

# Game Theory

## Auction

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# Introduction

- One of the most popular examples of static games of incomplete information is an auction.
- An auction is a mechanism of allocating goods.
- Advantages of auctions:
  - a simple way of determining the market-based prices
  - more flexible than setting a fixed price
  - can usually achieve efficiency by allocating the resources to those who value them most highly

# Examples of Auctions

- Consumption goods: antiques, art, wine
- Government: treasury bills, mineral rights, assets
- Stocks: IPOs
- Procurement auctions/Subcontracting: automobiles
- Internet: eBay, Amazon

# Types of Auctions

- Number of objects
  - A single object or many?
- Open vs. sealed-bid
  - Do you know the bids of other participants?
- One-sided vs. two-sided
  - Do buyers and sellers both submit bids, or just buyers?
- Private value vs. common value
  - Do the bidders have the same valuation for the object?

# Four Classical Auctions

- English: ascending, open
- Dutch: descending, open
- First-price, sealed-bid
- Second-price, sealed-bid (or Vickrey)

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# A Second-Price Sealed-Bid Auction

- Suppose there are  $n$  potential buyers (or bidders), with valuations  $v_1, \dots, v_n$  for an object.
- Let each  $v_i$  be drawn from some **distribution**  $F_i(\cdot)$ .
- Each bidder knows his own valuation but does not know other bidders' valuations.
- The bidders **simultaneously** submit bids  $b_i \in [0, \infty)$ .
- The highest bidder wins the object and pays the second highest bid, while the other bidders obtain nothing.
- If there are more than one winners, the object is allocated randomly among them.



# Payoff

- Let  $r_i$  be the highest bid of all players other than player  $i$ , where  $r_i = \max_{j \neq i} b_j$ .
- The bidder  $i$ 's payoff function is

$$u_i(b_i, b_{-i}; v_i) = \begin{cases} v_i - r_i, & \text{if } b_i > r_i; \\ \frac{v_i - r_i}{k}, & \text{if } b_i = r_i; \\ 0, & \text{otherwise,} \end{cases}$$

where  $k$  is the number of bids that equal  $b_i$ .

# Truth telling

- For each player  $i$ , the strategy of bidding his valuation (i.e.,  $s_i^*(v_i) = v_i$ ) **weakly dominates** all other strategies.
- Compare bidder  $i$ 's payoffs for different bids  $b_i$ .
- First, compare  $b_i^* = s_i^*(v_i) = v_i$  with  $b_i > v_i$ ,

$$u_i(b_i^*, b_{-i}; v_i) = \begin{cases} 0, & \\ 0, & \\ 0, & \\ v_i - r_i = 0, & \\ v_i - r_i, & \end{cases} \geq u_i(b_i, b_{-i}; v_i) = \begin{cases} 0, & \text{if } r_i > b_i, \\ \frac{v_i - r_i}{k}, & \text{if } r_i = b_i, \\ v_i - r_i, & \text{if } v_i < r_i < b_i, \\ \frac{v_i - r_i}{k} = 0, & \text{if } r_i = v_i, \\ v_i - r_i, & \text{if } r_i < v_i. \end{cases}$$

- Then  $b_i^* = v_i$  dominates  $b_i > v_i$ .

# Truth telling (Cont.)

- Second, compare  $b_i^* = v_i$  with  $b_i < v_i$ ,

$$u_i(b_i^*, b_{-i}; v_i) = \begin{cases} 0, & \text{if } r_i > v_i, \\ \frac{v_i - r_i}{k} = 0, & \text{if } r_i = v_i, \\ v_i - r_i, & \text{if } b_i < r_i < v_i, \\ v_i - r_i, & \text{if } r_i = b_i, \\ v_i - r_i, & \text{if } r_i < b_i. \end{cases} \geq u_i(b_i, b_{-i}; v_i) = \begin{cases} 0, & \text{if } r_i > v_i, \\ 0, & \text{if } r_i = v_i, \\ 0, & \text{if } b_i < r_i < v_i, \\ \frac{v_i - r_i}{k}, & \text{if } r_i = b_i, \\ v_i - r_i, & \text{if } r_i < b_i. \end{cases}$$

- Then  $b_i^* = v_i$  dominates  $b_i < v_i$ .

