Game Theory

Cooperative games

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Coalitional game

- A coalitional game (合作博弈) with transferable payoff (henceforth "coalitional game") $\langle N, v\rangle$ consists of
  - a finite set N of players,
  - a function $v\colon 2^N \setminus \{\emptyset\} \to \mathbb{R}$.
- Every member in $2^N \setminus \{\emptyset\}$ is called a coalition, and $v(S)$ is called the worth of the coalition S. The function is called the characteristic function (特征函数).

* Interpretation: $v(S)$ is a payoff that may be distributed in any way among the members of S.

- Convention: If $\{i_1, i_2, \ldots, i_j\}$ is a set of players, we will sometimes write $v(i_1, i_2, \ldots, i_j)$ rather than $v(\{i_1, i_2, \ldots, i_j\})$ for the worth of $\{i_1, i_2, \ldots, i_j\}$.

Coalitional game (Cont.)

- A coalitional game $\langle N, v\rangle$ is
  - monotonic if $T \subseteq S$ implies $v(S) \geq v(T)$;
  - cohesive if

$$v(N) \geq \sum_{k=1}^{K} v(S_k) \text{ for every partition } \{S_1, S_2, \ldots, S_K\} \text{ of } N;$$

  - super-additive if $S \cap T = \emptyset$ implies $v(S \cup T) \geq v(S) + v(T)$.
- It is clear that $\langle N, v\rangle$ is cohesive if it is super-additive.
- We will assume that the coalitional games are cohesive or super-additive.
- i and j, elements of N, are substitutes in v if for all S containing neither i nor j, $v(S \cup \{i\}) = v(S \cup \{j\})$.
- $i \in N$ is called a null player if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N$.

Core

- The core (核) is a solution concept for coalitional games that requires that no set of players be able to break away and take a joint action that makes all of them better off.
- Let $\langle N, v\rangle$ be a coalitional game.
  - A vector $(x_i)_{i\in S}$ of real numbers is an S-feasible payoff vector if $v(S) = \sum_{i\in S} x_i$.
  - We refer to an N-feasible payoff vector as a feasible payoff profile.
  - The core of the coalitional game $\langle N, v\rangle$ is the set of feasible payoff profiles $(x_i)_{i\in N}$ for which there is no coalition S and S-feasible payoff vector $(y_i)_{i\in S}$ for which $y_i > x_i$ for all $i \in S$.

Property

- If x is in the core, then x satisfies
  - (individual rational) $x_i \geq v(i)$ for all $i \in N$,
  - (group rational) $\sum_{i \in N} x_i = v(N)$.
-

Proposition

The core is the set of feasible payoff profiles $(x_i)_{i \in N}$ for which $\sum_{i \in S} x(i) \geq v(S)$ for every coalition S.

- The core is the set of payoff profiles satisfying a system of weak linear inequalities and hence is closed and convex.

Proof of "$\Leftarrow$"

- Suppose that $x = (x_i)_{i \in N}$ satisfies

$$\sum_{i \in N} x_i = v(N), \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all coalition } S.$$

- Assume x is not in the core, that is, there exist a coalition S and $y = (y_i)_{i \in S}$, such that $\sum_{i \in S} y_i = v(S)$ and $y_i > x_i$ for all $i \in S$.
- Then we have $\sum_{i \in S} y_i > \sum_{i \in S} x_i \geq v(S)$, a contradiction.

Proof of "$\Rightarrow$"

- Suppose that $x = (x_i)_{i \in N}$ does not satisfy

$$\sum_{i \in N} x_i = v(N), \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all coalition } S.$$

- If $\sum_{i \in N} x_i \neq v(N)$, x can not be in the core.
- Suppose, then, that there is a coalition S such that

$$\sum_{i \in S} x_i = v(S) - \epsilon,$$

where $\epsilon > 0$. For $i \in S$, define $z_i = x_i + \frac{\epsilon}{|S|}$.

- It is easily seen that $\sum_{i \in S} z_i = v(S)$ and $z_i > x_i$ for all $i \in S$. Hence x is not in the core.

Example: Two-person bargaining game

$N = \{1, 2\}$, $v(N) = 1$, and $v(1) = v(2) = 0$.

Answer.

