Game Theory

Dynamic games of complete information

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Motivating Example 1: Grenade Game

- Consider a two-move game between two players.
    - First, player 1 decides whether to give \$1000 to player 2.
    - Second, after observing the choice of player 1, player 2 chooses whether to explode a grenade that will kill both of them.

  Player 2 can threaten player 1 by saying "Give the money to me, otherwise I will explode the grenade to kill you!"
- Question: What should player 1 do in the first place? Is player 2's threat credible to player 1? What is the outcome of this simple game?

Motivating Example 2: The Farmer and The Snake

- On a winter evening, a farmer found a snake frozen with cold. The farmer wanted to save the snake, which would make himself happy. But he was worried if the snake would bite him after it was saved. Believing that the snake would be grateful, the farmer saved it. However, when the snake was recovered, it bit and killed the farmer immediately.
- Question: Why shouldn't the farmer save the snake?

Introduction

- The two examples differ from the games that we have studied before: players take actions sequentially, rather than simultaneously.
- These are examples of dynamic games.
- The central issue of dynamic games is credibility.
- We want to study dynamic games of complete information.
- Dynamic: sequential choice, or repeated play
- Complete information: each player's payoff function is common knowledge among all players

Introduction

- Two types of dynamic games of complete information:
  1. Dynamic games of complete and perfect information
  2. Dynamic games of complete and imperfect information
- In static games of complete information, we use normal-form representation to describe a game.
- Now we use extensive-form representation for dynamic games.
- In particular, we will draw game trees.

Games of Perfect Information

- Consider a two-player and two-stage game.
- Player 1 chooses an action L or R.
- Player 2 observes player 1's action and then chooses an action L' or R'.
- Each path (a combination of two actions) in the following tree is followed by two payoffs: the first for player 1 and the second for player 2.

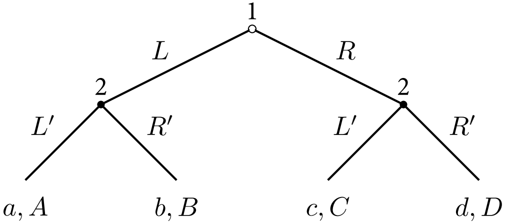


Figure: Extensive-form representation using a game tree

Games of Perfect Information

- The above game is an example of dynamic games of complete and perfect information.
- This type of games takes the following form:
  - Player 1 chooses an action a_1 from the feasible set A_1;
  - Player 2 observes a_1 and then chooses an action a_2 from the feasible set A_2;
  - Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.
- Note that
  - A_2 may depend on the action a_1, i.e., $A_2(a_1)$.
  - Some action a_1 may even end the game, so that $A_2(a_1)$ is an empty set (i.e., no choice of player 2).

Games of Perfect Information

- In Example 1:
- $A_1 = \{L, R\}$, where $L =$ "give \$1000" and $R =$ "don't give";
- $A_2(L) = A_2(R) = \{L', R'\}$, where $L' =$ "explode" and $R' =$ "don't explode".

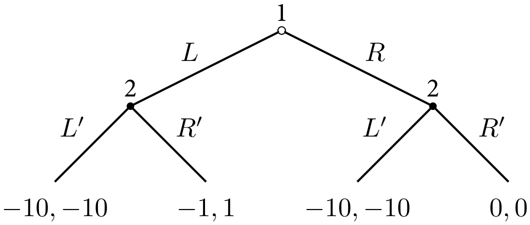


Figure: Game tree for example 1

Games of Perfect Information

- In Example 2:
- $A_1 = \{L, R\}$, where $L =$ “save” and $R =$ “don’t save”;
- $A_2(L) = \{L', R'\}$, where $L' =$ “bite” and $R' =$ “don’t bite”.

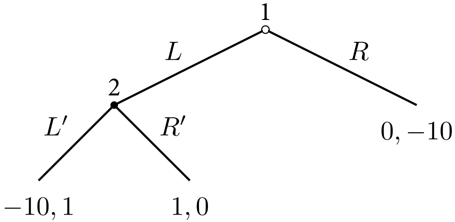


Figure: Game tree for example 2

Games of Perfect Information

- Some key features of dynamic games of complete and perfect information:
    1. the moves occur in sequence;
    2. all previous moves are observed before the next move is chosen;
    3. the players' payoffs from each combination of moves are common knowledge.
- How to solve this type of games?
- We use backwards induction (逆向归纳).