# Bayesian Nash equilibrium

- By the following proposition,  $(s_1^*, \dots, s_n^*)$  is a Bayesian Nash equilibrium, where  $s_i^*(v_i) = v_i$  for all  $i$ .

## Proposition

Consider a strategy profile  $(s_1^*, \dots, s_n^*)$  in a Bayesian game. Suppose for any player  $i$ , any  $t_i \in T_i$ ,  $a_i \in A_i$ , and  $a_{-i} \in A_{-i}$ ,

$$u_i(s_i^*(t_i), a_{-i}; t_i) \geq u_i(a_i, a_{-i}; t_i),$$

(i.e.,  $s_i^*(t_i)$  weakly dominates every  $a_i \in A_i$ ). Then  $(s_1^*, \dots, s_n^*)$  is a Bayesian Nash equilibrium.

# Bayesian Nash equilibrium: Proof

## Proof.

- Because  $s_{-i}^*(t_{-i}) \in A_{-i}$ , the weak dominance implies

$$u_i(s_i^*(t_i), s_{-i}^*(t_{-i}); t_i) \geq u_i(a_i, s_{-i}^*(t_{-i}); t_i)$$

for any  $t_i \in T_i$  and  $a_i \in A_i$ .

- Then  $s_i^*(t_i)$  solves

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i(a_i, s_{-i}^*(t_{-i}); t_i) p_i(t_{-i} | t_i),$$

for all  $t_i$  and for all  $i$ .

- Therefore,  $(s_1^*, \dots, s_n^*)$  is a Bayesian Nash equilibrium.



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# A First-Price Sealed-Bid Auction

- Suppose there are **two** bidders ( $i = 1, 2$ ).
- The bidders' valuations for an object are  $v_1$  and  $v_2$ , which are independently and uniformly distributed on  $[0, 1]$ .
- The **valuation**  $v_i$  is bidder  $i$ 's private information, which is unknown to the other bidder.
- Bidders submit their bids  $b_1$  and  $b_2$  **simultaneously**.
- The higher bidder wins the object and pays the **highest bid**, while the other obtains nothing.
- If there is a tie, the winner is determined by a flip of a coin.

# A First-Price Sealed-Bid Auction

Normal-form representation of this static Bayesian game

$G = \langle A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2 \rangle$ :

- $A_1 = A_2 = [0, \infty)$ , and each bid is  $b_i \in A_i$ ;
- $T_1 = T_2 = [0, 1]$ , and each valuation is  $v_i \in T_i$ ;
- Player  $i$  believes that  $v_j$  is uniformly distributed on  $[0, 1]$ ;
- The payoff  $u_i(b_i, b_j; v_i)$  is

$$u_i(b_i, b_j; v_i) = \begin{cases} v_i - b_i, & \text{if } b_i > b_j; \\ \frac{1}{2}(v_i - b_i), & \text{if } b_i = b_j; \\ 0, & \text{if } b_i < b_j. \end{cases}$$



# Formulation

- Bidder  $i$ 's **strategy** is a function  $b_i(v_i)$  from  $[0, 1]$  into  $[0, \infty)$
- $(b_1^*(v_1), b_2^*(v_2))$  is a **Bayesian Nash equilibrium** if and only if for  $i = 1, 2$  and each  $v_i \in [0, 1]$ ,  $b_i^*(v_i)$  solves

$$\begin{aligned} & \max_{b_i \geq 0} \mathbb{E}_{v_j} u_i(b_i, b_j^*(v_j); v_i) \\ &= \max_{b_i \geq 0} \left\{ (v_i - b_i) \text{Prob}\{b_i > b_j^*(v_j)\} + \frac{1}{2}(v_i - b_i) \text{Prob}\{b_i = b_j^*(v_j)\} \right\} \end{aligned}$$

# Linear Bayesian Nash equilibrium

- There may be many Bayesian Nash equilibria in this game.
- We just focus on equilibria in the form of **linear functions**:

$$b_1^*(v_1) = a_1 + c_1 v_1 \text{ and } b_2^*(v_2) = a_2 + c_2 v_2,$$

where  $c_i > 0$ , and  $0 \leq a_i < 1$  for  $i = 1, 2$ .

- To solve for the Bayesian Nash equilibria, we just need to find out  $a_i$  and  $c_i$  accordingly.

