# Game Theory

## Signaling games

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# Introduction

- Signaling games are a well-studied class of dynamic games of incomplete information.
- The concept of "signaling" refers to strategic models where informed agents take some observable actions before uninformed agents make their strategic decisions.
- Signaling games are a relatively simple setting in which to study
  - how players update beliefs based on observed actions (signals);
  - how players try to strategically reveal or conceal private information by their choice of actions.
- There are many applications of signaling games in economics (for example, Spence's job-market signaling model).

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# Timing

- A simple signaling game is a dynamic game of incomplete information involving two players: a Sender ($S$) and a Receiver ($R$).
- The timing of the game is as follows:
  1. Nature draws a type $t_i$ for the Sender from a set of feasible types $T = \{t_1, \ldots, t_I\}$ according to a probability distribution $P(t_i)$, where $P(t_i) > 0$ for every $i$ and $P(t_1) + \cdots + P(t_I) = 1$.
  2. The Sender observes $t_i$ and then chooses a message $m_j$ from a set of feasible messages $M = \{m_1, \ldots, m_J\}$.
  3. The Receiver observes $m_j$ (but not $t_i$) and then chooses an action $a_k$ from a set of feasible actions $A = \{a_1, \ldots, a_K\}$.
  4. Payoffs are given by $U_S(t_i, m_j, a_k)$ and $U_R(t_i, m_j, a_k)$.

# Strategy

- Consider the following signaling game:

$$T = \{t_1, t_2\}, A = \{a_1, a_2\}, P(t_1) = p, M = \{m_1, m_2\}.$$

- The Sender has four pure strategies:

$$(m_1, m_1), (m_1, m_2), (m_2, m_1), (m_2, m_2).$$

- The strategy $(m', m'')$ means the Sender of type $t_1$ chooses a message $m'$ and type $t_2$ chooses a message $m''$.

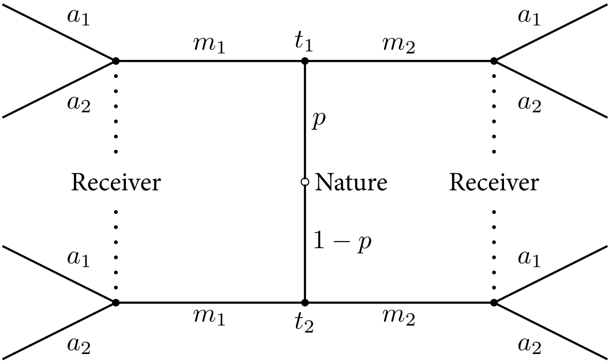
# Strategy

- Similarly, the Receiver has four pure strategies:

$$(a_1, a_1), (a_1, a_2), (a_2, a_1), (a_2, a_2).$$

- The strategy $(a', a'')$ means the Receiver plays $a'$ if the Sender chooses $m_1$ and plays $a''$ if the Sender chooses $m_2$.
- We call Sender's strategies $(m_1, m_1)$, $(m_2, m_2)$ to be pooling (because each type sends the same message), and $(m_1, m_2)$, $(m_2, m_1)$ to be separating (because each type sends a different message).

# Illustration



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# Signaling Requirement 1: Belief

We first translate the requirements for a perfect Bayesian equilibrium to the case of signaling games.

**Signaling Requirement 1**

After observing any message $m_j$ from $M$, the Receiver must have a belief about which types could have sent $m_j$.

Denote this belief by the probability distribution $\mu(t_i|m_j)$, where $\mu(t_i|m_j) \geq 0$ for each $t_i \in T$, and $\sum_{t_i \in T} \mu(t_i|m_j) = 1$.