Backwards Induction

- In the second stage, player 2 observes the action (say a_1) chosen by player 1 in the first stage, and then chooses an action by solving

$$\max_{a_2 \in A_2} u_2(a_1, a_2).$$

- Assume this optimization problem has a unique solution, denoted by $R_2(a_1)$. This is player 2's best response to player 1's action a_1.
- For example, $R_2(L) = R'$ and $R_2(R) = L'$.

Backwards Induction

- In the first stage, knowing player 2's best response, player 1's problem becomes
$$\max_{a_1 \in A_1} u_1(a_1, R_2(a_1)).$$
- Assume it also has a unique solution, denoted by a_1^*.
- For example, $a_1^* = R$ and $R_2(a_1^*) = L'$.
- We call $(a_1^*, R_2(a_1^*))$ the backwards-induction outcome of the game.

Backwards Induction

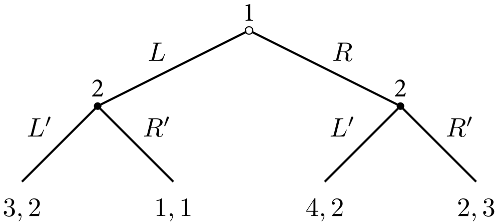
- In Example 1:
  - $R_2(L) = R_2(R) = R'$
  - $a_1^* = R$ and $R_2(a_1^*) = R'$
  - The backwards-induction outcome is (R, R').
- In Example 2:
  - $R_2(L) = L'$
  - $a_1^* = R$
  - The backwards-induction outcome is R.

Backwards Induction

- What is the relationship between a backwards-induction outcome and a Nash equilibrium?
- If both players choose their actions simultaneously, then the Nash equilibrium (a_1^{**}, a_2^{**}) is the intersection of two best responses, i.e., it solves
$$a_1^{**} = R_1(a_2^{**}),\ a_2^{**} = R_2(a_1^{**}).$$
- In the backwards-induction outcome, a_1^* is determined by maximizing $u_1(a_1, R_2(a_1))$, and we let $a_2^* = R_2(a_1^*)$.
- Since a_1^* may not maximize $u_1(a_1, a_2^*)$, the Nash equilibrium (a_1^{**}, a_2^{**}) can be different from the backwards-induction outcome (a_1^*, a_2^*).

Backwards Induction

- Consider the following game:



- $R_2(L) = L'$ and $R_2(R) = R'$
- The backwards-induction outcome is (L, L').

Backwards Induction

- Suppose both players choose actions simultaneously, then they play the following game:

		Player 2	
		L'	R'
Player 1	L	$3, 2$	$1, 1$
	R	$4, 2$	$2, 3$

- The Nash equilibrium is (R, R'), which differs from the backwards-induction outcome (L, L').
- The backwards-induction outcome in a dynamic game could be different from the Nash equilibrium of the corresponding game played simultaneously.

Stackelberg Model of Duopoly

- Consider a dominant firm moving first and a follower moving second.
- The game is played as follows:
  - Firm 1 chooses a quantity $q_1 \geq 0$.
  - Firm 2 observes q_1 and then chooses a quantity $q_2 \geq 0$.
  - The payoff of firm i is the profit

$$\pi_i(q_1, q_2) = q_i[P(Q) - c],$$

  where $Q = q_1 + q_2$ and

$$P(Q) = \begin{cases} a - Q, & \text{if } Q < a; \\ 0, & \text{if } Q \geq a. \end{cases}$$

- How to find the backwards-induction outcome?

Stackelberg Model of Duopoly

- First, find the best response function $R_2(q_1)$ for firm 2, i.e., for any given q_1, find q_2 that solves

$$\max_{q_2 \geq 0} \pi_2(q_1, q_2),$$

where

$$\pi_2(q_1, q_2) = \begin{cases} q_2(a - q_1 - q_2 - c), & \text{if } q_1 + q_2 < a; \\ -cq_2, & \text{if } q_1 + q_2 \geq a. \end{cases}$$

- Then we have

$$R_2(q_1) = \begin{cases} \frac{a-c-q_1}{2}, & \text{if } q_1 < a - c; \\ 0, & \text{if } q_1 \geq a - c. \end{cases}$$

- $R_2(q_1)$ is the same as that in the Cournot model.

