Game Theory

Static games with incomplete information

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A Motivating Example: Auction

- Suppose a seller wants to sell a product among a group of buyers.
- Each buyer is willing to pay v_i for the product, where v_i is buyer i's private information, i.e., only buyer i knows its valuation v_i, but not all other buyers or the seller.
- In order to sell the product, the seller runs an auction (first-price auction, second-price auction, etc.).
- Each buyer must bid for the product in order to be the winner.
- Question: What should each buyer do?

Introduction

- We have so far learned games of complete information, i.e., each player's payoff function is common knowledge among all players.
- In the auction example, each player's payoff function is no longer common knowledge $\Rightarrow$ buyer i's payoff function is not known by other buyers.
- This is an example of incomplete information games, in which at least one player is uncertain about another player's payoff function.
- Games of incomplete information are also called Bayesian games.
- Two types of Bayesian games: static vs. dynamic.

Cournot Competition under Asymmetric Information

- Consider the Cournot duopoly model with an inverse demand function $P = a - Q$, where $Q = q_1 + q_2$ and $a > 0$.
- Firm 1's cost function is $c_1(q_1) = cq_1$.
- Firm 2's cost function is

$$c_2(q_2) = \begin{cases} c_H q_2, & \text{with probability } \theta, \\ c_L q_2, & \text{with probability } 1 - \theta, \end{cases}$$

where $c_L < c_H$ and $0 < \theta < 1$.

Cournot Competition under Asymmetric Information (Cont.)

- Different from the standard Cournot model, the information is asymmetric:
  - Firm 1's cost function is known by both firms $\Rightarrow c_1(\cdot)$ is common knowledge.
  - Firm 2's cost function is completely known by itself, but not known by firm 1 $\Rightarrow c_2(\cdot)$ is not common knowledge.
  - Firm 2 only knows the distribution of firm 2's marginal cost, i.e., c_H with probability θ and c_L with probability $1-\theta$.
- What will be the quantities chosen by the firms?

Cournot Competition under Asymmetric Information (Cont.)

- Naturally, firm 2 may want to choose a different (and presumably lower) quantity if its marginal cost is high than if it is low.
- Firm 1 should rationally anticipate that firm 2 may tailer its quantity to its cost in this way.
- Let $q_2^*(c_H)$ and $q_2^*(c_L)$ denote firm 2's quantity choices when its marginal cost is c_H and c_L respectively, and let q_1^* denote firm 1's single choice of quantity.

Cournot Competition under Asymmetric Information (Cont.)

- If firm 2's cost is c_j ($j = L, H$), it will choose $q_2^*(c_j)$ to solve

$$\max_{q_2} (a - q_1^* - q_2 - c_j) q_2.$$

- Since firm 1 knows that firm 2's marginal cost is c_H with probability of θ and anticipates firm 2 to choose $q_2^*(c_j)$ depending on its cost, firm 1 chooses q_1^* to solve

$$\max_{q_1} \theta(a - q_1 - q_2^*(c_H) - c) q_1 + (1 - \theta)(a - q_1 - q_2^*(c_L) - c) q_1.$$

Cournot Competition under Asymmetric Information (Cont.)

- The (interior) first-order conditions (or best response functions) for the firms are

$$
\begin{aligned}
q_2^*(c_H) &= \frac{a - q_1^* - c_H}{2}, \\
q_2^*(c_L) &= \frac{a - q_1^* - c_L}{2}, \\
q_1^* &= \frac{a - \theta q_2^*(c_H) - (1-\theta) q_2^*(c_L) - c}{2}.
\end{aligned}
$$

Cournot Competition under Asymmetric Information (Cont.)

- The equilibrium of this game is $(q_1^*, (q_2^*(c_H), q_2^*(c_L)))$, where

$$q_1^* = \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3},$$
$$q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{1-\theta}{6}(c_H - c_L),$$
$$q_2^*(c_L) = \frac{a - 2c_L + c}{3} - \frac{\theta}{6}(c_H - c_L).$$

- We know $q_2^*(c_H) < q_2^*(c_L) \Rightarrow$ firm 2 produces less when its marginal cost increases.