# Signaling Requirement 2: Sequential rationality

## Signaling Requirement 2R

For each $m_j \in M$, the Receiver's action $a^*(m_j)$ must maximize the Receiver's expected utility, given the belief $\mu(t_i|m_j)$ about which types could have sent $m_j$. That is, $a^*(m_j)$ solves

$$\max_{a_k \in A} \sum_{t_i \in T} \mu(t_i|m_j) U_R(t_i, m_j, a_k).$$

## Signaling Requirement 2S

For each $t_i \in T$, the Sender's message $m^*(t_i)$ must maximize the Sender's utility, given the Receiver's strategy $a^*(m_j)$. That is, $m^*(t_i)$ solves

$$\max_{m_j \in M} U_S(t_i, m_j, a^*(m_j)).$$

# Signaling Requirement 2: Sequential rationality (Cont.)

- These two requirements imply that both the Receiver and the Sender act in an optimal way.
- Given the Sender's optimal strategy $m^*(t_i)$, i.e., $m^*$ is a function from $T$ into $M$, let $T_j = \{t_i \in T : m^*(t_i) = m_j\}$. $T_j$ is the set of all types sending the message $m_j$.
- The information set corresponding to $m_j$ is on the equilibrium path if $T_j \neq \emptyset$, and off the equilibrium path otherwise.

# Signaling Requirement 3: Rational belief

## Signaling Requirement 3

For each $m_j \in M$, if there exists $t_i \in T$ such that $m^*(t_i) = m_j$, i.e., $T_j \neq \emptyset$, then the Receiver's belief at the information set corresponding to $m_j$ must follow from Bayes' rule and the Sender's strategy:

$$\mu(t_i|m_j) = \frac{P(t_i)}{\sum_{t \in T_j} P(t)}, \forall t_i \in T_j.$$

# Perfect Bayesian Equilibria

**Definition**

A pure-strategy perfect Bayesian equilibrium in a signaling game is a pair of strategies $m^*(t_i)$ and $a^*(m_j)$ and a belief $\mu(t_i|m_j)$ satisfying Signaling Requirements (1), (2R), (2S), and (3).

- A strategy for the Sender is a function from the type space $T$ into the message space $M$; a strategy for the Receiver is a function from the message space $M$ into the action space $A$.
- For a perfect Bayesian equilibrium of a signaling game, if the Sender's strategy is pooling (or separating), then we call the equilibrium pooling (or separating), respectively.

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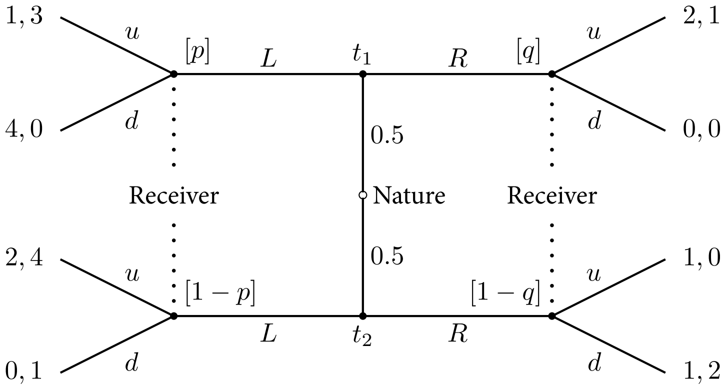
# Example

Find all pure-strategy perfect Bayesian equilibria in the following signaling game.




The first (the second) number is the payoff to the Sender (the Receiver).

# Formulation

- In this game,

$$T = \{t_1, t_2\}, P(t_1) = 0.5, M = \{L, R\}, A = \{u, d\}.$$

- The Sender's strategies are: $(L, L)$, $(L, R)$, $(R, L)$ and $(R, R)$, where $(m', m'')$ means that type $t_1$ chooses $m'$ and type $t_2$ chooses $m''$.
- The Receiver's strategies are: $(u, u)$, $(u, d)$, $(d, u)$, and $(d, d)$, where $(a', a'')$ means that the Receiver plays $a'$ following $L$ and $a''$ following $R$.
- We analyze the possibility of the four Sender's strategies to constitute perfect Bayesian equilibria.

# Case 1: PBE pooling on $L$

- Suppose the Sender adopts the strategy $(L, L)$.
- By Signaling Requirement 3, we have $p = 1 - p = 0.5$. Given this belief (or any belief) of the Receiver, the Receiver's best response to message $L$ is $u$, i.e., $a^*(L) = u$.
- For the message $R$, the Receiver's belief $q$ cannot be determined by Sender's strategy, and thus we can choose any belief $q$. Furthermore, both $a^*(R) = u$ and $a^*(R) = d$ are possible for some $q$. Indeed $a^*(R) = u$ iff $q \geq \frac{2}{3}$; and $a^*(R) = d$ iff $q \leq \frac{2}{3}$.
- We only need to see if sending $L$ is better than sending $R$ for both types $t_1$ and $t_2$.