Stackelberg Model of Duopoly

- Second, firm 1 knows $R_2(q_1)$ and solves

$$\max_{q_1 \geq 0} \pi_1(q_1, R_2(q_1)),$$

where

$$\pi_1(q_1, R_2(q_1)) = \begin{cases} q_1 \left[a - q_1 - \frac{a-q_1-c}{2} - c\right], & \text{if } q_1 < a - c; \\ q_1[a - q_1 - c], & \text{if } a - c \leq q_1 < a; \\ -cq_1, & \text{if } q_1 \geq a. \end{cases}$$

Stackelberg Model of Duopoly

- Clearly, for $q_1 > a - c$, firm 1's profit is always negative.
- Thus we only need to solve

$$\max_{q_1 \geq 0} q_1 \left[a - q_1 - \frac{a - q_1 - c}{2} - c\right] = \max_{q_1 \geq 0} \left[\frac{1}{2} q_1 (a - q_1 - c)\right],$$

which leads to the following first-order condition

$$a - c - 2q_1 = 0.$$

- The optimal choice of firm 1 is

$$q_1^* = \frac{a - c}{2}.$$

Stackelberg Model of Duopoly

- The quantity chosen by firm 2 is

$$q_2^* = R_2(q_1^*) = \frac{a-c}{4}.$$

- The market price is

$$P^* = a - q_1^* - q_2^* = c + \frac{a-c}{4}.$$

- Firms' profits and the total profit are

$$\pi_1^* = \frac{(a-c)^2}{8}, \ \pi_2^* = \frac{(a-c)^2}{16}, \text{ and } \Pi^* = \frac{3(a-c)^2}{16}.$$

Stackelberg Model of Duopoly

- Comparison between Cournot model and Stackelberg model:

Variable	Cournot Model	Stackelberg Model
q_1^*	$\frac{a-c}{3}$	$\frac{a-c}{2}$
q_2^*	$\frac{a-c}{3}$	$\frac{a-c}{4}$
π_1^*	$\frac{(a-c)^2}{9}$	$\frac{(a-c)^2}{8}$
π_2^*	$\frac{(a-c)^2}{9}$	$\frac{(a-c)^2}{16}$
Π^*	$\frac{2(a-c)^2}{9}$	$\frac{3(a-c)^2}{16}$
P^*	$c+\frac{a-c}{3}$	$c+\frac{a-c}{4}$

Games of Imperfect Information

- Consider the following simple two-stage game:
  - Players 1 and 2 simultaneously choose actions a_1 and a_2 from the feasible sets A_1 and A_2, respectively.
  - Players 3 and 4 observe the outcome of the first stage (a_1, a_2) and then simultaneously choose actions a_3 and a_4 from the feasible sets A_3 and A_4, respectively.
  - Payoffs are $u_i(a_1, a_2, a_3, a_4)$ for $i = 1, 2, 3, 4$.
- This game differs from the two-stage game with perfect information, since there are simultaneous moves within each stage.

Games of Imperfect Information

- We solve this game by using the idea of backwards induction.
- For each given (a_1, a_2), players 3 and 4 try to find the Nash equilibrium in stage 2.
- Assume the second-stage game has a unique Nash equilibrium

$$(a_3^*(a_1, a_2), a_4^*(a_1, a_2)).$$

- Then, player 1 and player 2 play a simultaneous-move game with payoffs

$$u_i(a_1, a_2, a_3^*(a_1, a_2), a_4^*(a_1, a_2)), \text{ for } i = 1, 2.$$

Games of Imperfect Information

- Suppose (a_1^*, a_2^*) is the unique Nash equilibrium of this simultaneous-move game.
- Then
$$(a_1^*, a_2^*, a_3^*(a_1^*, a_2^*), a_4^*(a_1^*, a_2^*))$$
is the subgame-perfect outcome of the two-stage game.

