Game Theory

Dynamic games of incomplete information

Xiang Sun

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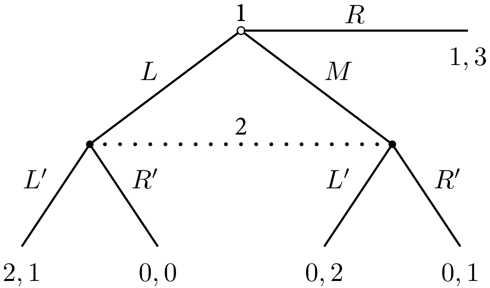
Motivating example

- Example 1:



- What are the pure-strategy Nash equilibria and subgame-perfect Nash equilibria in this game?

Motivating example (Cont.)

- The normal-form representation of the game is

		Player 2 L'	Player 2 R'
Player 1	L	$2, 1$	$0, 0$
Player 1	M	$0, 2$	$0, 1$
Player 1	R	$1, 3$	$1, 3$

- Two pure-strategy Nash equilibria:

$$(L, L') \text{ and } (R, R')$$

- Since the above game has no subgames, both (L, L') and (R, R') are subgame-perfect Nash equilibria.

Motivating example (Cont.)

However, (R, R') is based on a non-credible threat from player 2.

- On the one hand, if player 1 believes player 2's threat of playing R', then player 1 should choose R to end the game with payoff 1, which is larger than 0 by choosing L or M.
- On the other hand, if player 1 doesn't believe the threat and plays L or M, then when player 2 gets the move, he will indeed choose L', since L' strictly dominates R' for player 2.
- Thus, the threat of playing R' by player 2 is not credible.

Motivating example (Cont.)

- In Example 1, the equilibrium (R, R') is not reasonable as it depends on a non-credible threat.
- We need to strengthen the equilibrium concept to rule out some subgame-perfect Nash equilibria like (R, R').
- A stronger equilibrium concept $\Rightarrow$ perfect Bayesian equilibrium.

Belief

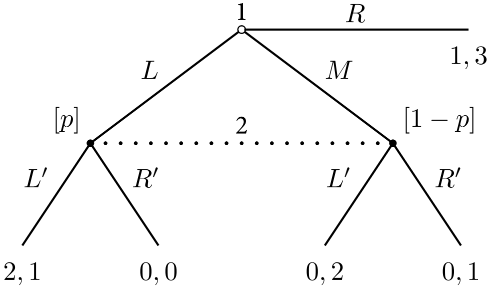
Requirement 1

At each information set, the player with the move must have a belief about which node in the information set has been reached by the play of the game.

- For a nonsingleton information set, a belief is a probability distribution over the nodes in the information set;
- For a singleton information set, a belief puts probability one on the single decision node.

Belief: Illustration

In Example 1, Requirement 1 implies that if player 2's nonsingleton information set is reached, player 2 must form a belief on which of the decision node has been reached, i.e., player 2 believes that player 1 has chosen L with probability p, and M with probability $1 - p$, where $p \in [0, 1]$.



Sequential rationality

Requirement 2

Given their beliefs, the players' strategies must be sequentially rational.

That is, at each information set, the action taken by the player with the move (and the player's subsequent strategy) must be optimal, given the player's belief at that information set and the other players' subsequent strategies (where a "subsequent strategy" is a complete plan of action covering every contingency that might arise after the given information set has been reached).

Sequential rationality: Illustration

- Given this belief, player 2's expected payoffs are
  - playing L': $p \cdot 1 + (1-p) \cdot 2 = 2 - p$
  - playing R': $p \cdot 0 + (1-p) \cdot 1 = 1 - p$
- Since R' is never optimal for any belief, (R, R') cannot satisfy Requirement 2.
- Requirements 1 and 2 together can already eliminate the equilibrium (R, R') that relies on a non-credible threat.
- Requirements 1 and 2 allow for arbitrary beliefs, including unreasonable ones. Further requirements on players' beliefs need to be introduced.

Equilibrium path

Definition

For a given equilibrium in a given extensive-form game, an information set is on the equilibrium path if it will be reached with positive probability if the game is played according to the equilibrium strategies, and is off the equilibrium path if it is definitely not to be reached if the game is played according to the equilibrium strategies.

- Here the "equilibrium" can mean Nash equilibrium, subgame-perfect Nash equilibrium, Bayesian Nash equilibrium or perfect Bayesian equilibrium.

