Game Theory

Static games of complete information

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Motivating Example 1: Prisoners' Dilemma

- Two suspects are arrested and charged with a crime. The police lack sufficient evidence to convict the suspects, unless at least one confesses. The suspects are held in separate cells and told that if only one confesses, the confessor will go free while the person does not confess will surely be convicted and given a 9-month jail sentence. If both confess, each will be sent to jail for 6 month. Finally, if neither confesses, both will be convicted of a minor offence and sentenced to jail for 1 month.
- Question: What should the suspects do?

Motivating Example 2: Battle of the Sexes

- Suppose a couple wanted to meet this evening, but did not reach an agreement on whether to attend an opera or a football match. The husband would most of all like to go to the football game, while the wife would prefer the opera. Moreover, both would prefer to go to the same place rather than different ones.
- Question: If they cannot communicate, where should they go?

Normal-form Games

- The two motivating examples can be considered as static games of complete information.
- Static: one-shot, simultaneous move
- Complete information: each player's payoff function is common knowledge among all players.
- How to formalize such a game? $\rightarrow$ normal-form representation
- The normal-form representation of a game specifies
  1. the players in the game;
  2. the strategies available to each player;
  3. the payoff received by each player for each combination of strategies that could be chosen by the players.

Normal-form Games

Definition

The normal-form (also called strategic-form) representation of an n-player game specifies the players' strategy sets/spaces $S_1, \ldots, S_n$ and their payoff functions $u_1, \ldots, u_n$. We denote this game by

$$G = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle.$$

Let $(s_1, \ldots, s_n)$ be a combination of strategies, one for each player. Then $u_i(s_1, \ldots, s_n)$ is the payoff to player i if for each $j = 1, \ldots, n$, player j chooses strategy s_j.

- The payoff of a player depends not only on his own action, but also on the actions of others $\rightarrow$ interdependence (or strategic interaction).

Normal-form Games

- For Example 1, the normal-form representation is $G = \langle S_1, S_2; u_1, u_2 \rangle$
- $S_1 = S_2 = \{D, C\}$, where D means "Defect", and C means "Confess"
- $u_1(D, D) = -1, u_1(D, C) = -9, u_1(C, D) = 0, u_1(C, C) = -6$
- $u_2(D, D) = -1, u_2(D, C) = 0, u_2(C, D) = -9, u_2(C, C) = -6$
- An alternative (but simple) way is to use a bi-matrix to represent the game.

Normal-form Games

- The payoffs of two players in Example 1 can be represented in the following bi-matrix:

		Prisoner 2	
		Defect	Confess
Prisoner 1	Defect	$-1, -1$	$-9, 0$
	Confess	$0, -9$	$-6, -6$

- Prisoner 1 is also called the row player, and Prisoner 2 the column player.
- Each entry of the bi-matrix has two numbers: the first number is the payoff of the row player and the second is that of the column player.

Normal-form Games

- In general, when there are only two players and each player has a finite number of strategies, then the payoff functions can be represented in a bi-matrix.
- The bi-matrix need not be symmetric, e.g.,

Player 2

		L	R
Player 1	U	$u_1(U,L), u_2(U,L)$	$u_1(U,R), u_2(U,R)$
	M	$u_1(M,L), u_2(M,L)$	$u_1(M,R), u_2(M,R)$
	D	$u_1(D,L), u_2(D,L)$	$u_1(D,R), u_2(D,R)$

- What if there are more than two players?

Normal-form Games

- The normal-form representation of Example 2 is $G = \{S_1, S_2; u_1, u_2\}$
- $S_1 = S_2 = \{\text{Opera}, \text{Football}\}$
- The payoff functions u_1 and u_2 are presented in the following bi-matrix:

		Wife: Opera	Wife: Football
Husband	Opera	1, 2	0, 0
	Football	0, 0	2, 1

- Husband is player 1, and wife is player 2.

Concepts of Strategies

- Important concepts:
  - Best response
  - (Strictly) dominated strategy
  - (Strictly) dominant strategy
- Some notations:

$$\begin{aligned} s &= (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \\ s_{-i} &= (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \\ S &= S_1 \times \cdots \times S_{i-1} \times S_i \times S_{i+1} \times \cdots \times S_n \\ S_{-i} &= S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n \end{aligned}$$

Best response

Definition

In a normal-form game $G = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$, the best response for player i to a combination of other players' strategies $s_{-i} \in S_{-i}$, denoted by $R_i(s_{-i})$, is referred to as the set of maximizers of

$$\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

- Remark: $R_i(s_{-i}) \subseteq S_i$ can be an empty set, a singleton, a finite set or an infinite set. We call R_i the best-response correspondence for player i.