# Case 1: PBE pooling on $L$ (Cont.)

- If $a^*(R) = u$, i.e., $(u, u)$ is the Receiver's strategy, then for type $t_1$, the Sender's payoff is 1 if $L$ is sent and 2 if $R$ is sent. Hence, sending $L$ is not optimal.
- If $a^*(R) = d$, i.e., $(u, d)$ is the Receiver's strategy, then for type $t_1$, the Sender's payoff is 1 if $L$ is sent and 0 if $R$ is sent, choosing $L$ is optimal; for type $t_2$, choosing $L$ is also optimal given $2 > 1$.
- Thus, $(L, L)$ is the Sender's best response to the Receiver's strategy $(u, d)$.
- Moreover, $(u, d)$ is also the Receiver's best response to the Sender's strategy $(L, L)$ if $q \leq \frac{2}{3}$.
- Therefore, $[(L, L), (u, d); p = \frac{1}{2}, q \leq \frac{2}{3}]$ is a pooling equilibrium.

# Case 1: PBE pooling on $L$ (Cont.)

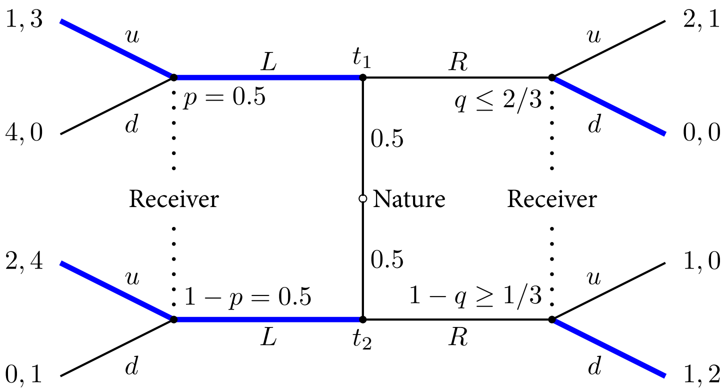



Figure: Pooling equilibrium: $[(L, L), (u, d); p = 0.5, q \leq \frac{2}{3}]$

# Case 2: PBE pooling on $R$

- Suppose the Sender adopts the strategy $(R, R)$.
- Then Signaling Requirement 3 implies that $q = 1 - q = \frac{1}{2}$. Given this belief, the Receiver's best response is to $R$ is $d$, i.e., $a^*(R) = d$, since $\frac{1}{2} < 1$.
- For the message $L$, we can choose any belief $p$. But we know for any $p$, the Receiver's best response to $L$ is $u$, i.e., $a^*(L) = u$.
- Given the Receiver's strategy $(u, d)$, for type $t_1$, the Sender's payoff is 0 if $R$ is sent and 1 if $L$ is sent, and thus $R$ is not optimal.
- Therefore, there is no equilibrium in which the Sender plays $(R, R)$.

# Case 3: Separation with $t_1$ playing $L$

- Suppose the Sender adopts the separating strategy $(L, R)$.
- Then, Signaling Requirement 3 implies $p = 1$ and $q = 0$. For these beliefs, we must have $a^*(L) = u$, and $a^*(R) = d$.
- Given the Receiver's strategy $(u, d)$, for type $t_2$, the Sender's payoff is 4 if $L$ is sent and 2 if $R$ is sent. Hence $R$ is not optimal.
- Therefore, there is no equilibrium in which the Sender plays $(L, R)$.

# Case 4: Separation with $t_1$ playing $R$

- Suppose the Sender adopts the separating strategy $(R, L)$.
- Then, Signaling Requirement 3 implies $p = 0$ and $q = 1$. For these beliefs, we have $a^*(L) = u$ and $a^*(R) = u$.
- Given the Receiver's strategy $(u, u)$, for type $t_1$, the Sender's payoff is 1 if $L$ is sent and 2 if $R$ is sent. Hence $R$ is optimal.
- For the Sender type $t_2$, the payoff is 2 if $L$ is sent and 1 if $R$ is sent. Hence $L$ is also optimal.
- Therefore, $[(R, L), (u, u); p = 0, q = 1]$ is a separating perfect Bayesian equilibrium.