# Assumptions

Rationale of the assumptions on  $a_i$  and  $c_i$ :

- $c_i > 0$ : a bidder with higher valuation is willing to bid higher.
- $a_i \geq 0$ : bids cannot be negative.
- $a_i \leq 1$ : for  $a_i > 1$ , bidder  $i$  can never end up with a positive payoff given  $v_i \in [0, 1]$ .

# Best response

- We need to determine each player's best response given the other's strategy.
- Suppose player  $j$  adopts a linear strategy  $b_j^*(v_j) = a_j + c_j v_j$  in equilibrium, where  $c_j > 0$ .
- We have

$$\text{Prob}(b_i = a_j + c_j v_j) = \text{Prob}\left(v_j = \frac{b_i - a_j}{c_j}\right) = 0.$$

- For any  $v_i \in [0, 1]$ , player  $i$ 's best response  $b_i(v_i)$  maximizes

$$(v_i - b_i) \text{Prob}(b_i > a_j + c_j v_j) = (v_i - b_i) \text{Prob}\left(v_j < \frac{b_i - a_j}{c_j}\right).$$

# Best response (Cont.)

- Since  $b_j^*(v_j) = a_j + c_j v_j \in [a_j, a_j + c_j]$ , we can restrict our attention to  $b_i \in [a_j, a_j + c_j]$  (i.e.,  $b_i < a_j$  is pointless, while  $b_i > a_j + c_j$  is not rational).
- Under the above restriction, we know

$$0 \leq \frac{b_i - a_j}{c_j} \leq 1.$$

- Player  $i$ 's best response solves

$$\max_{a_j \leq b_i \leq a_j + c_j} (v_i - b_i) \frac{b_i - a_j}{c_j}.$$

# A First-Price Sealed-Bid Auction

- The best response function of player  $i$  is

$$b_i(v_i) = \begin{cases} a_j, & \text{if } v_i \leq a_j; \\ \frac{1}{2}(v_i + a_j), & \text{if } a_j < v_i \leq a_j + 2c_j; \\ a_j + c_j, & \text{if } v_i > a_j + 2c_j. \end{cases}$$

# Best response (Cont.)

- We want the equilibrium bid to be a linear function on  $[0, 1]$ .
- There are three cases:

$$[0, 1] \subseteq \begin{cases} (-\infty, a_j] \\ [a_j, a_j + 2c_j] \\ [a_j + 2c_j, \infty) \end{cases}$$

# Selection

- Case 1 violates the assumption  $a_j < 1$ .
- Case 3 violates the assumptions  $a_j \geq 0$  and  $c_j > 0$ , which imply  $a_j + 2c_j > 0$ .
- Therefore, we have  $[0, 1] \subseteq [a_j, a_j + 2c_j]$ , and the best response is

$$b_i(v_i) = \frac{1}{2}(v_i + a_j).$$



# Selection (Cont.)

- In a Bayesian Nash equilibrium,

$$b_i^*(v_i) = a_i + c_i v_i = \frac{1}{2}(v_i + a_j)$$

for all  $v_i \in [0, 1]$ .

- Then we have

$$a_i = \frac{1}{2}a_j, \text{ and } c_i = \frac{1}{2}$$

for  $i, j = 1, 2$  and  $i \neq j$ .

# Linear Bayesian Nash equilibrium

- Therefore,

$$a_1 = a_2 = 0, \text{ and } c_1 = c_2 = \frac{1}{2}.$$

- The unique linear Bayesian Nash equilibrium is

$$b_1^*(v_1) = \frac{1}{2}v_1, \text{ and } b_2^*(v_2) = \frac{1}{2}v_2.$$

# Linear Bayesian Nash equilibrium (Cont.)

- Alternatively, if we can somehow guess that  $(b_1^*(v_1), b_2^*(v_2)) = (v_1/2, v_2/2)$  is a Bayesian Nash equilibrium, we can prove it directly.
- Suppose player  $j$  has adopted the strategy  $b_j^*(v_j) = v_j/2$ .
- Player  $i$ 's best response solves

$$\max_{b_i \in [0, 1/2]} (v_i - b_i) \text{Prob}(b_i > v_j/2) = \max_{b_i \in [0, 1/2]} 2(v_i - b_i)b_i$$

- For any  $v_i \in [0, 1]$ , the unique maximizer is  $b_i^* = v_i/2$ .
- Thus,  $(b_1^*(v_1), b_2^*(v_2)) = (v_1/2, v_2/2)$  is a Bayesian Nash equilibrium.