Strictly dominated strategy

Definition

In a normal-form game $G = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$, let $s_i', s_i'' \in S_i$. Strategy s_i' is strictly dominated by strategy s_i'' (or strategy s_i'' strictly dominates strategy s_i'), if for each feasible combination of the other players' strategies, player i's payoff from playing s_i' is strictly less than player i's payoff from playing s_i'', i.e.,

$$u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i}), \quad \forall s_{-i} \in S_{-i}.$$

We say s_i' is a strictly dominated strategy of player i.

- A rational player will never choose a strictly dominated strategy!

Strictly dominant strategy

Definition

In a normal-form game $G = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$, strategy $\tilde{s}_i \in S_i$ is a strictly dominant strategy of player i, if it strictly dominates any other strategies. Equivalently, if for each feasible combination of the other players' strategies, player i's payoff from playing $\tilde{s}_i$ is strictly larger than player i's payoff from playing any other strategies, i.e.,

$$u_i(\tilde{s}_i, s_{-i}) > u_i(\hat{s}_i, s_{-i}), \quad \forall s_{-i} \in S_{-i}, \forall \hat{s}_i \in S_i, \hat{s}_i \neq \tilde{s}_i.$$

- A rational player will always choose a strictly dominant strategy, if any.
- If a strictly dominant strategy exists, then it must be unique.

Example

- In Example 1:
  - Best response: $R_i(D) = R_i(C) = C$ for $i = 1, 2$
  - D is a strictly dominated strategy for both players.
  - C is a strictly dominant strategy for both players.
- In Example 2:
  - Best response: $R_i(O) = O$, and $R_i(F) = F$ for $i = 1, 2$
  - Neither player has any strictly dominated strategy.
  - Neither player has any strictly dominant strategy.

Relationship

The relationship between a strictly dominated (or dominant) strategy and a best response:

- Result 1: A strictly dominated strategy can never be a best response, i.e., if s_i' is a strictly dominated strategy of player i, then $s_i' \notin R_i(s_{-i})$ for all $s_{-i} \in S_{-i}$.
- Result 2: A strictly dominant strategy is always a best response, i.e., if $\tilde{s}_i$ is a strictly dominant strategy of player i, then $\tilde{s}_i \in R_i(s_{-i})$ for all $s_{-i} \in S_{-i}$.

IESDS

- How do we solve a game?
- We can use Iterated Elimination of Strictly Dominated Strategies (IESDS).
- Example 3:

Player 2

		L	M	R
Player 1	U	$1,0$	$1,2$	$0,1$
	D	$0,3$	$0,1$	$2,0$

IESDS

Step 1:

- Player 1 does not have a strictly dominated strategy.
- For Player 2, R is a strictly dominated strategy, which is strictly dominated by M. Hence player 2 will never choose R if he is rational.
- If player 1 knows that player 2 is rational, then he can eliminate R from player 2's strategy space by playing the following game:

		Player 2	
		L	M
Player 1	U	$1, 0$	$1, 2$
	D	$0, 3$	$0, 1$

IESDS

Step 2:

- Now player 1 has a strictly dominated strategy, which is strategy D.
- If player 2 also knows that i) player 1 knows that player 2 is rational, and ii) player 1 is rational, then he can also eliminate D.
- The game is further reduced to

		Player 2	
		L	M
Player 1	U	$1, 0$	$1, 2$

IESDS

Step 3:

- Again L is eliminated if player 1 knows that i) player 2 knows that player 1 knows that player 2 is rational, ii) player 2 knows that player 1 is rational, iii) player 2 is rational.
- (U, M) is the final outcome!

		Player 2
		M
Player 1	U	$1, 2$

IESDS

- Two main drawbacks of IESDS:
  - A key assumption: rationality of all players is common knowledge.
  - The prediction of IESDS may not be very precise, and sometimes it predicts nothing about the games.
- IESDS can do nothing with the following game:

		Player 2		
		L	C	R
Player 1	U	$0,4$	$4,0$	$5,3$
	M	$4,0$	$0,4$	$5,3$
	D	$3,5$	$3,5$	$6,6$

- We need to consider a much stronger solution concept to predict the outcomes of the games: Nash equilibrium!