Cournot Competition under Asymmetric Information (Cont.)

- Firm 2 has two payoff functions

$$\pi_2(q_1, q_2; c_L) = (a - q_1 - q_2 - c_L)q_2,$$
$$\pi_2(q_1, q_2; c_H) = (a - q_1 - q_2 - c_H)q_2.$$

- Firm 1 has only one payoff function

$$\pi_1(q_1, q_2; c) = (a - q_1 - q_2 - c)q_1.$$

- Firm 2 knows firm 1's payoff function, while firm 1 does not know firm 2's payoff functions but only knows the probability distribution.
- This is an example of (static) Bayesian games.

Static Bayesian Games

Consider a general static Bayesian game.

- Let player i's possible payoff function be $u_i(a_1, \ldots, a_n; t_i)$, where a_i is player i's action and t_i is called player i's type, which belongs to a set of possible types T_i (or type spaces).
- Player i's type t_i is his private information, and each type t_i corresponds to a different payoff function of player i.
- Let $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ be the types of other players and T_{-i} be the set of all t_{-i}.
- Player i is uncertain about other players' types, but only knows the probability distribution $p_i(t_{-i}|t_i)$ on T_{-i}, which is i's belief about other players' types, given i's knowledge of his own t_i.

Static Bayesian Games (Cont.)

Definition

The normal-form representation of an n-player static Bayesian game specifies players'

1. action spaces $A_1, \ldots, A_n$,
2. type spaces $T_1, \ldots, T_n$,
3. beliefs $p_1, \ldots, p_n$,
4. payoff functions $u_1, \ldots, u_n$.

We denote this game by

$$G = \langle A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n \rangle.$$

Illustration

In the Cournot game with asymmetric information,

- $A_1 = A_2 = [0, \infty)$;
- $T_1 = \{c\}$, and $T_2 = \{c_H, c_L\}$;
- $p_1(c_H|c) = \theta, p_1(c_L|c) = 1 - \theta$, and $p_2(c|c_H) = p_2(c|c_L) = 1$;
- Payoff functions are

$$\begin{aligned}\pi_1(q_1, q_2; c) &= (a - q_1 - q_2 - c)q_1, \\ \pi_2(q_1, q_2; c_L) &= (a - q_1 - q_2 - c_L)q_2, \\ \pi_2(q_1, q_2; c_H) &= (a - q_1 - q_2 - c_H)q_2.\end{aligned}$$

Timing

- The timing of a static Bayesian game:
  1. Nature draws a type vector $t = (t_1, \ldots, t_n)$, where $t_i \in T_i$;
  2. Nature reveals t_i to player i, but not to any other players;
  3. The players simultaneously choose actions, player i choosing $a_i \in A_i$;
  4. Payoffs $u_i(a_1, \ldots, a_n; t_i)$ are received.
- By introducing the frictional moves by nature in (1) and (2), we have described a game of incomplete information as a game of imperfect information.

Bayes' rule

- We often assume that the nature draws $t = (t_1, \ldots, t_n)$ according to the prior probability distribution $p(t)$, which is common knowledge.
- Then the belief $p_i(t_{-i}|t_i)$ can be computed by Bayes' rule

$$p_i(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{\sum_{t'_{-i} \in T_{-i}} p(t'_{-i}, t_i)}.$$

Two remarks

- First, there are games in which player i has private information not only about his or her own payoff function but also about another player's payoff function. We write player i's payoff function as $u_i(a_1, \ldots, a_n; t_1, \ldots, t_n)$. (interdependent)
- Second, we typically assume that players' types are independent (otherwise, correlated), i.e., $p_i(t_{-i}|t_i)$ does not depend on t_i, which can be denoted by $p_i(t_{-i})$. But $p_i(t_{-i})$ is still derived from the prior distribution $p(t)$.

Strategy

Definition

In the static Bayesian game $G = \langle A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n \rangle$, a strategy for player i is a function $s_i(t_i)$, i.e., $s_i \colon T_i \to A_i$. For given type t_i, $s_i(t_i)$ gives an action of player i.