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# A Double Auction

- Consider the case in which a buyer and a seller each have private information about their valuations.
- For example, the buyer is a firm and the seller is a worker. The firm knows the worker's marginal product and the worker knows his or her outside opportunity (Hall and Lazear, 1984).
- Consider a trading game called a double auction.

# A Double Auction

- The buyer's valuation for the good is  $v_b$ , and the seller's valuation is  $v_s$ .
- Both  $v_b$  and  $v_s$  are private information, and independently drawn from uniform distributions on  $[0, 1]$ .
- The seller names an asking price  $p_s$ , and the buyer simultaneously names an offer price  $p_b$ .
- If  $p_b \geq p_s$ , then trade occurs at a price  $p = \frac{p_b + p_s}{2}$ ; otherwise no trade occurs.

# Formulation

Normal-form representation of this static Bayesian game:

- $A_b = A_s = [0, \infty)$ , and  $p_k \in A_k$  for  $k = b, s$ ;
- $T_b = T_s = [0, 1]$ , and  $v_k \in T_k$  for  $k = b, s$ ;
- The buyer believes that  $v_s$  is uniformly distributed on  $[0, 1]$ , and likewise for the seller;
- The buyer's payoff is

$$u_b(p_b, p_s; v_b) = \begin{cases} v_b - p, & \text{if } p_b \geq p_s, \\ 0, & \text{otherwise,} \end{cases}$$

and the seller's payoff is

$$u_s(p_b, p_s; v_s) = \begin{cases} p - v_s, & \text{if } p_b \geq p_s, \\ 0, & \text{otherwise.} \end{cases}$$

# Bayesian Nash equilibrium

- A **strategy** for the buyer is a function  $p_b(v_b)$  specifying the price the buyer will offer for each of the buyer's possible valuations (likewise for the seller).
- A pair of strategies  $(p_b(v_b), p_s(v_s))$  is a **Bayesian Nash equilibrium** if the following two conditions hold.
- For each  $v_b \in [0, 1]$ ,  $p_b(v_b)$  solves

$$\max_{p_b} \left[ v_b - \frac{p_b + \mathbb{E}[p_s(v_s) | p_b \geq p_s(v_s)]}{2} \right] \text{Prob}(p_b \geq p_s(v_s)),$$

where  $\mathbb{E}[p_s(v_s) | p_b \geq p_s(v_s)]$  is the expected price the seller will name, conditional on the price being no higher than the buyer's offer  $p_b$ .



# Bayesian Nash equilibrium (Cont.)

- For each  $v_s \in [0, 1]$ ,  $p_s(v_s)$  solves

$$\max_{p_s} \left[ \frac{p_s + \mathbb{E}[p_b(v_b) | p_b(v_b) \geq p_s]}{2} - v_s \right] \text{Prob}(p_b(v_b) \geq p_s),$$

where  $\mathbb{E}[p_b(v_b) | p_b(v_b) \geq p_s]$  is the expected price the buyer will offer, conditional on the offer being no smaller than the seller's demand of price  $p_s$ .

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# One-price equilibrium

- There are many Bayesian Nash equilibria.
- Consider the following **one-price equilibrium** in which trade occurs at a single price (if at all).
- The buyer's strategy is

$$p_b^*(v_b) = \begin{cases} x, & \text{if } v_b \geq x, \\ 0, & \text{otherwise,} \end{cases}$$

and the seller's strategy is

$$p_s^*(v_s) = \begin{cases} x, & \text{if } v_s \leq x, \\ 1, & \text{otherwise.} \end{cases}$$

# One-price equilibrium (Cont.)

- Is it a Bayesian Nash equilibrium?
- Given the buyer's strategy, consider whether the seller with type  $v_s$  would want to deviate
  - For  $v_s > x$ , the seller would prefer no trading to trading at  $x$ .
  - For  $v_s < x$ , the seller would prefer trading at  $x$  to no trading.
  - For  $v_s = x$ , the seller is indifferent between trading at  $x$  and no trading, and will not deviate either.
- An analogous argument applies for the buyer.

