

Game Theory

Repeated games

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A Motivating Example

- Consider the following Prisoners' Dilemma problem:

		Player 2	
		L_2	R_2
Player 1	L_1	1, 1	5, 0
	R_1	0, 5	4, 4

- If the game is played once, the unique Nash equilibrium is (L_1, L_2) .
- What if the game is played **more than once**? Will the cooperative outcome (R_1, R_2) be achieved through **repeated interactions**?

Introduction

- Long-term (or repeated) interactions are very common.
- Examples:
 - Firms are engaged in competition over time.
 - Most employment relationships last for a long time.
 - Countries compete over tariffs years by years.
- In a long-term relationship, one must consider **how his/her current behavior will influence others' behavior in the future, or how threats or promises about future behavior can affect current behavior.**
- In these dynamic situations, one might care about “**reputation**”, which is often used to describe how a person's **past actions** affect **future beliefs and behavior.**

Introduction

- We use **repeated games** (重复博弈) to study such interactions among players.
- In repeated games, we are interested in how **repeated interactions** among players would affect their behavior.
- Two types of repeated games:
 - finitely repeated games
 - infinitely repeated games
- The results predicted by these two types of games differ dramatically.

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Example

- Consider the following repeated game (i.e., two-stage Prisoners' Dilemma game):
 - The two players play the simultaneous-move game twice;
 - Each player **observes the outcome** of the first play before the second game begins;
 - The payoff of each player in the whole game is simply the sum of two payoffs in both stages (i.e., no discounting).
- This game is an example of the two-stage imperfect information games that we have learned before.

Example (Cont.)

- We can use backwards induction to solve the game.
- In stage 2, the unique Nash equilibrium is (L_1, L_2) , in which each player receives 1.
- In stage 1, the two players play the following equivalent game:

		Player 2	
		L_2	R_2
Player 1	L_1	2, 2	6, 1
	R_1	1, 6	5, 5

- Hence, (L_1, L_2) is the unique Nash equilibrium in stage 1.
- The subgame-perfect outcome: (L_1, L_2) is played in both periods.
- What is the subgame-perfect Nash equilibrium?

Finitely Repeated Games

- Let $G = \langle A_1, \dots, A_n; u_1, \dots, u_n \rangle$ denote a static game of complete information in which players 1 through n simultaneously choose actions a_1 through a_n from the action spaces A_1 through A_n , and the payoffs are $u_1(a_1, \dots, a_n)$ through $u_n(a_1, \dots, a_n)$.
- The game G is called the **stage game** (阶段博弈) of the repeated game.

Definition

Given a stage game G , let $G(T)$ denote the **finitely repeated game** in which G is played T times, with the outcomes of all preceding plays observed before the next play begins. The payoffs for $G(T)$ are simply the sum of the payoffs from the T stage games.

Subgame-perfect Nash equilibrium

Proposition

If the stage game G has a **unique** Nash equilibrium then, for any finite T , the repeated game $G(T)$ has a unique subgame-perfect outcome: the Nash equilibrium of G is played in every stage.

- In the Prisoners' Dilemma example, the unique outcome in each period is (L_1, L_2) regardless of how many times the game is played.
- The result in the above proposition can be extended even if G itself is a dynamic game of complete information.

Multiple Nash equilibria

- What if the stage game G has **multiple** Nash equilibria?
- Then there may be subgame-perfect outcomes of the repeated game $G(T)$ in which, for any $t < T$, the outcome of stage t is not a Nash equilibrium of G .
- Consider the following game:

		Player 2		
		L_2	M_2	R_2
Player 1	L_1	1, 1	5, 0	0, 0
	M_1	0, 5	4, 4	0, 0
	R_1	0, 0	0, 0	3, 3

- There are two Nash equilibria: (L_1, L_2) and (R_1, R_2) .

Multiple Nash equilibria (Cont.)

- Suppose the game is repeated twice.
- Then it is **possible** that the first-stage outcome is neither (L_1, L_2) nor (R_1, R_2) in a subgame-perfect Nash equilibrium.
- Consider, for example, player i 's strategy:
 - play M_i in the first stage;
 - play R_i if the first-stage outcome is (M_1, M_2) ; otherwise, play L_i .
- It can be verified that the strategy profile constitutes a subgame-perfect Nash equilibrium, in which the first-stage outcome is (M_1, M_2) .
- R_i serves as a reward and L_i serves as a punishment.

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Infinitely Repeated Games

What happens if the Prisoners' Dilemma game is played **forever**?