(x_1, x_2) is in the core if and only if

$$x_1 \geq 0, x_2 \geq 0, \text{ and } x_1 + x_2 = 1.$$

$\square$

Example: Three-person bargaining game

$N = \{1, 2, 3\}$, $v(N) = 1$ and $v(S) = 0$ for all $S \subsetneqq N$.

Answer.

- (x_1, x_2, x_3) is in the core if and only if

$$x_1 + x_2 + x_3 = v(N) = 1, \text{ and } \sum_{i \in S} x_i \geq v(S) = 0 \text{ for all } S \subsetneqq N.$$

- The core is therefore the set

$$\{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \geq 0, x_1 + x_2 + x_3 = 1\}.$$

$\square$

Example: Market with two sellers and a buyer

$N = \{1, 2, 3\}$, $v(N) = v(1, 2) = v(1, 3) = 1$, and $v(S) = 0$ for all other $S \subseteq N$.

Answer.

- x is in the core if and only if

$$x_1+x_2+x_3 = 1,\ x_1+x_2 \geq 1,\ x_1+x_3 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

- Hence the core is $\{(1, 0, 0)\}$.

$\square$

Example: Three-person majority game

Suppose that three players can obtain one unit of payoff, any two of them can obtain 1 independently of the actions of the third, and each player alone can obtain nothing, independently of the actions of the remaining two players.
$N = \{1, 2, 3\}$, $v(N) = v(1, 2) = v(1, 3) = v(2, 3) = 1$ and $v(i) = 0$ for all $i \in N$.

Answer.

For x to be in the core, we need $x_1 + x_2 + x_3 = 1$, $x_i \geq 0$ for all $i \in N$, $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$ and $x_2 + x_3 \geq 1$. There exists no x satisfying these condition, so the core is empty. □

Example: A majority game

A group of n players, where $n \geq 3$ is odd, has one unit to divide among its members. A coalition consisting of a majority of the players can divide the unit among its members as it wishes. This situation is modeled by the coalitional game $\langle N, v \rangle$ in which $|N| = n$ and

$$v(S) = \begin{cases} 1, & \text{if } |S| \geq \frac{n}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Example: A majority game (Cont.)

Answer.

- The game has an empty core by the following argument. Assume that x is in the core. If $|S| = n-1$ then $v(S) = 1$ so that $\sum_{i \in S} x_i \geq 1$. Since there are n coalitions of size $n-1$ we thus have

$$\sum_{\{S\colon |S|=n-1\}} \sum_{s \in S} x_i \geq n.$$

- On the other hand, we have

$$\sum_{\{S\colon |S|=n-1\}} \sum_{i \in S} x_i = \sum_{i \in N} \sum_{\{S\colon |S|=n-1, S \ni i\}} x_i = \sum_{i \in N} (n-1) x_i = n-1,$$

a contradiction. □

Notation

- Denote by $\mathcal{C}$ the set of all coalitions, and for any coalition S denote by $\mathbb{R}^{|S|}$ the $|S|$-dimensional Euclidian space in which the dimensions are indexed by the members of S.
- Denote by $\mathbf{1}_S \in \mathbb{R}^{|N|}$ the characteristic vector of S given by

$$(\mathbf{1}_S)_i = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Balanced collection of weights

- A collection $(\lambda_S)_{S\in\mathcal{C}}$ of numbers in $[0,1]$ is a balanced collection of weights if for every player i the sum of λ_S over all the coalitions that contain i is 1:
$$\sum_{S\in\mathcal{C}} \lambda_S \mathbf{1}_S = \mathbf{1}_N.$$
- Example 1: the collection (λ_S) in which $\lambda_N = 1$ and $\lambda_S = 0$ for all other S is a balanced collection of weights.
- Example 2: let $|N| = 3$. Then the collection (λ_S) in which $\lambda_S = \frac{1}{2}$ if $|S| = 2$ and $\lambda_S = 0$ otherwise is a balanced collection of weights; so too is the collection (λ_S) in which $\lambda_S = 1$ if $|S| = 1$ and $\lambda_S = 0$ otherwise.

Balanced game

A game $\langle N, v \rangle$ is balanced if

$$\sum_{S \in \mathcal{C}} \lambda_S v(S) \leq v(N) \text{ for every balanced collection of weights.}$$

- Each player has one unit of time, which he must distribute among all the coalitions of which he is a member.
- In order for a coalition S to be active for the fraction of time λ_S, all its members must be active in S for this fraction of time, in which case the coalition yields the payoff $\lambda_S v(S)$.