Player i's strategy space S_i is the set of all functions from T_i into A_i.

- In the previous example, $(q_2^*(c_H), q_2^*(c_L))$ is a strategy for firm 2, while q_1^* is a strategy for firm 1.

Bayesian Nash Equilibrium

Definition

In the static Bayesian game $G = \langle A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n \rangle$, the strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ are a (pure-strategy) Bayesian Nash equilibrium if for each player i and for each of i's types $t_i \in T_i$, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i(s_{-i}^*(t_{-i}), a_i; t_i) \cdot p_i(t_{-i} | t_i).$$

- In a general finite static Bayesian game (finite players, finite actions, and finite types), a Bayesian Nash equilibrium exists, perhaps in mixed strategies.

Bayesian Nash Equilibrium

- In a Bayesian Nash equilibrium, each player's strategy is a best response to other players' strategies.
- In other words, no player wants to change his or her strategy unilaterally given other players' equilibrium strategies, even if the change involves only one action by one type.
- A Bayesian Nash equilibrium is simply a Nash equilibrium in a Bayesian game.
- In the Cournot game with asymmetric information, the strategies $(q_1^*, (q_2^*(c_H), q_2^*(c_L)))$ are a Bayesian Nash equilibrium since neither firm 1 nor firm 2 wants to deviate from its equilibrium strategy.

Application: A Trading Game

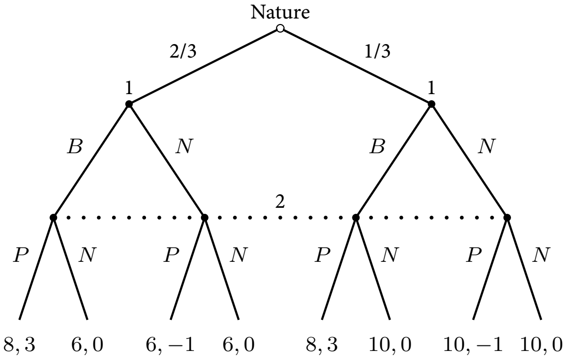
- Suppose a seller can procure a product at a cost of $c = 1$.
- A buyer wants to buy the good, and is willing to pay $v_0 = 12$.
- The buyer can also purchase the good from other places, where the valuation is his private information.
- The seller knows the distribution of the valuation for the outside option is either $v = 10$ or $v = 14$, each with a probability of $\frac{2}{3}$ and $\frac{1}{3}$, respectively.
- The price of the good is $P = 4$, which is exogenous and independent of where the buyer makes a purchase.
- All c, v_0 and P are common knowledge among both players.

Application: A Trading Game

- The seller decides whether to procure the good, and the buyer simultaneously decides whether to order the good from the seller.
- If the seller procures the good, its payoff is $P - c$ if the buyer makes a purchase, and $-c$ otherwise.
- If the seller does not procure the good, its payoff is zero regardless of the buyer's choice.
- The buyer's payoff is $v_0 - P$ if he buys from the seller, and $v - P$ otherwise.
- What should the seller and the buyer do?

Application: A Trading Game

- The extensive-form representation of the game:



- Player 1 is the buyer and player 2 is the seller.

Application: A Trading Game

- Normal-form representation of the game:
  - Action spaces: $A_1 = \{B, N\}$ and $A_2 = \{P, N\}$;
  - Type spaces: $T_1 = \{10, 14\}$ and $T_2 = \{1\}$;
  - Beliefs: the buyer's belief on the seller's type is 1 on $\{1\}$, and the seller's belief on the buyer's types is $\frac{2}{3}$ on 10 and $\frac{1}{3}$ on 14;
  - Payoffs are given as above.
- Strategy spaces: $S_1 = \{BB, BN, NB, NN\}$ and $S_2 = \{P, N\}$
  - The meaning of BN: the buyer with outside option 10 chooses "to buy" and with outside option 14 chooses "not to buy".