Present value

Definition

Let π_t be the payoff in stage t . Given the discount factor $\delta \in (0, 1)$, the **present value** of the infinite sequence of payoffs π_1, π_2, \dots is

$$\pi_1 + \delta\pi_2 + \delta^2\pi_3 + \dots = \sum_{t=1}^{\infty} \delta^{t-1} \pi_t.$$

Infinitely Repeated Games

- Recall $G = \langle A_1, \dots, A_n; u_1, \dots, u_n \rangle$ is the stage game of repeated games.

Definition

Given a stage game G , let $G(\infty, \delta)$ denote the **infinitely repeated game** in which G is played forever and players share the discount factor δ . For each t , the outcomes of the $t - 1$ preceding plays are **observed** before the t -th stage begins. Each player's payoff in $G(\infty, \delta)$ is the **present value** of the player's payoffs from the infinite sequence of stage games.

Infinitely Repeated Games

Consider the following infinitely repeated game of Prisoners' Dilemma:

- In the first stage, the two players play the stage game G and receive payoffs $\pi_{1,1}$ and $\pi_{2,1}$;
- In stage t , the players observe the actions chosen in the preceding $t - 1$ stages, and then play G to receive $\pi_{1,t}$ and $\pi_{2,t}$;
- The payoff of the infinitely repeated game is the present value of the sequence of payoffs: $\sum_{t=1}^{\infty} \delta^{t-1} \pi_{i,t}$ for player $i = 1, 2$.

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Strategies in Infinitely Repeated Games

- There are infinitely many strategies for the players.
- Some common strategies:
 - 1 noncooperative strategy:
 - play L_i in every stage
 - 2 (grim) trigger strategy:
 - play R_i in the first stage;
 - in stage t , if the outcome of all $t - 1$ preceding stages has been (R_1, R_2) , then play R_i ; otherwise, play L_i
 - 3 tit-for-tat (or tit for two tats) strategy
 - 4 carrot-and-stick strategy (or two-phase strategy)

Strategies in Infinitely Repeated Games

- We focus on the first two strategies.
- If both players adopt the noncooperative strategy, (L_1, L_2) is repeated forever.
- Using a trigger strategy, player i cooperates until someone fails to cooperate, which triggers a switch to noncooperation forever.
- If both players adopt the trigger strategy, then the outcome of the infinitely repeated game is (R_1, R_2) in every stage.
- Question: Is it a Nash equilibrium in the infinitely repeated game where both players adopt the trigger strategy (i.e., cooperation is achieved)?

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Nash Equilibria

Claim

Both players adopting the noncooperative strategy is a Nash equilibrium.

Proof.

- Assume player i plays L_i in every stage.
- Then player j 's best response is also “to play L_j in every stage”.



Nash Equilibria

Claim

Both players adopting the trigger strategy is a Nash equilibrium if and only if $\delta \geq \frac{1}{4}$.

Nash Equilibria: Proof

- Assume player i has adopted the trigger strategy. We seek to show player j 's best response is also to adopt the trigger strategy.
- Case 1: The outcome in a previous stage is not (R_1, R_2) . Since player i plays L_i forever, player j 's best response is also to play L_j forever.
- Case 2: In the first stage or in a stage where all the preceding outcomes have been (R_1, R_2) , if player j plays the trigger strategy, then he should play R_j in this stage, and the outcome from this stage onwards will be (R_1, R_2) in every stage. Thus player j 's payoff from this stage onwards is

$$\sum_{t=1}^{\infty} 4 \times \delta^{t-1} = \frac{4}{1 - \delta}.$$

Nash Equilibria: Proof (Cont.)

- If player j plays L_j in this stage, player i still plays R_i in this stage but L_i forever from the next stage. Thus player j will also play L_j from the next stage onwards. This means player j 's payoff from this stage onwards is

$$5 + \sum_{t=1}^{\infty} \delta^t = 5 + \frac{\delta}{1 - \delta}.$$

- Therefore, playing the trigger strategy in this case is optimal iff

$$\frac{4}{1 - \delta} \geq 5 + \frac{\delta}{1 - \delta} \Leftrightarrow \delta \geq 1/4.$$

- Summarizing Cases 1 and 2, the trigger strategies constitute a Nash equilibrium for the game iff $\delta \geq 1/4$.

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Subgame-perfect Nash Equilibrium

Claim

The trigger-strategy Nash equilibrium in the infinitely repeated Prisoners' Dilemma game is subgame perfect.