Interpretation

- The condition $\sum_{S \in \mathcal{C}} \lambda_S \mathbf{1}_S = \mathbf{1}_N$ is a feasibility condition (for every individual the sum of its amounts of his time he spends with each coalition must equal exactly the amount of time he is endowed with).
- A game is balanced if there is no feasible allocation of time that yields the players more than $v(N)$.

Bondareva-Shapley theorem

Bondareva-Shapley theorem

A coalitional game has a non-empty core if and only if it is balanced.

Proof of "$\Rightarrow$".

- Let x be a payoff profile in the core of $\langle N, v\rangle$ and $(\lambda_S)_{S\in\mathcal{C}}$ a balanced collection of weights.
- Then

$$\sum_{S\in\mathcal{C}} \lambda_S v(S) \leq \sum_{S\in\mathcal{C}} \lambda_S \sum_{i\in S} x_i = \sum_{i\in N} x_i \sum_{S\ni i} \lambda_S = \sum_{i\in N} x_i = v(N),$$

so that $\langle N, v\rangle$ is balanced.

$\square$

Proof of "$\Leftarrow$"

- Assume that $\langle N, v\rangle$ is balanced. Then there is no balanced collection $(\lambda_S)_{S\in\mathcal{C}}$ of weights for which

$$\sum_{S\in\mathcal{C}} \lambda_S v(S) > v(N).$$

- Therefore the convex set $\{(\mathbf{1}_N, v(N)+\epsilon) \in \mathbb{R}^{|N|+1} : \epsilon > 0\}$ is disjoint from the convex cone

$$\left\{ y \in \mathbb{R}^{|N|+1} \mid y = \sum_{S\in\mathcal{C}} \lambda_S(\mathbf{1}_S, v(S)) \text{ where } \lambda_S \geq 0 \text{ for all } S\in\mathcal{C} \right\},$$

since if not then $\mathbf{1}_N = \sum_{S\in\mathcal{C}} \lambda_S \mathbf{1}_S$, so that $(\lambda_S)_{S\in\mathcal{C}}$ is a balanced collection of weights and $\sum_{S\in\mathcal{C}} \lambda_S v(S) > v(N)$.

Proof of "$\Leftarrow$" (Cont.)

- Thus by hyperplane separating theorem there is a non-zero vector $(\alpha_N, \alpha) \in \mathbb{R}^{|N|} \times \mathbb{R}$ such that

$$(\alpha_N, \alpha) \cdot y \geq 0 > (\alpha_N, \alpha) \cdot (\mathbf{1}_N, v(N) + \epsilon)$$

  for all y in the cone and all $\epsilon > 0$.
- Since $(\mathbf{1}_N, v(N))$ is in the cone, we have $\alpha < 0$.
- Now let $x = \frac{\alpha_N}{-\alpha}$.
- Since $(\mathbf{1}_S, v(S))$ is in the cone for all $S \in \mathcal{C}$, we have $x(S) = x \cdot \mathbf{1}_S \geq v(S)$ for all $S \in \mathcal{C}$, and $v(N) \geq \mathbf{1}_N x = \sum_{i \in N} x_i$.
- Thus $v(N) = \sum_{i \in N} x_i$, so that x is in the core of $\langle N, v \rangle$.

Shapley value

- A coalition of players cooperates, and obtains a certain overall gain from that cooperation.
- Since some players may contribute more to the coalition than others or may possess different bargaining power (for example threatening to destroy the whole surplus), what final distribution of generated surplus among the players should arise in any particular game?
- Or phrased differently: how important is each player to the overall cooperation, and what payoff can he or she reasonably expect?
- The Shapley value provides one possible answer to this question.

Shapley value

Given a coalitional game $\langle N, v\rangle$ where $N = \{1, 2, \ldots, n\}$, the Shapley value is an n-vector, denoted by $\phi(v) = \big(\phi_1(v), \phi_2(v), \ldots, \phi_n(v)\big)$, satisfying the following conditions:

S1. Symmetry condition: if i and j are substitutes in v, then $\phi_i(v) = \phi_j(v)$.

S2. Null player condition: if i is a null player, then $\phi_i(v) = 0$.

S3. Efficiency condition: $\sum_{i \in N} \phi_i(v) = v(N)$.