Application: A Trading Game

- Alternatively, we can use the following bi-matrix to represent the game:

Buyer

Seller	BB	BN	NB	NN
P	$3, 8, 8$	$5/3, 8, 10$	$1/3, 6, 8$	$-1, 6, 10$
N	$0, 6, 10$	$0, 6, 10$	$0, 6, 10$	$0, 6, 10$

- For example, consider the outcome (P, BN):
  - the buyer with type 10 receives $v_0 - P = 8$, and with type 14 receives $v - P = 10$;
  - the seller's expected payoff is $3 \times \frac{2}{3} - 1 \times \frac{1}{3} = \frac{5}{3}$.
- In particular, we can consider two types of the buyer as two players and we can solve the Bayesian Nash equilibria in the above (like three-player) normal-form representation of the game.

Application: A Trading Game

- We first find out the best response functions for each of the "three players" (the seller and each type of the buyer).

		Buyer			
		BB	BN	NB	NN
Seller	P	$\underline{3}, \underline{8}, 8$	$\underline{5/3}, \underline{8}, \underline{10}$	$\underline{1/3}, 6, 8$	$-1, 6, \underline{10}$
	N	$0, \underline{6}, \underline{10}$	$0, \underline{6}, \underline{10}$	$0, \underline{6}, \underline{10}$	$\underline{0}, \underline{6}, \underline{10}$

- Two Bayesian Nash equilibria: (P, BN) and (N, NN).

Application: Mixed Strategies Revisited

- Consider the game of battle of the sexes

		Wife: Opera	Wife: Football
Husband	Opera	1, 2	0, 0
	Football	0, 0	2, 1

- There are three possible Nash equilibria: (Opera, Opera), (Football, Football) and ($\frac{1}{3}$Opera+$\frac{2}{3}$Football, $\frac{2}{3}$Opera+$\frac{1}{3}$Football).
- In the mixed-strategy Nash equilibrium, the husband plays Opera with probability $\frac{1}{3}$ and Football with probability $\frac{2}{3}$, while the wife plays Opera with probability $\frac{2}{3}$ and Football with probability $\frac{1}{3}$.

Application: Mixed Strategies Revisited

- Suppose the couple are uncertain about the payoffs for each other.
- Consider the following payoff matrix

		Wife: Opera	Wife: Football
Husband	Opera	$1, 2 + t_w$	$0, 0$
	Football	$0, 0$	$2 + t_h, 1$

- Here t_w is privately known by the wife, while t_h is privately known by the husband.
- Assume that t_w and t_h are independently drawn from a uniform distribution on $[0, x]$, where $x > 0$.

Application: Mixed Strategies Revisited

- The normal-form representation of this static Bayesian game is $G = \langle A_h, A_w; T_h, T_p; p_h, p_w; u_h, u_w \rangle$:
  - $A_h = A_w = \{\text{Opera, Football}\}$;
  - $T_h = T_w = [0, x]$;
  - The husband believes that t_w (the wife believes that t_h) is uniformly distributed on $[0, x]$;
  - u_h and u_w are given before.
- What are players' strategies?

Application: Mixed Strategies Revisited

- We can construct a Bayesian Nash equilibrium (s_h^*, s_w^*), where

$$s_h^* = \begin{cases} \text{Football}, & \text{if } t_h > \bar{t}_h, \\ \text{Opera}, & \text{if } t_h \le \bar{t}_h, \end{cases} \text{ and } s_w^* = \begin{cases} \text{Opera}, & \text{if } t_w > \bar{t}_w, \\ \text{Football}, & \text{if } t_w \le \bar{t}_w. \end{cases}$$

- Note $\bar{t}_h$ and $\bar{t}_w$ are two critical values, which need to be determined.
- In the Bayesian Nash equilibrium, the husband will choose Football if t_h exceeds the critical value $\bar{t}_h$, and choose Opera otherwise.