Equilibrium path (Cont.)

- In Example 1, consider player 2's nonsingleton information set.
- For the equilibrium (L, L'), the nonsingleton information set is on the equilibrium path, while there is no information set off the equilibrium path.
- For the equilibrium (R, R'), the nonsingleton information set is off the equilibrium path, and there is no information set on the equilibrium path.

Belief is rational

Requirement 3

At information sets on the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies.

- In Example 1, for the equilibrium (L, L'), Requirement 3 implies that player 2's belief must be $p = 1$.
- Consider a hypothetical situation: the game has a mixed-strategy equilibrium in which player 1 plays L with probability q_1, M with probability q_2, and R with probability $1 - q_1 - q_2$. Requirement 3 would force player 2's belief to be

$$p = \text{Prob}(L \text{ is played } | L \text{ or } M \text{ is played}) = \frac{q_1}{q_1 + q_2}.$$

Belief is rational+

Requirement 4

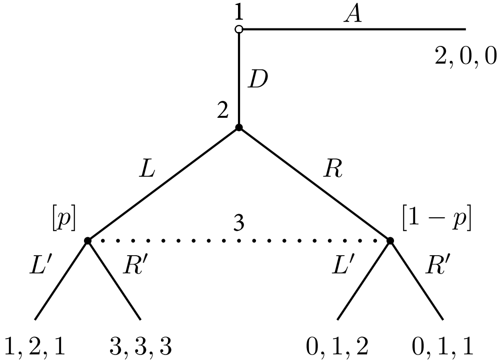
At information sets off the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies where possible.

- In Example 1, for the equilibrium (R, R'), Requirement 4 does not put any restrictions on player 2's belief p.

Definition

A perfect Bayesian equilibrium consists of strategies and beliefs satisfying Requirements 1–4.

Example 2



What are the (pure-strategy) Nash equilibria and subgame-perfect Nash equilibria of this game? Are they also perfect Bayesian equilibria?

Perfect Bayesian Equilibrium

- The normal-form representation of the game:

	L	R
A	$\underline{2}, \underline{0}, \underline{0}$	$\underline{2}, \underline{0}, \underline{0}$
D	$1, \underline{2}, 1$	$0, 1, \underline{2}$

Player 3 chooses L'

	L	R
A	$2, \underline{0}, \underline{0}$	$\underline{2}, \underline{0}, \underline{0}$
D	$\underline{3}, \underline{3}, \underline{3}$	$0, 1, 1$

Player 3 chooses R'

- Player 1 chooses the row, player 2 chooses the column and player 3 chooses the matrix.
- Four pure-strategy Nash equilibria:

$$(A, L, L'), (A, R, L'), (A, R, R'), (D, L, R').$$

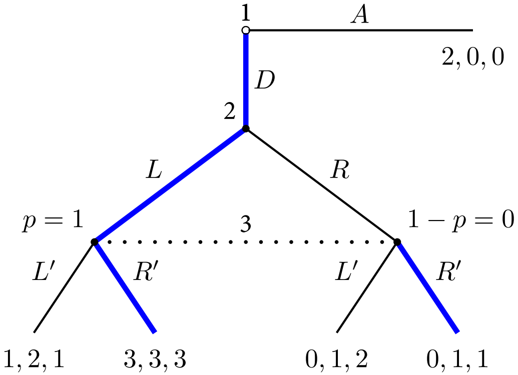
Perfect Bayesian Equilibrium

- The game has a unique subgame (beginning at player 2's singleton information set), and the unique Nash equilibrium of this subgame is (L, R').
- Hence, the unique subgame-perfect Nash equilibrium of the game is (D, L, R').
- The other three Nash equilibria are not subgame-perfect.
- Check whether each equilibrium is a perfect Bayesian equilibrium.

Perfect Bayesian Equilibrium

- Consider the subgame-perfect Nash equilibrium (D, L, R').
- These strategies and the belief $p = 1$ for player 3 satisfy Requirements 1–3.
- They also satisfy Requirement 4, since there is no information set off the equilibrium path
- Then the strategies (D, L, R') and the belief $p = 1$ indeed constitute a perfect Bayesian equilibrium.