Subgame-perfect Nash Equilibrium: Proof

- In an infinitely repeated game, a **subgame** is characterized by its previous history. The subgames can be grouped as follows:
 - 1 Subgames whose previous histories are always a finite sequence of (R_1, R_2) .
 - 2 Subgames whose previous histories contain other outcomes different from (R_1, R_2) .
- For a subgame in Case (i), the players' strategies in such a subgame are again the trigger strategies, which is a Nash equilibrium for the whole game and thus for the subgame as well.
- For a subgame in Case (ii), the players' strategies are simply to repeat (L_1, L_2) all the time in the subgame, which is also a Nash equilibrium.

Subgame-perfect Nash Equilibrium: Proof (Cont.)

- We can also show directly that trigger strategies constitute a subgame-perfect Nash equilibrium.
- Alternatively, we can use an approach based on the following result:
- **One-deviation principle** (单阶段偏离原则): A strategy profile is a subgame-perfect Nash equilibrium if and only if, for each player i and for each subgame, **no single deviation** would raise player i 's payoff in the subgame.

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Collusion between Cournot Duopolists

- In the Cournot model, the unique **Nash equilibrium** involving each firm producing $q_c = \frac{a-c}{3}$, and earning a profit of $\pi_c = \frac{(a-c)^2}{9}$.
- If there is a monopolist, then the monopoly quantity is $q_m = \frac{a-c}{2}$ and profit is $\pi_m = \frac{(a-c)^2}{4}$.
- If the two firms can **collude** to produce $\frac{q_m}{2}$ each, then they jointly produce the monopoly quantity q_m . Each of them obtains a profit of $\frac{\pi_m}{2} = \frac{(a-c)^2}{8}$.
- If firm i produces $\frac{q_m}{2}$, then the **best response** for firm j is to produce $q_d = \frac{3(a-c)}{8}$. In this case, firm i 's profit is $\frac{3(a-c)^2}{32}$, while firm j 's profit is $\pi_d = \frac{9(a-c)^2}{64}$.

Collusion between Cournot Duopolists (Cont.)

- Consider the infinitely repeated game based on the Cournot stage game when both firms have the discount factor $0 < \delta < 1$.
- Trigger strategy:
 - produce half of the monopoly quantity $\frac{q_m}{2}$, in the first period.
 - in period t , produce $\frac{q_m}{2}$ if both firms have produced $\frac{q_m}{2}$ in all the preceding $t - 1$ periods; otherwise, produce the Cournot quantity q_c .
- Here the cooperative output is $\frac{q_m}{2}$ and noncooperative output is q_c .
- Question: Is the collusive outcome sustained?

Trigger-strategy SPE

Claim

For the infinitely repeated game with the Cournot stage game, both firms playing the trigger strategy is a subgame-perfect Nash equilibrium if and only if $\delta \geq \frac{9}{17}$.

Proof.

- Suppose firm i has adopted the trigger strategy, we need to show firm j 's best response is also to play the trigger strategy in any subgame.
- There are again two types of subgames to be checked.

Trigger-strategy SPE: Proof

- First, if a quantity other than $\frac{q_m}{2}$ has been chosen by any firm before the current period, then firm i chooses q_c from this period onwards. The best response for firm j is also to choose q_c from this period onwards. Thus, playing the trigger strategy is optimal in this subgame.
- Second, in period t , if the outcomes of all previous periods are $(\frac{q_m}{2}, \frac{q_m}{2})$, firm j 's payoff from this period onwards if it chooses the trigger strategy is

$$\frac{\pi_m}{2(1-\delta)}.$$

Trigger-strategy SPE: Proof (Cont.)

- If firm j deviates from the trigger strategy by choosing a quantity other than $\frac{q_m}{2}$, then firm i produces $\frac{q_m}{2}$ in this period, but q_c from period $t + 1$ onwards. Thus, it is optimal for firm j to produce q_d in this period and q_c from period $t + 1$ onwards. Thus, firm j 's present value of the payoffs from period t onwards is

$$\pi_d + \frac{\delta}{1-\delta} \pi_c.$$

- Therefore, trigger strategy is the best response for firm j to firm i 's trigger strategy iff

$$\frac{\pi_m}{2(1-\delta)} \geq \pi_d + \frac{\delta}{1-\delta} \pi_c \Leftrightarrow \delta \geq \frac{\pi_d - \frac{\pi_m}{2}}{\pi_d - \pi_c} = \frac{9}{17}.$$