Application: Mixed Strategies Revisited

- Given the wife's strategy, the husband's expected payoffs of choosing Opera and Football are

$$\begin{aligned} u_h(\text{Opera}, s_w^*|t_h) &= \Pr(s_w^* = \text{Opera}) \cdot 1 + \Pr(s_w^* = \text{Football}) \cdot 0 \\ &= \left(1 - \frac{\bar{t}_w}{x}\right) \cdot 1 + \frac{\bar{t}_w}{x} \cdot 0 = 1 - \frac{\bar{t}_w}{x}, \end{aligned}$$

and

$$u_h(\text{Football}, s_w^*|t_h) = \left(1 - \frac{\bar{t}_w}{x}\right) \cdot 0 + \frac{\bar{t}_w}{x} \cdot (2 + t_h) = \frac{\bar{t}_w}{x}(2 + t_h).$$

- Thus, choosing Opera is optimal iff

$$1 - \frac{\bar{t}_w}{x} \geq \frac{\bar{t}_w}{x}(2 + t_h) \Leftrightarrow t_h \leq \frac{x}{\bar{t}_w} - 3 = \bar{t}_h. \tag{1}$$

Application: Mixed Strategies Revisited

- Similarly, given the husband's strategy, the wife's expected payoffs of playing Opera and Football are

$$u_w(\text{Opera}, s_h^*|t_w) = \frac{\bar{t}_h}{x} \cdot (2 + t_w) + \left(1 - \frac{\bar{t}_h}{x}\right) \cdot 0 = \frac{\bar{t}_h}{x}(2 + t_w),$$

and

$$u_w(\text{Football}, s_h^*|t_w) = \frac{\bar{t}_h}{x} \cdot 0 + \left(1 - \frac{\bar{t}_h}{x}\right) \cdot 1 = 1 - \frac{\bar{t}_h}{x}.$$

- Thus, choosing Football is optimal iff

$$1 - \frac{\bar{t}_h}{x} \geq \frac{\bar{t}_h}{x}(2 + t_w) \Leftrightarrow t_w \leq \frac{x}{\bar{t}_h} - 3 = \bar{t}_w. \tag{2}$$

Application: Mixed Strategies Revisited

- Solving (1) and (2) simultaneously, we obtain $\bar{t}_h = \bar{t}_w = \frac{\sqrt{9+4x}-3}{2}$.
- In equilibrium, the husband plays Opera with probability p^* and Football with probability $1-p^*$, while the wife plays Football with probability p^* and Opera with probability $1-p^*$, where

$$p^* = \frac{\bar{t}_h}{x} = \frac{\bar{t}_w}{x} = \frac{2}{\sqrt{9+4x}+3}.$$

- When $x \to 0$, we get that $p^* \to \frac{1}{3}$.
- As the incomplete information disappears, the players' behavior in this pure-strategy Bayesian Nash equilibrium approaches their behavior in the mixed-strategy Nash equilibrium in the original game of complete information.

Question 1

Find all the pure-strategy Bayesian Nash equilibria in the following static Bayesian game:

1. Nature determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.

	L	R
T	$1, 1$	$0, 0$
B	$0, 0$	$0, 0$

Game 1

	L	R
T	$0, 0$	$0, 0$
B	$0, 0$	$2, 2$

Game 2

2. Player 1 learns whether nature has drawn Game 1 or Game 2, but Player 2 does not.
3. Player 1 chooses either T or B; Player 2 simultaneously chooses either L or R.
4. Payoffs are given by the game drawn by nature.

Question 2

Consider the following static Bayesian game.

- Nature selects Game 1 with probability $1/3$, Game 2 with probability $1/3$ and Game 3 with probability $1/3$.
- Player I learns whether Nature has selected Game 1 or not; Player II learns whether Nature has selected Game 2 or not.
- Players I and II simultaneously choose their actions: Player I either T or B, and Player II either L or R.
- Payoffs are given by the game selected by Nature.

Game 1

	L	R
T	$0, 0$	$6, -1$
B	$-1, 6$	$4, 4$

Game 2

	L	R
T	$1, 3$	$0, 0$
B	$0, 0$	$3, 1$

Game 3

	L	R
T	$2, -2$	$-2, 2$
B	$-2, 2$	$2, -2$

All of this is common knowledge. Find all the pure-strategy Bayesian Nash equilibria.