# Case 4: Separation with $t_1$ playing $R$ (Cont.)

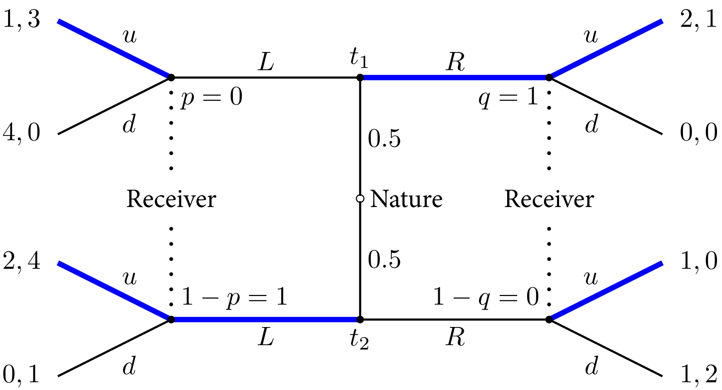



Figure: Separating equilibrium: $[(R, L), (u, u); p = 0, q = 1]$

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# How to find PBE

How to find (pure-strategy) perfect Bayesian equilibria in signaling games:

1. Start with a strategy of the Sender (pooling or separating);
2. If possible, calculate the beliefs of the Receiver using Bayes' rules. Otherwise, choose arbitrary beliefs;
3. Given the beliefs, find out the best response of the Receiver;
4. Check whether the Sender's strategy is a best response to the Receiver's strategy.

# How to find PBE (Cont.)

- Consider an alternative way to find perfect Bayesian equilibria.
- We first find Bayeisan Nash equilibria, and then check which equilibria are perfect Bayesian equilibria.
- Consider the following bi-matrix to represent the game:

Receiver

| Sender | $(u,u)$ | $(u,d)$ | $(d,u)$ | $(d,d)$ |
|---|---|---|---|---|
| $(L,L)$ | $1, \underline{2}, \underline{3.5}$ | $\underline{1}, \underline{2}, \underline{3.5}$ | $\underline{4}, 0, 0.5$ | $4, 0, 0.5$ |
| $(L,R)$ | $1, 1, 1.5$ | $\underline{1}, 1, \underline{2.5}$ | $\underline{4}, \underline{1}, 0$ | $\underline{4}, \underline{1}, 1$ |
| $(R,L)$ | $\underline{2}, \underline{2}, \underline{2.5}$ | $0, \underline{2}, 2$ | $2, 0, 1$ | $0, 0, 0.5$ |
| $(R,R)$ | $\underline{2}, 1, 0.5$ | $0, 1, \underline{1}$ | $2, \underline{1}, 0.5$ | $0, \underline{1}, \underline{1}$ |

- Two (pure-strategy) Bayesian Nash equilibria: $((L,L),(u,d))$ and $((R,L),(u,u))$

# How to find PBE (Cont.)

- To check whether they are perfect Bayesian equilibria, we only need to find beliefs, satisfying all four Signaling Requirements.
- For $(L, L)$, Bayes' rule requires $p = \frac{1}{2}$ and there is no requirement for $q$. Given the belief, $a^*(L) = u$, and $a^*(R) = d$ iff $q \leq \frac{2}{3}$. Thus $(u, d)$ is a best response to $(L, L)$ iff $p = \frac{1}{2}$ and $q \leq \frac{2}{3}$.
- For $(R, L)$, Bayes' rule requires $p = 0$ and $q = 1$. Given this belief, $a^*(L) = u$ and $a^*(R) = u$. Thus $(u, u)$ is a best response to $(R, L)$.
- Therefore, $[(L, L), (u, d); p = \frac{1}{2}, q \leq \frac{2}{3}]$ and $[(R, L), (u, u); p = 0, q = 1]$ are two perfect Bayesian equilibria.