S4. Additivity condition: $\phi_i(v + w) = \phi_i(v) + \phi_i(w)$.

$\phi_i(v)$ is interpreted as the power of player i in the coalitional game $\langle N, v\rangle$, or what it is worth to i to participate in the game $\langle N, v\rangle$.

Shapley theorem

Shapley theorem

Shapley value is uniquely determined:

$$\begin{aligned}\phi_i(v) &= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!}[v(S \cup \{i\}) - v(S)] \\ &= \frac{1}{n} \sum_{s=0}^{n-1} \frac{1}{\binom{n-1}{s}} \sum_{S \subseteq N \setminus \{i\}, |S|=s} [v(S \cup \{i\}) - v(S)].\end{aligned}$$

Interpretation

- Imagine the coalition being formed one actor at a time, with each actor demanding their contribution $v(S \cup \{i\}) - v(S)$ as a fair compensation.
- Then for each actor take the average of this contribution over the possible different permutations in which the coalition can be formed.

Computation

Let $\gamma(s) = \frac{s!(n-s-1)!}{n!}$. Then we have

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \gamma(|S|)[v(S \cup \{i\}) - v(S)].$$

Example: Two-person bargaining game

- $N = \{1, 2\}$, $v(1, 2) = 1$, $v(1) = v(2) = 0$.
- Since $n = 2$, we have $\gamma(0) = \gamma(1) = \frac{1}{2}$.
- For player 1, we have

S	$\emptyset$	$\{2\}$
$v(S \cup \{1\}) - v(S)$	0	1

Hence, $\phi_1(v) = 0\frac{1}{2} + 1\frac{1}{2} = \frac{1}{2}$.

Example: Two-person bargaining game (Cont.)

- For player 2, we have

S	$\emptyset$	$\{1\}$
$v(S \cup \{1\}) - v(S)$	0	1

Hence, $\phi_2(v) = 0\frac{1}{2} + 1\frac{1}{2} = \frac{1}{2}$.

- For player 2, we can get $\phi_2(v) = \frac{1}{2}$ by efficiency condition directly.

Example: Three-person majority game

- $N = \{1, 2, 3\}, v(1) = v(2) = v(3) = 0,$
  $v(1, 2) = v(1, 3) = v(2, 3) = v(N) = 1$
- Since $n = 3$, we have $\gamma(0) = \gamma(2) = \frac{1}{3}$, and $\gamma(1) = \frac{1}{6}$.
- For player 1, we have

S	$\emptyset$	$\{2\}$	$\{3\}$	$\{2,3\}$
$v(S \cup \{1\}) - v(S)$	0	1	1	0

Hence, $\phi_1(v) = 0\frac{1}{3} + 1\frac{1}{6} + 1\frac{1}{6} + 0\frac{1}{3} = \frac{1}{3}$.

Example: Three-person majority game (Cont.)

- For player 2, we have

S	$\emptyset$	$\{1\}$	$\{3\}$	$\{1,3\}$
$v(S \cup \{2\}) - v(S)$	0	1	1	0

  Hence, $\phi_2(v) = 0\frac{1}{3} + 1\frac{1}{6} + 1\frac{1}{6} + 0\frac{1}{3} = \frac{1}{3}$.
- For player 3, we can get $\phi_3(v) = \frac{1}{3}$ by efficiency condition directly.

Example: Market with two sellers and one buyer

- $N = \{1, 2, 3\}$, $v(1, 2, 3) = v(1, 2) = v(1, 3) = 1$, and $v(S) = 0$ for all other $S \subseteq N$.
- Since $n = 3$, we have $\gamma(0) = \gamma(2) = \frac{1}{3}$, and $\gamma(1) = \frac{1}{6}$.
- For player 1, we have

S	$\emptyset$	$\{2\}$	$\{3\}$	$\{2,3\}$
$v(S \cup \{1\}) - v(S)$	0	1	1	1

Hence, $\phi_1(v) = 0\frac{1}{3} + 1\frac{1}{6} + 1\frac{1}{6} + 1\frac{1}{3} = \frac{2}{3}$.

Example: Market with two sellers and one buyer (Cont.)