Two-phase strategy

- What happens if players are less patient, i.e., $\delta < \frac{9}{17}$? Are there any other strategies that can support the collusive outcome as a subgame-perfect Nash equilibrium?
- Consider the **two-phase (or carrot-and-stick) strategy**:
 - in the first period, produce half of the monopoly quantity $\frac{q_m}{2}$;
 - in period t , produce $\frac{q_m}{2}$ if both firms produce $\frac{q_m}{2}$ or both firms produce x in period $t - 1$; otherwise, produce x .
- This strategy involve a **(one-period) punishment phase** in which the firm produces x and a **(potentially infinite) collusive phase** in which the firm produces $\frac{q_m}{2}$.
- Such a strategy punishes
 - a firm for deviating from the collusive phase
 - a firm for deviating from the punishment phase

Two-phase strategy SPE

- If both firms produce x , the profit of each firm is denoted by $\pi(x) = (a - 2x - c)x$, where $\frac{x}{a-c} \leq \frac{1}{2}$.
- If firm i produces x , the **best response** of firm j is to produce $q_{dp} = \frac{a-x-c}{2}$ and the corresponding profit is denoted by $\pi_{dp}(x) = \frac{(a-x-c)^2}{4}$.
- There are two types of subgames:
 - **collusive subgames**: the outcome of previous period is either $(\frac{q_m}{2}, \frac{q_m}{2})$ or (x, x) ;
 - **punishment subgames**: the outcome of previous period is neither $(\frac{q_m}{2}, \frac{q_m}{2})$ nor (x, x) .

Two-phase strategy SPE

- To show both firms adopting the two-phase strategy is a subgame-perfect Nash equilibrium, we use the one-deviation principle.
- Suppose firm i has adopted the two-phase strategy.
- In **collusive subgames**, if firm j also adopts the two-phase strategy, its payoff is

$$\left(1 + \delta + \frac{\delta^2}{1-\delta}\right) \frac{1}{2}\pi_m.$$

- If firm j deviates in this period only, then firm i still chooses $\frac{q_m}{2}$ in this period but x in the next period. Then firm j would choose q_d in this period and x in the next period. The payoff from deviation is

$$\pi_d + \delta\pi(x) + \frac{\delta^2}{1-\delta} \frac{1}{2}\pi_m.$$

Two-phase strategy SPE: Proof

- Thus, choosing the two-phase strategy is optimal iff

$$(1 + \delta)\frac{1}{2}\pi_m \geq \pi_d + \delta\pi(x). \quad (1)$$

- In **punishment subgames**, it is optimal to choose the two-phase strategy for firm j iff

$$\pi(x) + \delta\frac{1}{2}\pi_m \geq \pi_{dp}(x) + \delta\pi(x). \quad (2)$$

- Both firms adopting the two-phase strategy is a subgame-perfect Nash equilibrium iff (1) and (2) hold.

Two-phase strategy SPE: Proof (Cont.)

- The two conditions (1) and (2) can be rewritten as

$$\delta \left(\frac{1}{2} \pi_m - \pi(x) \right) \geq \pi_d - \frac{1}{2} \pi_m, \quad (3)$$

$$\delta \left(\frac{1}{2} \pi_m - \pi(x) \right) \geq \pi_{dp}(x) - \pi(x). \quad (4)$$

- Intuitions: The gain this period from deviating must not exceed the discounted value of the loss next period from punishment.

Two-phase strategy SPE (Cont.)

- Consider the case $\delta = \frac{1}{2} < \frac{9}{17}$.
- Condition (3) is satisfied iff $\frac{x}{a-c} \leq \frac{1}{8}$ or $\frac{x}{a-c} \geq \frac{3}{8}$.
- Condition (4) is satisfied iff $\frac{3}{10} \leq \frac{x}{a-c} \leq \frac{1}{2}$.
- Thus, two-phase strategies constitute a subgame-perfect Nash equilibrium in the game iff $\frac{3}{8}(a-c) \leq x \leq \frac{1}{2}(a-c)$.

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Subgame-perfect Nash Equilibrium

- In the Prisoners' Dilemma example, the cooperative outcome, which cannot be achieved in stage game or in any finitely repeated game, can be sustained if the stage game is played forever.
- The condition is that the discount factor is sufficiently large (or players are sufficiently patient).
- **Folk theorem** (无名氏定理): Cooperative equilibria which do not exist in static games can be achieved in repeated games.