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# Cheap-Talk Games

- Cheap-talk games are analogous to signaling games, but the Sender's messages are just talk, i.e., costless, non-binding, nonverifiable claims.
- Cheap talk cannot be informative in some cases (for example, Spence's job-market signaling model).
- There are situations where cheap talk can convey some information (although may not be fully precise), for example, Stein (1989), Matthews (1989), Austen-Smith (1990).
- In general, cheap talk can be informative under certain conditions.

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# Cheap-Talk Games

The timing of the simplest cheap-talk game is identical to the timing of the simplest signaling game (only payoff functions differ):

1. Nature draws a type $t_i$ for the Sender from a set of feasible types $T = \{t_1, \ldots, t_I\}$ according to a probability distribution $P(t_i)$, where $P(t_i) > 0$ for every $i$ and $P(t_1) + \cdots + P(t_I) = 1$.
2. The Sender observes $t_i$ and then chooses a message $m_j$ from a set of feasible messages $M = \{m_1, \ldots, m_J\}$.
3. The Receiver observes $m_j$ (but not $t_i$) and then chooses an action $a_k$ from a set of feasible actions $A = \{a_1, \ldots, a_K\}$.
4. Payoffs are given by $U_S(t_i, a_k)$ and $U_R(t_i, a_k)$ (independent of $m_j$).

# Cheap-Talk Games

- The key feature of the cheap-talk game is that the message has no direct effect on the payoffs of the Sender and the Receiver.
- The message can only be informative by changing the Receiver's belief about the Sender's type.
- Since anything can be said (i.e., $M$ can be a very large set), it is typically assumed that $M = T$.
- The definition of perfect Bayesian equilibrium in a cheap-talk game is identical to that in a signaling game.
- One key difference between these two games is that there always exists a pooling equilibrium in a cheap-talk game.

# Pooling equilibrium

- The following is a pooling equilibrium:

$$m^*(t_i) = t^*, \ \mu(t_i \mid m_j) = P(t_i), \ a^*(m_j) = a^*$$

for all $t_i \in T$ and $m_j \in M$, where $t^*$ is any message, and $a^*$ solves

$$\max_{a_k \in A} \sum_{t_i \in T} P(t_i) U_R(t_i, a_k).$$

- In this pooling equilibrium, the Sender of all types sends the same message $t^*$, while the Receiver keeps the prior belief of all messages and takes an action optimally according to the belief.
- We call it a babbling equilibrium, which is not informative.
- An interesting question is whether there exists any non-pooling equilibrium in which communication can be effective.

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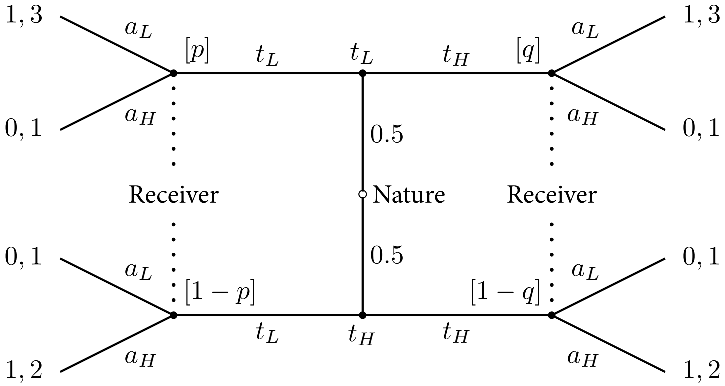
# Example

Find all pure-strategy perfect Bayesian equilibria of the following game.



# Example (Cont.)

- Note that the above signaling game is indeed a cheap-talk game, since neither the Sender's payoff nor the Receiver's payoff depends on the signals.
- Clearly, there are two pooling equilibria:

$$[(t_L, t_L), (a_L, a_L); p = \tfrac{1}{2}, q \geq \tfrac{1}{3}],$$

and

$$[(t_H, t_H), (a_L, a_L); p \geq \tfrac{1}{3}, q = \tfrac{1}{2}].$$

- There also exists a separating equilibrium:

$$[(t_L, t_H), (a_L, a_H); p = 1, q = 0].$$

# Two-type, two-action cheap-talk game

- Consider a two-type, two-action example:

$$T = \{t_L, t_H\},\ P(t_L) = p,\ A = \{a_L, a_H\},\ M = T.$$

- We use the following matrix to represent the payoffs: the first (second) number is the payoff to the Sender (Receiver).

| | $t_L$ | $t_H$ |
|---|---|---|
| $a_L$ | $x, 1$ | $y, 0$ |
| $a_H$ | $z, 0$ | $w, 1$ |

It is independent of messages.