- For player 2, we have

S	$\emptyset$	$\{1\}$	$\{3\}$	$\{1,3\}$
$v(S \cup \{2\}) - v(S)$	0	1	0	0

  Hence, $\phi_2(v) = 0\frac{1}{3} + 1\frac{1}{6} + 0\frac{1}{6} + 0\frac{1}{3} = \frac{1}{6}$.
- For player 3, we can get $\phi_3(v) = \frac{1}{6}$ by efficiency condition directly.
- Note that the core allocation in the example above $(1, 0, 0)$ differs considerably from the Shapley value $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$. One can interpret that zero payoff to players 2 and 3 in the core allocation as the result of cutthroat competition between them.

Core vs. Shapley value

- Core: solution concept that assigns the set of payoffs that cannot be improved upon by any coalition.
- Shapley value: solution concept that assigns the average of marginal contributions to coalitions.

Bargaining problem

Bargaining problem

A two-person bargaining problem, denoted by $\langle U, d\rangle$, consists of

- U is the set of possible agreements in terms of utilities that they yield to 1 and 2. An element of U is a pair $u = (u_1, u_2)$.
- d is a pair (d_1, d_2), called the disagreement point or threat point.

If agreement $u = (u_1, u_2) \in U$ is reached, then 1 gets utility u_1 and 2 gets utility u_2. If no agreement is reached then 1 gets utility d_1 and 2 gets utility d_2.

The set of two-person bargaining games is denoted by W.

Convention

Assume that

- U is compact and convex.
- U contains a point y for which $y_i > d_i$ for $i = 1, 2$, that is, bargaining is worthwhile for both the players.

Nash bargaining solution

The Nash bargaining solution is a mapping $f\colon W \to \mathbb{R}^2$ that associates a unique element $f(U,d)$ with the game $\langle U,d\rangle$, satisfying the following axioms:

N1. Feasibility: $f(U,d) \in U$.

N2. Individual rationality: $f(U,d) \geq d$ for all $\langle U,d\rangle \in W$.

N3. Pareto optimality: $f(U,d)$ is Pareto optimal. That is, there does not exist a point $(u_1,u_2) \in U$ such that

$$u_1 \geq f_1(U,d), u_2 \geq f_2(U,d), (u_1,u_2) \neq f(U,d).$$

Nash bargaining solution (Cont.)

N4. Symmetry: If $\langle U, d\rangle \in W$ satisfies $d_1 = d_2$ and $(x_1, x_2) \in U$ implies $(x_2, x_1) \in U$, then $f_1(U, d) = f_2(U, d)$.

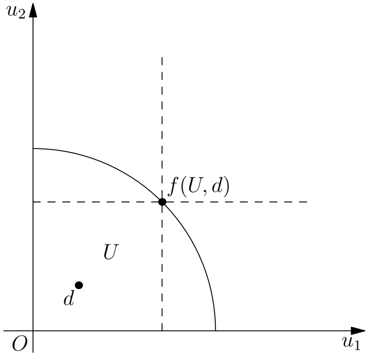
N5. Invariance under linear transformations: Let $a_1, a_2 > 0$, $b_1, b_2 \in \mathbb{R}$, and $\langle U, d\rangle, \langle U', d'\rangle \in W$ where $d'_i = a_i d_i + b_i$, $i = 1, 2$, and $U' = \{x \in \mathbb{R}^2 \mid x_i = a_i y_i + b_i, i = 1, 2, y \in U\}$. Then $f_i(U'_i, d'_i) = a_i f_i(U, d) + b_i$, $i = 1, 2$.

N6. Independence of irrelevant alternatives: If $\langle U, d\rangle, \langle U', d'\rangle \in W$, $d = d'$, $U \subseteq U'$, and $f(U', d') \in U$, then $f(U, d) = f(U', d')$.

The interpretation is that, given any bargaining problem $\langle U, d\rangle$, the solution function tells us that the agreement $u = f(U, d)$ will be reached.

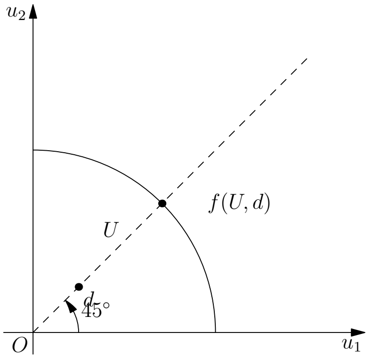
Pareto optimality

There are no points in U that are "North-East" of $f(U, d)$.