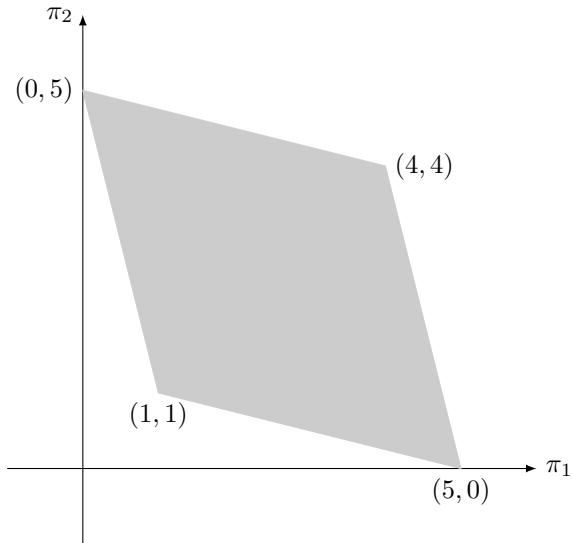
Feasible Payoff

Definition

The payoffs (x_1, \dots, x_n) are **feasible** in the stage game G if they are a convex combination (i.e., a weighted average, where the weights are all nonnegative and sum to one) of the pure-strategy payoffs of G .

- In the Prisoners' Dilemma example, all pure-strategy payoffs $(1, 1)$, $(0, 5)$, $(4, 4)$ and $(5, 0)$ are feasible.
- The payoffs $(2.5, 2.5)$ are also feasible, which can be achieved if player i adopts the mixed-strategy $\frac{1}{2}L_i + \frac{1}{2}R_i$ for $i = 1, 2$.
- All feasible payoffs are depicted in the shaded region of Figure 1.

Feasible Payoff



Average Payoff

Definition

Given the discount factor δ , the **average payoff** of the infinite sequence of payoffs π_1, π_2, \dots is

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_t.$$

- Both present value and average payoff can present a player's payoff in an infinitely repeated game.
- Average payoff is directly **comparable** to the payoffs from the stage game.

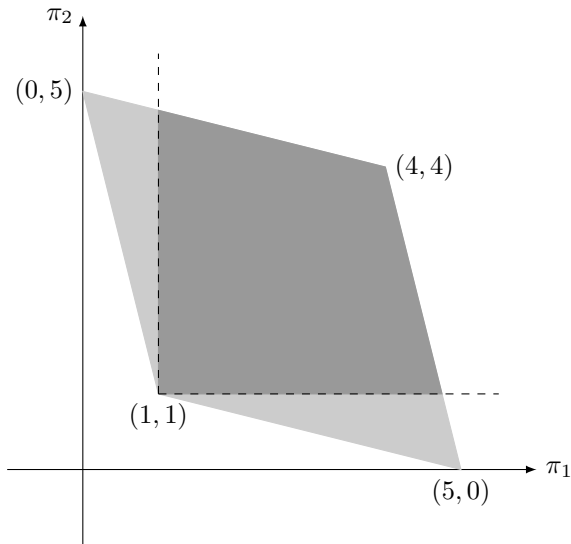
Friedman Theorem

Theorem (Friedman 1971)

Let G be a finite, static game of complete information. Let (e_1, \dots, e_n) denote the payoffs from a Nash equilibrium of G , and let (x_1, \dots, x_n) denote any feasible payoffs from G , where $x_i > e_i$ for each player i . If the discount factor δ is sufficiently close to one, then there exists a subgame-perfect Nash equilibrium in the infinitely repeated game $G(\infty, \delta)$ that achieves (x_1, \dots, x_n) as the average payoff.

- Friedman theorem is part of the Folk theorem.
- Fudenberg and Maskin (1986) have shown that the above result can be extended if the equilibrium payoffs are replaced by reservation payoffs.

Friedman Theorem



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Question

Let the game given below be the stage game of an infinitely repeated game where δ_1 and δ_2 are the discount factors for Players 1 and 2.

	L	R
A	1, 2	5, 0
B	1, 8	4, 6

- 1 Determine the ranges of δ_1 and δ_2 for which the trigger strategies for both players are a Nash equilibrium. The trigger strategy for Player 1 (2) is to play B (R) if all preceding actions are (B, R) ; to play A (L) otherwise.
- 2 Show that the Nash equilibrium in part (i) is also a subgame-perfect Nash equilibrium.
- 3 Show that playing (A, L) in every stage is a subgame-perfect Nash equilibrium.
- 4 In (i), is the trigger strategy a Nash equilibrium if to play B (L) instead of A (L)?