Perfect Bayesian Equilibrium

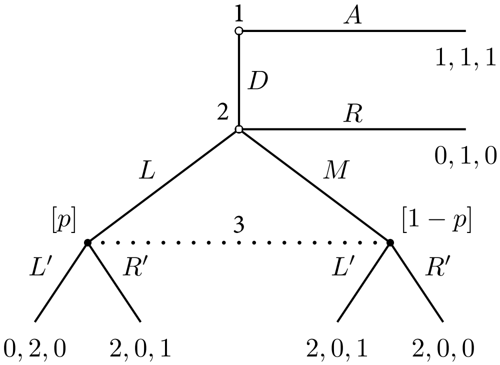


Perfect Bayesian equilibrium in Example 2: $((D, L, R'); p = 1)$

Perfect Bayesian Equilibrium

- The other three Nash equilibria do not satisfy all Requirements 1–4.
- For example, consider the Nash equilibrium (A, L, L').
- Requirement 4 implies that for player 3's nonsingleton information set off the equilibrium path, player 3's belief must be $p = 1$.
- Requirement 2 then implies that for $p = 1$, player 3 must choose R' rather than L'.
- Therefore, the strategies (A, L, L') and the belief $p = 1$ do not satisfy Requirements 1 to 4, and they are not a perfect Bayesian equilibrium.

Example 3



Perfect Bayesian Equilibrium

- Three pure-strategy Nash equilibria:

$$(A, L, L'), (A, R, L'), (A, R, R').$$

- Consider the strategies (A, L, L') and the belief $p \leq \frac{1}{2}$, which satisfy Requirements 1–3.
- Requirement 4 implies that for player 3's information set off the equilibrium path, the belief must be $p = 1$, which contradicts $p \leq \frac{1}{2}$.
- Therefore, there exists no belief together with the strategies (A, L, L') that constitutes a perfect Bayesian equilibrium.

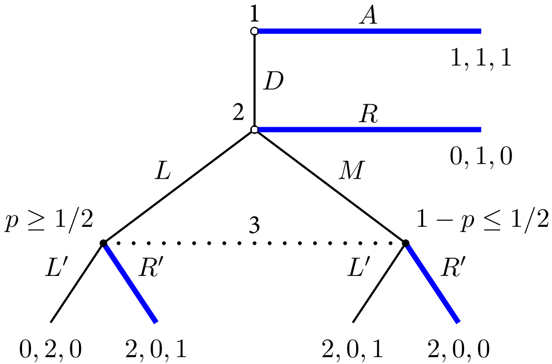
Perfect Bayesian Equilibrium

- Consider strategies (A, R, L') and the belief $p \leq \frac{1}{2}$.
- They satisfy Requirement 4, which puts no restrictions on player 3's belief at the information set off the equilibrium path.
- They also satisfy Requirements 1 and 3.
- However, at player 2's singleton information set, player 2 should choose L rather than R given player 3's equilibrium strategy, which implies that Requirement 2 is violated.
- Thus, strategies (A, R, L') and the belief $p \leq \frac{1}{2}$ do not constitute a perfect Bayesian equilibrium.

Perfect Bayesian Equilibrium

- Consider the strategies (A, R, R') and the belief $p \geq \frac{1}{2}$.
- They satisfy all Requirements 1–4, and thus constitute a perfect Bayesian equilibrium.

Perfect Bayesian Equilibrium



Perfect Bayesian equilibrium in Example 3: $((A, R, R'); p \geq \frac{1}{2})$

Perfect Bayesian Equilibrium

- The procedure to determine whether a given equilibrium is a perfect Bayesian equilibrium:
  1. Determine a belief for each information set by Bayes' rule;
  2. Check whether the equilibrium is optimal given each belief determined in (1) and the subsequent strategies.
- A perfect Bayesian equilibrium consists not only strategies but also beliefs of players, and it requires each player's strategy to be optimal given his or her reasonable beliefs.

Relationship between Different Equilibrium Concepts

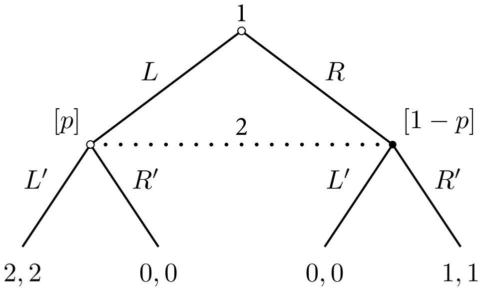
- Perfect Bayesian equilibrium is a stronger equilibrium concept that refines different types of equilibria.
- On the one hand, it refines Bayesian Nash equilibrium (in the same way as subgame-perfect Nash equilibrium refines Nash equilibrium).
- On the other hand, it strengthens subgame-perfect Nash equilibrium by explicitly analyzing beliefs.
- In addition, while a Nash equilibrium requires that no player chooses a strictly dominated strategy, a perfect Bayesian equilibrium requires no player's strategy to be strictly dominated beginning at any information set.