- Note that the above matrix differs from the normal-form representation of the game.

# Two-type, two-action cheap-talk game (Cont.)

- Consider the following separating equilibrium:
  - the Sender's strategy: $[m^*(t_L) = t_L, m^*(t_H) = t_H]$;
  - the Receiver's beliefs: $\mu(t_L|t_L) = 1$ and $\mu(t_L|t_H) = 0$;
  - the Receiver's strategy: $[a^*(t_L) = a_L, a^*(t_H) = a_H]$.
- In the above equilibrium, each type of the Sender tells the truth.
- It can be shown that the separating equilibrium exists iff $x \geq z$ and $y \leq w$.
- In other words, the Sender's and the Receiver's interests perfectly align.
- In general, Crawford and Sobel (1982) have shown that more communication can occur through cheap talk when players' preferences are more closely aligned.

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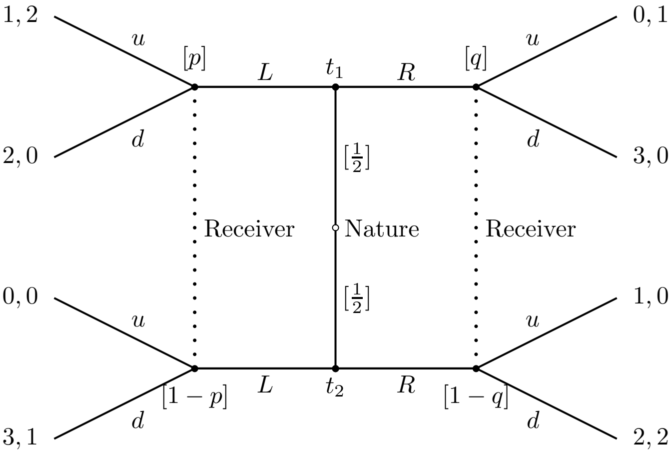
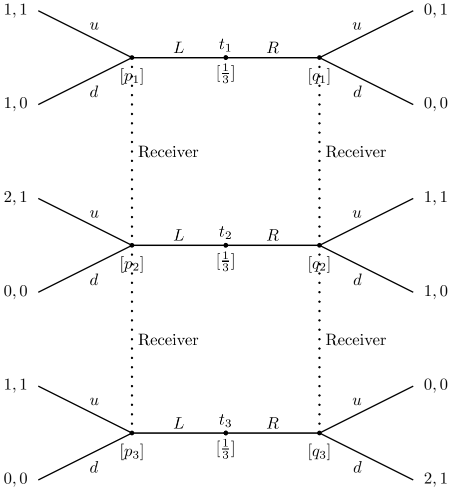
# Question 2

Find all perfect Bayesian equilibria in the following signaling game.



# Question 3

Find all perfect Bayesian equilibria in the following signaling game.



# Question 4

Two partners must dissolve their partnership. Partner 1 currently owns share $s$ of the partnership, partner 2 owns share $1 - s$. the partners agree to play the following game: partner 1 names a price, $p$, for the whole partnership, and partner 2 then chooses either to buy 1's share for $ps$ or to sell his or her share to 1 for $p(1 - s)$. Suppose it is common knowledge that the partners' valuations for owning the whole partnership are independently and uniformly distributed on $[0, 1]$, but that each partner's valuation is private information. What is the perfect Bayesian equilibrium?

# Question 5

A buyer and a seller have valuations $v_b$ and $v_s$. It is common knowledge that there are gains from trade (i.e., that $v_b > v_s$), but the size of the gains is private information, as follows: the seller's valuation is uniformly distributed on $[0, 1]$; the buyer's valuation $v_b = k \cdot v_s$, where $k > 1$ is common knowledge; the seller knows $v_s$ (and hence $v_b$) but the buyer does not know $v_b$ (or $v_s$). Suppose the buyer makes a single offer, $p$, which the seller either accepts or rejects. What is the perfect Bayesian equilibrium when $k < 2$? When $k > 2$?