Nash Equilibrium

Definition

In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, the strategies $(s_1^*, \ldots, s_n^*)$ are a Nash equilibrium if,

$$s_i^* \in R_i(s_{-i}^*), \quad \forall i = 1, \ldots, n.$$

Equivalently,

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*), \quad \forall i = 1, \ldots, n.$$

Then s_i^* is the equilibrium strategy of player i.

Nash Equilibrium

- Interpretation
  - Each player's strategy must be a best response, given other players' equilibrium strategies.
  - No single player wants to deviate unilaterally $\rightarrow$ strategically stable or self-enforcing
- How to find a Nash equilibrium (NE)?
  - For a bi-matrix game, underline the payoff to each player's best response for any given other players' strategies.
  - If you find all payoffs in a single entry are underlined, then this is a Nash equilibrium.

Nash Equilibrium: Example

- Example 4:

Player 2 (columns), Player 1 (rows)

	L	C	R
U	$0, \underline{4}$	$\underline{4}, 0$	$5, 3$
M	$\underline{4}, 0$	$0, \underline{4}$	$5, 3$
D	$3, 5$	$3, 5$	$\underline{6}, \underline{6}$

There exists a unique NE: (D, R).

- Prisoners' Dilemma:

	Defect	Confess
Defect	$-1, -1$	$-9, \underline{0}$
Confess	$\underline{0}, -9$	$\underline{-6}, \underline{-6}$

Nash Equilibrium: Example (Cont.)

- Battle of the Sexes:

	Opera	Football
Opera	$\underline{1}, \underline{2}$	$0, 0$
Football	$0, 0$	$\underline{2}, \underline{1}$

- Hawk-Dove:

	Dove	Hawk
Dove	$3, 3$	$\underline{1}, \underline{4}$
Hawk	$\underline{4}, \underline{1}$	$0, 0$

- Matching Pennies:

	Head	Tail
Head	$-1, \underline{1}$	$\underline{1}, -1$
Tail	$\underline{1}, -1$	$-1, \underline{1}$

Issues on Nash Equilibrium

- A Nash equilibrium needs not to be Pareto optimal, for example, prisoners' dilemma. More generally, Nash equilibrium does not rule out the possibility that a subset of players can deviate jointly in a way that makes every player in the subset better off.
- The Nash equilibrium implicitly assumes that players know that each player is to play the equilibrium strategy. Given this knowledge, no player wants to deviate. So, there is a sort of circularity in this concept—the players behave in the way because they are supposed to behave in this way.

NE vs. IESDS

What is the relationship between Nash equilibrium and IESDS?

Proposition 1

In an n-player normal-form game $G = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$, if the strategies $(s_1^*, \ldots, s_n^*)$ are a Nash equilibrium, then they survive iterated elimination of strictly dominated strategies.

Proof of Proposition 1

We use proof by contradiction.

- Suppose s_i^* is the first of the strategies $(s_1^*, \ldots, s_n^*)$ to be eliminated for being strictly dominated. Then there must exist a strategy s_i'' that has not yet been eliminated from S_i that strictly dominates s_i^*, i.e.,

$$u_i(s_i^*, s_{-i}) < u_i(s_i'', s_{-i})$$

for all strategies $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ that have not been eliminated from the other players' strategy spaces.

- Since s_i^* is the first equilibrium strategy to be eliminated, we have

$$u_i(s_i^*, s_{-i}^*) < u_i(s_i'', s_{-i}^*),$$

which contradicts the definition of NE, which requires that s_i^* is a best response to s_{-i}^*.

Implications of Proposition 1

- Any Nash equilibrium can survive IESDS, and must be an outcome of IESDS, i.e.,

$$\{\text{Nash equilibria}\} \subseteq \{\text{Outcomes of IESDS}\}$$

- Nash equilibrium is a stronger solution concept than IESDS.
- Nash equilibrium does not require that rationality is common knowledge.

Implications of Proposition 1: Example

- Example 5:

Player 2 (columns), Player 1 (rows)

	L	M	R
U	$0,0$	$1,2$	$0,1$
D	$1,3$	$0,1$	$2,0$

- IESDS has 4 outcomes:

$$\{(U,L),(U,M),(D,L),(D,M)\}.$$

- There are only 2 NEs:

$$\{(U,M),(D,L)\}.$$

NE vs. IESDS

Proposition 2

Consider an n-player normal-form game $G = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$, which is finite. If iterated elimination of strictly dominated strategies eliminates all but the strategies $(s_1^*, \ldots, s_n^*)$, then these strategies are the unique Nash equilibrium of the game.