Relationship between Different Equilibrium Concepts

Perfect Bayesian equilibrium corresponds to

- Nash equilibrium (with appropriate beliefs) in static games of complete information;
- Bayesian Nash equilibrium in static games of incomplete information;
- subgame-perfect Nash equilibrium (with appropriate beliefs) in dynamic games of complete and perfect information (and also many dynamic games of complete but imperfect information).

Example 4



Three perfect Bayesian equilibria:

$$((L, L'); p = 1), ((R, R'); p = 0), \left(\left(\tfrac{1}{3}L + \tfrac{2}{3}R, \tfrac{1}{3}L' + \tfrac{2}{3}R'\right); p = \tfrac{1}{3}\right).$$

Example 4 (Cont.)

- The normal-form representation of the game is

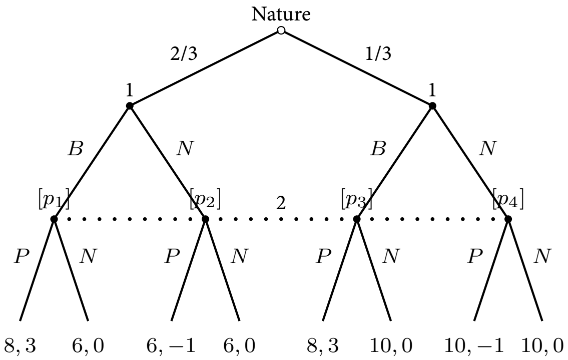
		Player 2	
		L'	R'
Player 1	L	$2, 2$	$0, 0$
	M	$0, 0$	$1, 1$

- Three Nash equilibria:

$$(L, L'), (R, R'), \left(\tfrac{1}{3}L + \tfrac{2}{3}R, \tfrac{1}{3}L' + \tfrac{2}{3}R'\right)$$

- Each Nash equilibrium (together with a correct belief) corresponds to a perfect Bayesian equilibrium in this static game of complete information.

Example 5



Two Bayesian Nash equilibria: (BN, P) and (NN, N)

Example 5 (Cont.)

- Two perfect Bayesian equilibria:

$$\big((BN, P); p_1 = \tfrac{2}{3}, p_4 = \tfrac{1}{3}\big), \big((NN, N); p_2 = \tfrac{2}{3}, p_4 = \tfrac{1}{3}\big).$$

- Consider the first equilibrium, for example.
- For the strategy *BN* chosen by player 1, Requirement 3 implies that the belief is $p_1 = \frac{2}{3}$ and $p_4 = \frac{1}{3}$.
- Given this belief, it is optimal for player 2 to choose *P*.
- Given player 2's strategy *P*, it is optimal for player 1 type 1 to choose *B*, and type 2 to choose *N*.

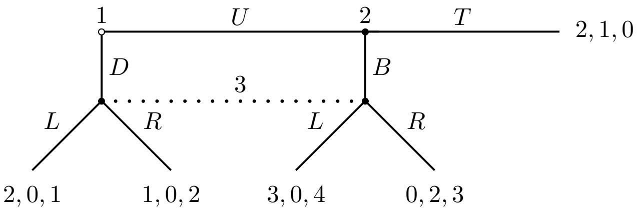
Question 1

Consider the following game between three Players:

- Player 1 moves first. He has two actions: U and D. Action U gives the next move to Player 2, action D gives the next move to Player 3.
- If Player 2 is given the move he also has two actions: T and B. Action T ends the game, action B gives the move to Player 3.
- If Player 3 is given the move he also has two actions: L and R. Both actions end the game. Player 3 does not know whether the move was given to him by Player 1 or Player 2.

Question 1 (Cont.)

The extensive form is given as follows:



where the payoff vector (x, y, z) means that Player 1 receives utility x, Player 2 receives utility y and Player 3 receives utility z.

1. Find all Nash equilibria.
2. Find all subgame-perfect Nash equilibria.
3. Find all perfect Bayesian equilibria.