# One-price equilibrium: Trading area

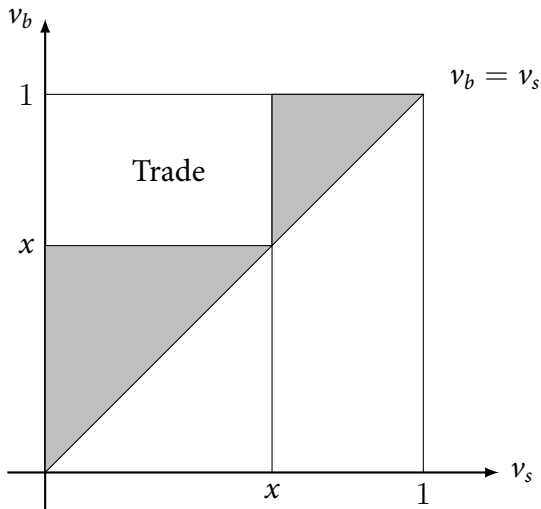


Figure: Outcome in one-price equilibrium

# One-price equilibrium: Trading area (Cont.)

- In equilibrium, trade occurs when  $v_s \leq x$  and  $v_b \geq x$  (as shown in Figure 1).
- However, trade would be efficient for all pairs of  $(v_s, v_b)$  such that  $v_b \geq v_s$ , but does not occur in the two shaded regions.

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# Linear Bayesian Nash equilibrium

- Now we consider a **linear** Bayesian Nash equilibrium.
- Suppose the seller's strategy is  $p_s^*(v_s) = a_s + c_s v_s$ , and then  $p_s$  is uniformly distributed on  $[a_s, a_s + c_s]$ .
- Given seller's strategy  $p_s(v_s)$ , buyer's expected payoff is

$$\begin{aligned}
 & \mathbb{E}_{v_s} \pi_b[p_b, p_s(v_s) \mid v_b] \\
 &= \int_{a_s \leq p_s(v_s) \leq p_b} v_b - \frac{p_b + p_s(v_s)}{2} dv_s + \int_{p_b < p_s(v_s) \leq a_s + c_s} 0 dv_s \\
 &= \int_{a_s \leq u \leq p_b} v_b - \frac{p_b + u}{2} d\frac{u}{c_s} = \frac{p_b - a_s}{c_s} \left( v_b - \frac{3}{4}p_b - \frac{1}{4}a_s \right).
 \end{aligned}$$



# Best response

- Maximizing  $E_{v_s} \pi_b[p_b, p_s(v_s) \mid v_b]$  yields buyer's best response

$$p_b(v_b) = \frac{2}{3}v_b + \frac{a_s}{3},$$

which implies  $c_b = \frac{2}{3}$  and  $a_b = \frac{a_s}{3}$ .

- Thus, if the seller plays a linear strategy, the buyer's best response is also linear.

# Best response (Cont.)

- Analogously, suppose the buyer's strategy is  $p_b^*(v_b) = a_b + c_b v_b$ , and then  $p_b$  is uniformly distributed on  $[a_b, a_b + c_b]$ .
- Then the seller's choice becomes

$$\max_{p_s} \left[ \frac{1}{2} \left( p_s + \frac{p_s + a_b + c_b}{2} - v_s \right) \right] \frac{a_b + c_b - p_s}{c_b}.$$

- The best response function of the seller is

$$p_s = \frac{2}{3} v_s + \frac{1}{3} (a_b + c_b).$$

# Solving equilibrium

- In equilibrium, the buyer's strategy must be a best response to the seller's strategy, i.e.,

$$c_b = \frac{2}{3}, \text{ and } a_b = \frac{a_s}{3}.$$

- Analogously, the seller's strategy must be a best response to the buyer's strategy, i.e.,

