{-i}) = 0$, her expected utility is again zero.

Hence, a truthful report is optimal.

The proof of dominant strategy incentive compatibility also shows that a buyer's utility is always non-negative if she wins the auction. If she loses the auction, her utility is zero. Therefore, the mechanism also satisfies ex post individual rationality.

The lowest type, $\underline{\theta}$, either loses the auction, and has utility zero, or wins the auction and has to pay $\underline{\theta}$, in which case utility is also zero. □

17 Remark: We don't claim that canonical auctions are the only dominant strategy incentive compatible and ex post individually rational direct mechanisms, but we show that this class is rich enough to include some of the mechanisms that we identified previously as expected revenue maximizing and as expected welfare maximizing.

We show that for any such functions, an allocation rule that is constructed as in the expected revenue maximizing auction can be supplemented with transfer rules that make the mechanism dominant strategy incentive compatible and ex post individually rational. The advantage of generalizing the result in this way is that we can use it to also show the implementability of allocation rules other than the expected revenue maximizing.

3 VCG/Pivot mechanism

18 To implement the efficient (welfare maximizing) allocation rule $q\colon \Theta \to \Delta$, the transfer rule in canonical auction can be determined as follows: taking $\psi_i(\theta_i)$ to be θ_i and

$$t_i(\theta) = \begin{cases} \max_{j \neq i} \theta_j, & \text{if } \theta_i > \max_{j \neq i} \theta_j, \\ \frac{1}{n}\theta_i, & \text{if } \theta_i \text{ is one of the } n \text{ largest ones in } \theta, \\ 0, & \text{otherwise.} \end{cases}$$

19 Consider another transfer rule $t^{\text{pivot}}\colon \Theta \to \Delta$:

$$t_i^{\text{pivot}}(\theta) = -\sum_{j \neq i} q_j(\theta)\theta_j + \max_{p_{-i}} \sum_{j \neq i} p_j \theta_j.$$

This transfer rule t^{pivot} is the same as t obtained above:

- If $\theta_i > \max_{j \neq i} \theta_j$, then $t_i^{\text{pivot}}(\theta) = 0 - \max_{j \neq i} \theta_j$.
- If $\theta_i = \max_{j \neq i} \theta_j$, then $t_i^{\text{pivot}}(\theta) = -\frac{n-1}{n} \max_{j \neq i} \theta_j + \max_{j \neq i} \theta_j = \frac{1}{n}\theta_i$.

- If $\theta_i < \max_{j\neq i}\theta_j$, then $t_i^{\text{pivot}}(\theta) = -\max_{j\neq i}\theta_j + \max_{j\neq i}\theta_j = 0$.

20
- The term $\max_{p_{-i}}\sum_{j\neq i} p_j\theta_j$ maximizes the sum of everyone else's value if i were ignored.
- The term $\sum_{j\neq i} q_j(\theta)\theta_j$ is the maximum sum of other agents' value when i is taken into account.

Agent i get paid everyone else's value under the allocation that is actually chosen and get charged everyone's value in the world where you do not participate. That is, agent i pays her social cost.

21
- If i's presence makes no difference in maximizing choice of q in two cases, then $t_i(\theta) = 0$, that is, agents who do not affect the outcome pay 0.
- Otherwise, we can think of i as being pivotal, and then t_i represents the loss in value that is imposed on the other agents due to the change in decision that results from i's presence in society.

22 Proposition: (q, t^{pivot}) is DSIC.

Proof. (1) Consider any agent i and take θ_{-i} as given.

(2) If agent i is of type θ_i, and reports that she is of type θ_i', then her utility is:

$$\begin{aligned} q_i(\theta_i', \theta_{-i})\theta_i - t_i^{\text{pivot}}(\theta_i', \theta_{-i}) &= q_i(\theta_i', \theta_{-i})\theta_i + \sum_{j\neq i} q_j(\theta_i', \theta_{-i})\theta_j - \max_{p_{-i}}\sum_{j\neq i} p_j\theta_j \\ &= \sum_{j\in I} q_j(\theta_i', \theta_{-i})\theta_j - \underbrace{\max_{p_{-i}}\sum_{j\neq i} p_j\theta_j}_{h_i(\theta_{-i})}. \end{aligned}$$

(3) Note that $h_i(\theta_{-i})$ is not changed by agent i's report. Only the first expression matters for i's incentives.

(4) Since q is efficient, we have

$$\sum_{j\in I} q_j(\theta_i, \theta_{-i})\theta_j \geq \sum_{j\in I} q_j(\theta_i', \theta_{-i})\theta_j$$

for all θ_i'.

(5) Therefore, it is optimal for agent i to report her true type.