Bank Runs

- Two investors have each deposited $\$5$ millions with a bank. The bank has invested these deposits in a long-term project.
- If the bank is forced to liquidate its investment before the project matures, a total of $\$8$ millions can be recovered.
- If the bank allows the investment to reach maturity, the project will pay out a total of $\$16$ millions.
- There are two dates at which the investors can make withdrawals at the bank: Date 1 is before the bank's investment matures and Date 2 is after.
- Suppose there is no discounting.

Bank Runs

- Players' payoffs in date 1:

	Withdraw	Don't
Withdraw	4, 4	5, 3
Don't	3, 5	next stage

- Players' payoffs in date 2:

	Withdraw	Don't
Withdraw	8, 8	11, 5
Don't	5, 11	8, 8

Bank Runs

- We work backwards:
- At date 2, in the unique Nash equilibrium, both withdraw and each obtains $8.
- At date 1, they play the following game:

	Withdraw	Don't
Withdraw	4, 4	5, 3
Don't	3, 5	8, 8

- There are 2 pure-strategy Nash equilibria of this game:
    1. Both withdraw and each obtains $4;
    2. Both don't and each obtains $8.

Bank Runs

- There are 2 subgame-perfect outcomes of the original two-stage game:
  1. Both withdraw at date 1 to obtain \$4 $\rightarrow$ the case of bank run
  2. Both don't withdraw at date 1 but do at date 2, and obtain \$8
- Although there are two possible subgame-perfect outcomes, only the second one is efficient.
- This model does not predict when bank runs will occur, but does show that they can occur as an equilibrium outcome.

Normal-form Representation of Games

- In static games, we consider normal-form representation to describe a game.

Definition

The normal-form representation of a game specifies

1. the players in the game;
2. the strategies available to each player;
3. the payoff received by each player for each combination of strategies that could be chosen by the players.

Extensive-form Representation of Games

- In dynamic games, we need to use extensive-form representation.

Definition

The extensive-form representation of a game specifies:

(1) the players in the game;

(2a) when each player has the move;

(2b) what each player can do at each of his or her opportunities to move;

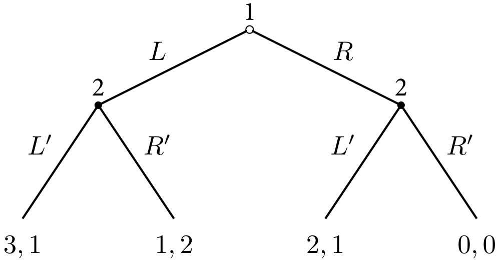
(2c) what each player knows at each of his or her opportunities to move;

(3) the payoffs received by each player for each combination of moves that could be chosen by the players.

- Note that (2a) to (2c) describe strategies of each player in detail.

Extensive-form Representation of Games

- We use game trees for extensive-form games.
- Example 1:



Extensive-form Representation of Games

- In Example 1, the game tree begins with a decision node for player 1, which is also the initial node of the game.
- After player 1's choice (L or R) is made, player 2's decision node is reached. And player 2 needs to decide whether to choose L' or R'.
- A terminal node is reached after player 2's move (i.e., the game ends), and payoffs of players are realized.

Information Set

- A dynamic game of complete and perfect information is a game in which the players move in sequence, all previous moves are observed before the next move is chosen, and payoffs are common knowledge.
- Such games can be easily represented by a game tree.
- For games with imperfect information, some previous moves are not observed by the player with the current move.
- To present this kind of ignorance of previous moves and to describe what each player knows at each of his/her move, we introduce the notion of a player's information set.

Information Set

Definition

An information set for a player is a collection of decision nodes satisfying:

(i) The player needs to move at every node in the information set.

(ii) When the play of the game reaches a node in the information set, the player with the move does not know which node in the set has (or has not) been reached.

- (ii) implies that the player must have the same set of feasible actions at each decision node in an information set, otherwise the player could infer from the set of actions available that some node(s) had or had not been reached.

Information Set

- In an extensive-form game, a collection of decision nodes, which constitutes an information set, is connected by a dotted line.
- We can use information set to differentiate perfect and imperfect information.
- A game is of perfect information if every information set is a singleton, and of imperfect information if there is at least one non-singleton information set.