$$c_s = \frac{2}{3}, \text{ and } a_s = \frac{a_b + c_b}{3}.$$

# Linear Bayesian Nash equilibrium

- The linear equilibrium strategies are

$$p_b^*(v_b) = \frac{2}{3}v_b + \frac{1}{12},$$

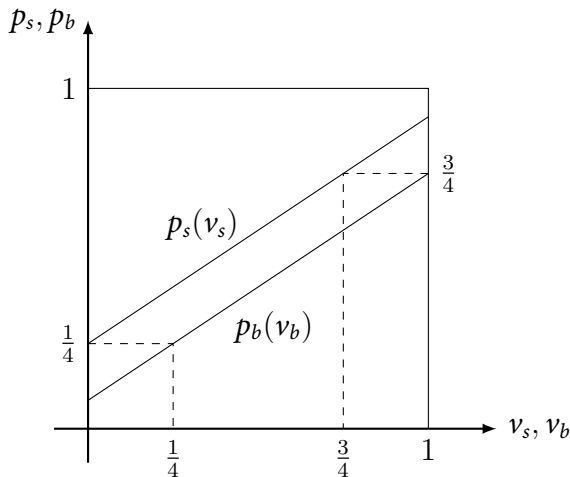
and

$$p_s^*(v_s) = \frac{2}{3}v_s + \frac{1}{4}.$$

- Trade occurs if and only if

$$v_b \geq v_s + \frac{1}{4}.$$

# Linear Bayesian Nash equilibrium: Outcome



**Figure:** Strategies of players in the linear equilibrium

# Linear Bayesian Nash equilibrium: Trading area

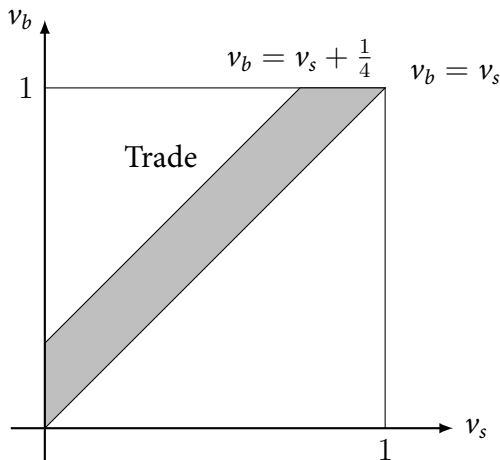


Figure: Outcome in the linear equilibrium

# Linear Bayesian Nash equilibrium: Trading area (Cont.)

- In both equilibria, **efficient** trade does not always occur, but the most valuable trade (i.e.,  $v_b = 1$  and  $v_s = 0$ ) does occur.
- The one-price equilibrium misses some valuable trades (such as  $v_b = 1$  and  $v_s = x - \epsilon$ , where  $\epsilon$  is small), and achieves some trades that are worth next to nothing (such as  $v_b = x + \epsilon$  and  $v_s = x - \epsilon$ ).
- The linear equilibrium misses all trades that are worth next to nothing, but achieves all trades worth at least  $\frac{1}{4}$ .
- The linear equilibrium dominates the one-price equilibrium in terms of the expected gains that the players receive.

# Inefficiency

- Myerson and Satterwaite (1983) show that for the uniform distributions, the linear equilibrium yields the highest expected gains for the players among **all** possible Bayesian Nash equilibria in the double auction.
- This implies that there is no Bayesian Nash equilibrium of the double auction in which trade occurs if and only if it is efficient.
- The latter result can be quite general.



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## Question 3

Consider a first-price, sealed-bid auction in which the bidders' valuations are independently and uniformly distributed on  $[0, 1]$ . Show that if there are  $n$  bidders, then the strategy of bidding  $\frac{n-1}{n}$  times one's valuation is a symmetric Bayesian Nash equilibrium of this auction.

## Question 4

There are  $n \geq 2$  players. Each player  $i$  must simultaneously decide whether to join a team ( $x_i = 1$ ) or not ( $x_i = 0$ ); hence  $z = \sum_{i=1}^n x_i$  is the size of the team. If player  $i$  does not join (so that  $x_i = 0$ ) then  $i$  receives a payoff of zero. If player  $i$  joins the team (so that  $x_i = 1$ ) then  $i$  pays a cost of  $c_i$ . If all  $n$  players join the team (so that  $z = n$ ) then each player enjoys a benefit of  $v$ . Hence player  $i$ 's payoff is  $u_i = v - c_i$  when  $z = n$ , and  $u_i = -x_i c_i$  when  $z < n$ . Suppose that  $v > c_i > 0$ .

- ① Suppose that the costs  $c_1, \dots, c_n$  are common knowledge. Find all pure-strategy Nash equilibria.
- ② Now, suppose that information is incomplete. Player  $i$ 's cost realization  $c_i$  is known only to  $i$ ; players' costs are drawn independently from the same uniform distribution:  $c_i \sim U[0, \bar{c}]$ . Find the symmetric Bayesian Nash equilibrium.<sup>1</sup>

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<sup>1</sup>Symmetric Bayesian Nash equilibrium is a strategy profile, in which each Player chooses the same strategy.