□

23 Proposition: (q, t^{pivot}) is ex post IR.

Proof. Exercise. □

4 General VCG mechanism

24 Setup:

- There are N agents. The set of agents is denoted by $I = \{1, 2, \ldots, N\}$.
- The set of potential social decisions is denoted by D.
- Agent i's information is represented by a type θ_i which lies in a set Θ_i. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, and $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_N$.
- Agents have preferences over decisions that are represented by a utility function. Agent i's utility if decision d is chosen, and agent i pays transfer t_i is:

$$v_i(d, \theta_i) - t_i.$$

25 A decision/allocation rule is a mapping $d\colon \Theta \to D$.

A decision/allocation rule $d(\cdot)$ is efficient if

$$\sum_i v_i\big(d(\theta), \theta_i\big) \geq \sum_i v_i\big(d', \theta_i\big) \text{ for all } \theta \in \Theta \text{ and } d' \in D,$$

that is,

$$d(\theta) \in \arg\max_{d' \in D} \sum_i v_i(d', \theta_i) \text{ for all } \theta \in \Theta.$$

26 Agent i's transfer function is a mapping $t_i\colon \Theta \to \mathbb{R}$. $t_i(\theta)$ represents the payment that i receives based on the announcement of types θ. Let $t(\theta) = \big(t_1(\theta), t_2(\theta), \ldots, t_N(\theta)\big)$.

A transfer function t is said to be feasible if $\sum_i t_i(\theta) \geq 0$ for all θ.

A transfer function t is said to be balanced if $\sum_i t_i(\theta) = 0$ for all θ. (d, t) satisfies budget balance if the transfer function is balanced.

27 Definition: A direct mechanism (d, t^{VCG}) is called a Vickrey-Clarke-Groves mechanism if d is an efficient decision rule, and if for every i there is a function

$$h_i\colon \Theta_{-i} \to \mathbb{R},$$

such that

$$t_i^{\mathrm{VCG}}(\theta) = -\sum_{j \neq i} v_j\big(d(\theta), \theta_j\big) + h_i(\theta_{-i}) \text{ for all } \theta \in \Theta.$$

28 In a VCG mechanism each agent i is paid the sum of the other agents' value from the implemented alternative whereby utilities are calculated using the agents' reported types. This is the first term in the formula. This term aligned agent i's interests with utilitarian welfare.

The second term is a constant that depends on the other agents' reported types, and that does not affect agent i's incentives. This constant can be used to raise the overall revenue from the mechanism.

29 Proposition: VCG mechanisms are dominant strategy incentive compatible.

Proof. (1) Consider any agent i and take θ_{-i} as given.

(2) If agent i is of type θ_i, and reports that she is of type θ_i', then her utility is:

$$v_i\big(d(\theta_i', \theta_{-i}), \theta_i\big) + \sum_{j \neq i} v_j\big(d(\theta_i', \theta_{-i}), \theta_j\big) - h_i(\theta_{-i}) = \sum_{j \in I} v_j\big(d(\theta_i', \theta_{-i}), \theta_j\big) - h_i(\theta_{-i}).$$

(3) Note that $h_i(\theta_{-i})$ is not changed by agent i's report. Only the first expression matters for i's incentives.

(4) Since d is efficient, we have

$$\sum_{j \in I} v_j\big(d(\theta_i, \theta_{-i}), \theta_j\big) \geq \sum_{j \in I} v_j\big(d(\theta_i', \theta_{-i}), \theta_j\big)$$

for all θ_i'.

(5) Therefore, it is optimal for agent i to report her true type.

□

30 Remark: Every efficient social choice function can be truthfully implemented in a dominant strategy by a VCG mechanism.

31 Proposition: Suppose $\Theta_i = [\underline{\theta}_i, \bar{\theta}_i]$, $v_i(d, \cdot)$ is differentiable, and there exists $B > 0$ such that for all d and θ, $|\frac{\mathrm{d}v_i}{\mathrm{d}\theta_i}(d, \theta_i)| \leq B$. Suppose that (d, t) is a dominant strategy incentive compatible mechanism, and suppose that d is efficient. Then (d, t) is a VCG mechanism.

Proof. Every dominant strategy incentive compatible mechanism that implements an efficient decision rule d must involve the same transfers as the VCG mechanism up to additive constants $c_i(\theta_{-i})$ that may be added to any agent i's transfers. But adding such constants to a VCG mechanism yields by the definition of VCG mechanisms another VCG mechanism. □

32 Proposition: Suppose that for every i, the set Θ_i is a convex subset of a finite-dimensional Euclidean space. Moreover, assume that for every i the function $v_i(d, \theta_i)$ is a convex function of θ_i. Suppose that (d, t) is a dominant strategy incentive compatible mechanism, and suppose that d is efficient. Then (d, t) is a VCG mechanism.