Information Set

- Let's consider a two-player simultaneous-move (static) game as follows:
  1. Player 1 chooses $a_1 \in A_1$;
  2. Player 2 does not observe player 1's move but chooses an $a_2 \in A_2$;
  3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.
- We need an information set to describe player 2's ignorance of player 1's actions.
- The above static game of complete information can be represented as a dynamic game of complete but imperfect information.

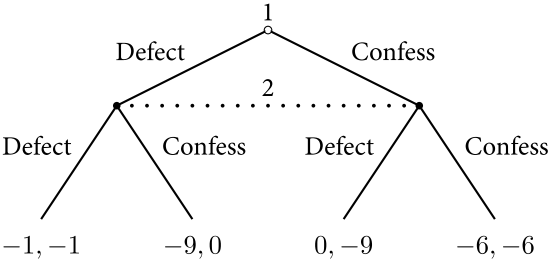
Information Set

- Example 2: Prisoners' Dilemma
- The normal-form representation is

		Prisoner 2	
		Defect	Confess
Prisoner 1	Defect	$-1, -1$	$-9, 0$
	Confess	$0, -9$	$-6, -6$

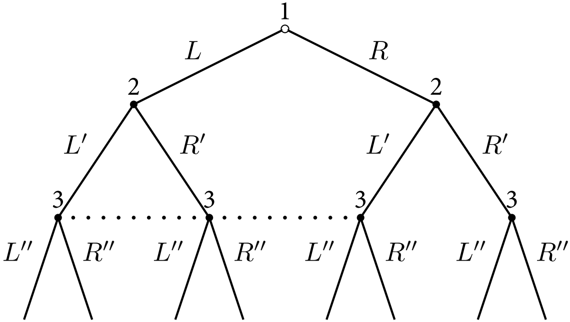
Information Set

- Example 2:
- The extensive-form representation is:



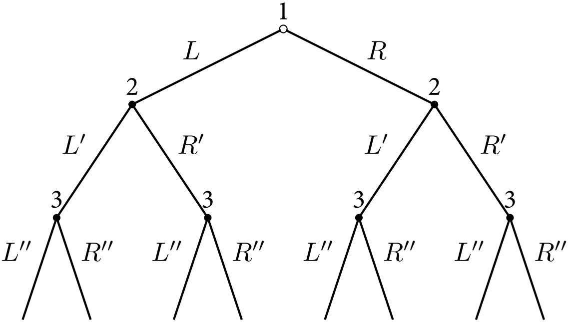
Information Set

- Example 3:
- Player 3 has a non-singleton information set and a singleton information set.



Information Set

- Example 4:
- Player 3 has 4 singleton information sets.



Strategy

Definition

A strategy for a player is a complete plan of actions. It specifies a feasible action for the player in every contingency in which the player might be called on to act.

- An equivalent definition: A player's strategy is a function which assigns an action to each information set (not each decision node) belonging to the player.
- An action and a strategy do not make a big difference in static games, while they do in dynamic games.

Strategy

- In Example 1:
- Player 1 has 2 actions (and also 2 strategies): L and R.
- Player 2 has 2 actions: L' and R', but 4 strategies:

$$(L', L'); (L', R'); (R', L'); (R', R').$$

- For example, the strategy (L', R') means:
  - if player 1 plays L, then player 2 plays L';
  - if player 1 plays R, then player 2 plays R'.

Strategy

- In Example 2:
- Both players have two actions and also two strategies: Defect and Confess.
- In Example 3:
- Player 1 has two strategies: L and R.
- Player 2 has four strategies:

$$(L', L'); (L', R'); (R', L'); (R', R').$$

- Player 3 has four strategies

$$(L'', L''); (L'', R''); (R'', L''); (R'', R'').$$

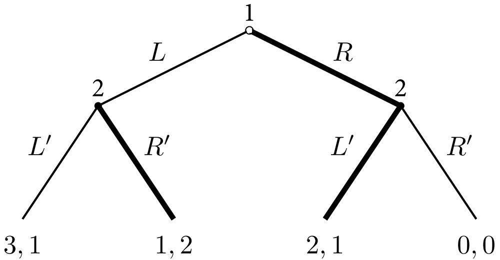
Strategy

- In Example 4:
- Player 3 has 16 strategies.
- For instance, the strategy (L'', R'', R'', L'') means:
  - if player 1 plays L and player 2 plays L', then player 3 plays L'';
  - if player 1 plays L and player 2 plays R', then player 3 plays R'';
  - if player 1 plays R and player 2 plays L', then player 3 plays R'';
  - if player 1 plays R and player 2 plays R', then player 3 plays L''.