Symmetry

Suppose that $\langle U, d\rangle$ is such that U is symmetric around the 45-degree line and $d_1 = d_2$, then $f_1(U, d) = f_2(U, d)$, that is, when everything in $\langle U, d\rangle$ is symmetric, the point $f(U, d)$ is itself on the 45-degree line.



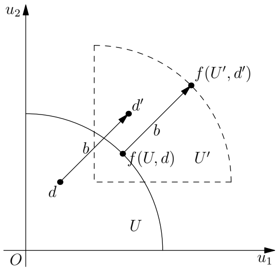
Invariance under linear transformations

Suppose we have two bargaining problems $\langle U, d\rangle$ and $\langle U', d'\rangle$ with the following property. For some vector $b = (b_1, b_2)$,

$$d' = d + b, \quad U' = U + b.$$

Then invariance under linear transformations imposes that

$$f(U', d') = f(U, d) + b,$$



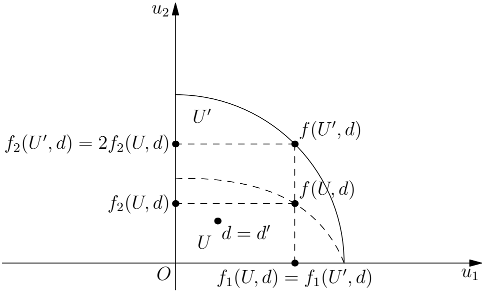
Invariance under linear transformations (Cont.)

Suppose we have two bargaining problems $\langle U, d\rangle$ and $\langle U', d'\rangle$ with $d = (0, 0)$ and the following property.

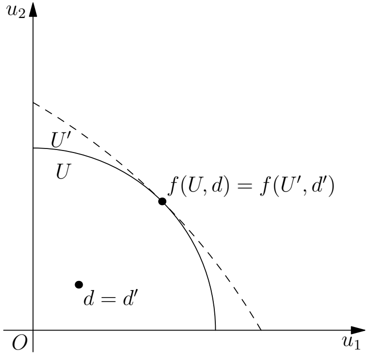
$$U'_1 = k_1 U_1, \quad U'_2 = k_2 U_2.$$

Then invariance under linear transformations imposes that

$$f_1(U', d) = k_1 f_1(U, d), \quad f_2(U', d) = k_2 f_2(U, d).$$



Independence of irrelevant alternatives



Nash bargaining solution

- Theorem: A game $\langle U, d\rangle \in W$ has a unique Nash solution $u^* = f(U, d)$ satisfying Conditions N1–N6. Furthermore, the solution u^* satisfies Conditions N1–N6 if and only if

$$(u_1^* - d_1)(u_2^* - d_2) > (u_1 - d_1)(u_2 - d_2)$$

  for all $u \in U$, $u \geq d$, and $u \neq u^*$.
- Remark:
  - Existence of an optimal solution: Since the set U is compact and the objective function is continuous, there exists an optimal solution.
  - Uniqueness of the optimal solution: The objective function is strictly quasi-concave. Therefore, maximization problem has a unique optimal solution.

Example

Find the Shapley values of the game with $N = \{1, 2\}$ and the characteristic function v. Now consider the bargaining game where $U = \{(u_1, u_2) \mid u_1 + u_2 = v(N), u_1 \geq v(\{1\}), u_2 \geq v(\{2\})\}$ and $d = (v(\{1\}), v(\{2\}))$. Find the bargaining solution of the game (U, d).

Answer.

- Since $n = 2$, we have $\gamma(0) = \gamma(1) = \frac{1}{2}$. Denote $v = v(N)$, $v_1 = v(\{1\})$ and $v_2 = v(\{2\})$.
- Shapley value: For player i,

S	$\emptyset$	$\{j\}$
$v(S \cup \{i\}) - v(S)$	v_i	$v - v_j$

Hence the Shapley value for player i is $\frac{v_i + v - v_j}{2}$.

□

Example (Cont.)

Answer (Cont.)

- To get the Nash bargaining solution, we solve the following problem
$$\max_{u_1+u_2=v, u_1 \geq v_1, u_2 \geq v_2} (u_1 - v_1)(u_2 - v_2).$$
The solution is $u_i^* = \frac{v_i + v - v_j}{2}$. Note that we need to check whether $u_i^* \geq v_i$.
- Hence, both Nash bargaining solution and the Shapley value give the same result.

□