4.1 Pivot mechanism

33 One version of VCG mechanism is called the pivot mechanism, where

$$h_i(\theta_{-i}) = \max_{d \in D} \sum_{j \neq i} v_j(d, \theta_j).$$

34 In the pivot mechanism, i's transfer becomes

$$t_i^{\text{pivot}}(\theta) = -\sum_{j \neq i} v_j\big(d(\theta), \theta_j\big) + \max_{d \in D} \sum_{j \neq i} v_j(d, \theta_j).$$

This transfer is always non-negative, and so the pivot mechanism is always feasible.

35
- The term $\max_{d \in D} \sum_{j \neq i} v_j(d, \theta_j)$ maximizes the sum of everyone else's value if i were ignored.
- The term $\sum_{j \neq i} v_j\big(d(\theta), \theta_j\big)$ is the maximum sum of other agents' value when i is taken into account.

Agent i get paid everyone else's value under the allocation that is actually chosen, i.e., $\sum_{j \neq i} v_j\big(d(\theta), \theta_j\big)$, and get charged everyone's value in the world where you do not participate. That is, agent i pays her social cost.

36
- If i's presence makes no difference in maximizing choice of d in two cases, then $t_i(\theta) = 0$, that is, agents who do not affect the outcome pay 0.
- Otherwise, we can think of i as being pivotal, and then t_i represents the loss in value that is imposed on the other agents due to the change in decision that results from i's presence in society.

37 Definition: A social choice function (d, t) is ex post individually rational if for each agent i, for each θ_i and θ_{-i},

$$v_i\big(d(\theta_i, \theta_{-i}), \theta_i\big) - t_i(\theta_i, \theta_{-i}) \geq 0.$$

38 Proposition: If the function v_i is always nonnegative, the pivot mechanism is ex post individually rational.

Proof. Routine. □

39 Proposition (Uniqueness of VCG transfers): Suppose $\Theta_i = [\underline{\theta}_i, \bar{\theta}_i]$, $v(d, \cdot)$ is differentiable, and there exists $B > 0$ such that for all d and θ $|v_\theta(d, \theta)| \leq B$. If (d, t) is dominant strategy incentive compatible and d is efficient, then there exists $h_i \colon \Theta_{-i} \to \mathbb{R}$, such that

$$t_i(\theta) = t_i^{\text{pivot}}(\theta) + h_i(\theta_{-i}) \text{ for all } \theta \in \Theta.$$

Proof. By payoff equivalence, for two dominant strategy incentive compatible mechanisms (d,t) and (d,t^{pivot}), we have

$$t_i(\theta) = t_i^{\text{pivot}}(\theta) + h_i(\theta_{-i}).$$

□

4.2 Balancing the budget

40 Theorem: There exists a VCG mechanism that satisfies budget balance if and only if for every i there is a function $g_i \colon \Theta_{-i} \to \mathbb{R}$ such that

$$\sum_{i=1}^{N} v_i\big(d(\theta),\theta_i\big) = \sum_{i=1}^{N} g_i(\theta_{-i}) \text{ for all } \theta \in \Theta.$$

41 *Proof of necessity.*

(1) Suppose that a VCG mechanism $\big(d(\cdot), t^{\text{VCG}}(\cdot)\big)$ is budget balanced, then we have

$$\sum_{i=1}^{N}\left[h_i(\theta_{-i}) - \sum_{j\neq i} v_j\big(d(\theta),\theta_j\big)\right] = \sum_{i=1}^{N} t_i^{\text{VCG}}(\theta) = 0.$$

(2) This equality is equivalent to

$$\sum_{i=1}^{N} h_i(\theta_{-i}) = \sum_{i=1}^{N}\sum_{j\neq i} v_j\big(d(\theta),\theta_j\big) = (N-1)\sum_{i=1}^{N} v_i\big(d(\theta),\theta_i\big).$$

(3) Hence, if we set for every i and θ_{-i},

$$g_i(\theta_{-i}) = \frac{h_i(\theta_{-i})}{N-1},$$

we have obtained the desired form for the function $\sum_{i=1}^{N} v_i\big(d(\theta),\theta_i\big)$.

□

42 *Proof of sufficiency.*

(1) Suppose that $\sum_{i=1}^{N} v_i\big(d(\theta),\theta_i\big)$ has the form described in the statement.

(2) For every i and every θ_{-i} we consider the VCG mechanism with

$$h_i(\theta_{-i}) \triangleq (N-1) g_i(\theta_{-i}).$$

(3) Then for every θ, the sum of agents' payments is

$$\sum_{i=1}^{N} t_i^{\text{VCG}}(\theta) = \sum_{i=1}^{N}\left[h_i(\theta_{-i}) - \sum_{j\neq i} v_j\big(d(\theta),\theta_j\big)\right] = (N-1)\left[\sum_{i=1}^{N} g_i(\theta_{-i}) - \sum_{i=1}^{N} v_i\big(d(\theta),\theta_i\big)\right] = 0.$$

□

5 Homework

- Reading: Börgers 4.2, 7.2, 7.3, 7.5