Strategy

- In the Cournot model of duopoly, firm i's action and strategy is the same, i.e., $q_i \geq 0$.
- In the Stackelberg model, the action and strategy for firm 1 (the leader) is again $q_1 \geq 0$.
- How about firm 2 (the follower)? How many information sets does firm 2 have?
- Firm 2's action is $q_2 \geq 0$, but its strategy is $q_2(q_1) \geq 0$ for any $q_1 \geq 0$.

Problems of NE



- There are two NE: $(L, R'R')$ and $(R, R'L')$.
- $(L, R'R')$ has a problem: No matter which action is chosen by Player 1, player 2 must choose L' at the right node.
- Interpretation: Player 2 tells player 1: if you choose R, I will choose R' (threat), then each of us will get 0.
  This threat is non-creditable: Player 1 should not believe that player 2 will choose R' after observing R.

Subgame-Perfect Nash Equilibrium

Definition

A subgame in an extensive-form game

(a) begins at a decision node n that is a singleton information set (but is not the game's initial node);

(b) includes all the decision and terminal nodes following node n in the game tree (but no nodes that do not follow n);

(c) does not cut any information sets (i.e., if a decision node n' follows n in the game tree, then all other nodes in the information set containing n' must also follow n, and so must be included in the subgame).

Subgame-Perfect Nash Equilibrium

- Example 1 has 2 subgames.
- Example 2 has no subgame (since player 2's decision nodes are in the same non-singleton information set).
- Example 3 has only 1 subgame, beginning at player 3's decision node following R and R'. (The subtrees beginning at player 2's decision nodes violate (c)).
- Example 4 has 6 subgames.

Subgame-Perfect Nash Equilibrium

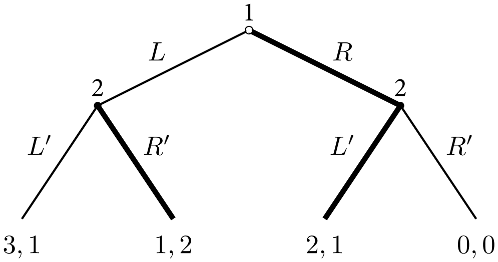
Definition (Selten, 1965)

A Nash equilibrium is subgame-perfect if the players' strategies constitute a Nash equilibrium in every subgame.

- It can be shown that any finite dynamic game of complete information has a subgame-perfect Nash equilibrium, perhaps in mixed-strategies.
- To find subgame-perfect Nash equilibria, we first need to find Nash equilibria in each subgame, and then use backwards-induction to solve for the whole game.

Subgame-Perfect Nash Equilibrium

- In Example 1, there are two subgames: in the left subgame, the Nash equilibrium involves the player 2 choosing R'; in the right subgame, the Nash equilibrium involves the player 2 choosing L'.
- The subgame-perfect Nash equilibrium is $(R, (R', L'))$.
- We can use thick lines to represent the equilibrium paths.



Subgame-Perfect Nash Equilibrium

- Subgame-perfect Nash equilibrium is closely related to two previous concepts:
  1. backwards-induction outcome
  2. subgame-perfect outcome
- What's the difference between an equilibrium and an outcome?
- An equilibrium is a collection of players' strategy profiles, while an outcome is a collection of players' actions.

Subgame-Perfect Nash Equilibrium

- Consider the following two-stage game of complete and perfect information:
  1. Player 1 chooses an action $a_1 \in A_1$;
  2. Player 2 observes a_1 and then chooses an action $a_2 \in A_2$;
  3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.
- The best response $R_2(a_1)$ solves $\max_{a_2 \in A_2} u_2(a_1, a_2)$.
- a_1^* solves $\max_{a_1 \in A_1} u_1(a_1, R_2(a_1))$.

Subgame-Perfect Nash Equilibrium

- The backwards-induction outcome is $(a_1^*, R_2(a_1^*))$.
- The subgame-perfect Nash equilibrium is $(a_1^*, R_2(\cdot))$.
- Note that $R_2(a_1^*)$ is an action, while $R_2(\cdot)$ is a strategy for player 2.
- In Example 1:
- (R, L') is the backwards-induction outcome, while $(R, (R', L'))$ is the subgame-perfect Nash equilibrium.
- In the Stackelberg model:
- The backwards-induction outcome is (q_1^*, q_2^*), where $q_1^* = \frac{a-c}{2}$ and $q_2^* = \frac{a-c}{4}$, while the subgame-perfect Nash equilibrium is $(q_1^*, R_2(q_1))$, where $R_2(q_1) = \frac{a-c-q_1}{2}$.

Subgame-Perfect Nash Equilibrium

- Consider the following two-stage game of complete but imperfect information:
  - Players 1 and 2 simultaneously choose actions a_1 and a_2 from the feasible sets A_1 and A_2, respectively.
  - Players 3 and 4 observe the outcome of the first stage (a_1, a_2) and then simultaneously choose actions a_3 and a_4 from the feasible sets A_3 and A_4, respectively.
  - Payoffs are $u_i(a_1, a_2, a_3, a_4)$ for $i = 1, 2, 3, 4$.
- For each given (a_1, a_2), players 3 and 4 play the Nash equilibrium in stage 2

$$(a_3^*(a_1, a_2), a_4^*(a_1, a_2)).$$

Subgame-Perfect Nash Equilibrium

- Then, player 1 and player 2 play a simultaneous-move game with payoffs
$$u_i(a_1, a_2, a_3^*(a_1, a_2), a_4^*(a_1, a_2)), i = 1, 2$$
- Suppose (a_1^*, a_2^*) is the unique Nash equilibrium in stage 1.
- Then the subgame-perfect outcome is
$$(a_1^*, a_2^*, a_3^*(a_1^*, a_2^*), a_4^*(a_1^*, a_2^*)).$$
- The subgame-perfect Nash equilibrium is
$$(a_1^*, a_2^*, a_3^*(a_1, a_2), a_4^*(a_1, a_2)).$$

Nash Equilibrium vs. Subgame-Perfect Nash Equilibrium

- A Nash equilibrium may not be subgame-perfect.
- In Example 1, the normal-form representation is

		Player 2			
		(L', L')	(L', R')	(R', L')	(R', R')
Player 1	L	$3, 1$	$3, 1$	$1, 2$	$1, 2$
	R	$2, 1$	$0, 0$	$2, 1$	$0, 0$

- Two Nash equilibria: $(L, (R', R'))$ and $(R, (R', L'))$
- Only one subgame-perfect Nash equilibrium: $(R, (R', L'))$

Nash Equilibrium vs. Subgame-Perfect Nash Equilibrium

- The Nash equilibrium $(R, (R', L'))$ is subgame-perfect, because R' and L' are the optimal strategies in the left and right subgames, respectively, where player 2 is the only player.
- On the other hand, the Nash equilibrium $(L, (R', R'))$ is not subgame-perfect, because when player 1 chooses R, R' is not optimal to player 2 in the right subgame, i.e., R' is not a Nash equilibrium in that subgame.
- One can think the strategy (R', R') by player 2 as a threat to player 1.

Nash Equilibrium vs. Subgame-Perfect Nash Equilibrium

- Nash equilibria that rely on non-credible threats or promises can be eliminated by the requirement of subgame perfection.
- Subgame-perfect Nash equilibrium is a refinement of Nash equilibrium, i.e.,

$$\{\text{Subgame-perfect Nash equilibria}\} \subseteq \{\text{Nash equilibria}\}$$

Application: Sequential Bargaining Game

- Suppose players 1 and 2 are bargaining over one dollar.
- They discount payoffs received a period later by a discount factor δ, where $0 < \delta < 1$.
- Consider the following three-period bargaining game:
  - (1a) In the first period, player 1 proposes $s_1(1)$ for himself and $s_2(1)$ for player 2.
  - (1b) Player 2 either accepts the offer to end the game or rejects the offer to continue the game.
  - (2a) In the second period, player 2 proposes $s_1(2)$ for player 1 and $s_2(2)$ for himself.
  - (2b) Player 1 either accepts the offer to end the game or rejects the offer to continue the game.
  - (3) In the third period, player 1 receives a share s of the dollar, leaving $1 - s$ to player 2.

Application: Sequential Bargaining Game

- Let $s_1(3) = s$ and $s_2(3) = 1 - s$.
- In general, in period t, $s_1(t)$ and $s_2(t)$ are offered to players 1 and 2. The offers satisfy
$$s_1(t) + s_2(t) = 1.$$
- The present value of payoff to player i is $\delta^{t-1} s_i(t)$ if the bargaining is ended in period t.
- We use backwards induction to solve the game.

Application: Sequential Bargaining Game

- In the second period, player 2 is at the move. Because the payoff to player 1 in period 3 is s, player 2 will offer $s_1(2) = \delta s$ to player 1 and $s_2(2) = 1 - \delta s$ to himself. Player 1 accepts the offer.
- In the first period, player 1 will offer $\delta(1 - \delta s)$ to player 2 and $1 - \delta(1 - \delta s)$ to himself, and player 2 will accept the offer. Then, the game ends.

Application: Sequential Bargaining Game

- The backwards-induction outcome of the three-period bargaining game:
- Player 1 offers the settlement

$$
\begin{aligned}
s_1^*(1) &= 1 - \delta(1 - \delta s), \\
s_2^*(1) &= \delta(1 - \delta s).
\end{aligned}
$$

- Player 2 accepts the offer, and the game ends.

Question 1

Denote by G the following game:

		Player 3	
		A	B
Player 2	A	$1, 1$	$5, 0$
	B	$0, 5$	$4, 4$

Game 1:

1. Player 1 chooses p from the set $\{0, 2\}$.
2. Players 2 and 3 observe p and then choose actions in G.
3. For $i \in \{2, 3\}$, Player i's payoff is the payoff from G plus the amount paid by Player 1, which is p if Player i played B and 0 if he played A. Player 1's payoff is δ minus the total amount he paid to Players 2 and 3, where

$$\delta = \begin{cases} 5, & \text{if both players 2 and 3 played } B, \\ 0, & \text{otherwise.} \end{cases}$$

Game 2: Same as Game 1, except that in stage 2 players 2 and 3 do not observe player 1's choice p. (Hint: write the game in a tri-matrix.)
Find the subgame-perfect outcome for each game.

Question 2

Three oligopolists operate in a market with inverse demand given by $P(Q) = a - Q$, where $Q = q_1 + q_2 + q_3$ and q_i is the quantity produced by firm i. Each firm has a constant marginal cost of production, c, and no fixed cost. The firms choose their quantities as follows: (1) firm 1 chooses $q_1 \geq 0$; (2) firms 2 and 3 observe q_1 and then simultaneously choose q_2 and q_3, respectively. What is the subgame-perfect outcome?

Question 3

Players 1 and 2 are bargaining over one dollar in two periods: In the first period, Player 1 proposes s_1 for himself and $1 - s_1$ for player 2. In the second period, player 2 decides whether to accept the offer or to reject the offer. If player 2 accepts the offer, the payoff are s_1 for player 1 and $1 - s_1$ for player 2. If player 2 rejects the offer, the payoff are zero for both players.

1. Describe all strategies of player 1 and player 2.
2. Find some (as many as you can) Nash equilibria.
3. Find a subgame-perfect Nash equilibrium of the game (write down your proof).
4. Find some Nash equilibria which are not subgame-perfect (write down your proof).

Question 4

Suppose the players in the infinite-horizon bargaining game have different discount factors: δ_1 for Player 1 and δ_2 for Player 2. Adapt the argument in the lecture to show that in the backwards-induction outcome, Player 1 offers the settlement

$$\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

to Player 2, who accepts.