Proof of Proposition 2

- By Proposition 1, Nash equilibrium strategies can never be eliminated in IESDS. Since $(s_1^*, \ldots, s_n^*)$ are the only strategies which are not eliminated, s_i^* is thus the only possible equilibrium strategy for player i. Hence, we cannot find two different Nash equilibria.
- It remains to show that $(s_1^*, \ldots, s_n^*)$ are indeed a Nash equilibrium.
- We use proof by contradiction. Suppose s_i^* is not a best response of player i to s_{-i}^*.
- Let the relevant best response be b_i (which must exist since the game is finite), i.e.,

$$\max_{s_i \in S_i} u_i(s_i, s_{-i}^*) = u_i(b_i, s_{-i}^*) > u_i(s_i^*, s_{-i}^*).$$

  But b_i must be strictly dominated by some strategy t_i at some stage of the process of iterated elimination.

Proof of Proposition 2 (Cont.)

- So we have
$$u_i(b_i, s_{-i}) < u_i(t_i, s_{-i})$$
for all strategies (s_{-i}) that have not been eliminated from other players' strategy spaces.
- Since s^*_{-i} have not been eliminated, we have
$$u_i(b_i, s^*_{-i}) < u_i(t_i, s^*_{-i}),$$
which contradicts the fact that b_i is a best response to s^*_{-i}.

Cournot Model of Duopoly

- Suppose two firms (1 and 2) produce a homogeneous good, and compete in quantities.
- Let q_i be the quantity produced by firm i, where $i = 1, 2$.
- The aggregate quantity of the good is denoted by $Q = q_1 + q_2$.
- The inverse demand of the good is

$$P(Q) = \begin{cases} a - Q, & \text{if } Q < a, \\ 0, & \text{if } Q \geq a. \end{cases}$$

- The cost function of firm i is $C_i(q_i) = cq_i$, where $0 < c < a$.
- Question: How much should each firm produce?

Cournot Model of Duopoly (Cont.)

We first need to translate the problem into a normal-form game.

1. Players: the two firms
2. Strategies: $S_i = [0, \infty)$ for $i = 1, 2$ (any q_i is a strategy of firm i)
3. Payoffs:

$$\pi_i(q_i, q_j) = \begin{cases} q_i[a - (q_i + q_j) - c], & \text{if } q_i + q_j < a, \\ -cq_i, & \text{if } q_i + q_j \geq a. \end{cases}$$

Cournot Model of Duopoly (Cont.)

- The pair of quantities (q_1^*, q_2^*) is a Nash equilibrium if for each firm i that q_i^* solves
$$\max_{0 \leq q_i < \infty} \pi_i(q_i, q_j^*).$$
- Equivalently,
$$q_i^* \in R_i(q_j^*),$$
where $i, j = 1, 2$ and $i \neq j$.

Cournot Model of Duopoly (Cont.)

- To solve for the Nash equilibrium, we first need to find the best response correspondence of each player.
- Consider the following two cases:
- Case 1: When $q_j > a - c$, player i's payoff is

$$\pi_i(q_i, q_j) \begin{cases} < 0, & \text{if } q_i > 0, \\ = 0, & \text{if } q_i = 0, \end{cases}$$

which is clearly maximized at $q_i = 0$. Thus, the best response of firm i is $R_i(q_j) = 0$.

Cournot Model of Duopoly (Cont.)

- Case 2: When $0 \leq q_j \leq a - c$, player i's payoff is

$$\pi_i(q_i, q_j) \begin{cases} < 0, & \text{if } q_i > a - c - q_j, \\ = q_i[a - (q_i + q_j) - c], & \text{if } q_i \leq a - c - q_j. \end{cases}$$

  The optimal q_i is determined by the following first-order condition

$$a - q_j - c - 2q_i = 0.$$

  Thus, the best response is $R_i(q_j) = \frac{1}{2}(a - q_j - c)$.

- In sum, the best response correspondence (or function) of player i is

$$R_i(q_j) = \begin{cases} \frac{1}{2}(a - q_j - c), & \text{if } 0 \leq q_j \leq a - c, \\ 0, & \text{if } q_j > a - c. \end{cases}$$

Cournot Model of Duopoly (Cont.)

- The Nash equilibrium (q_1^*, q_2^*) is the intersection of two best response correspondences, which imply that

$$q_1^* = R_1(q_2^*) \text{ and } q_2^* = R_2(q_1^*).$$

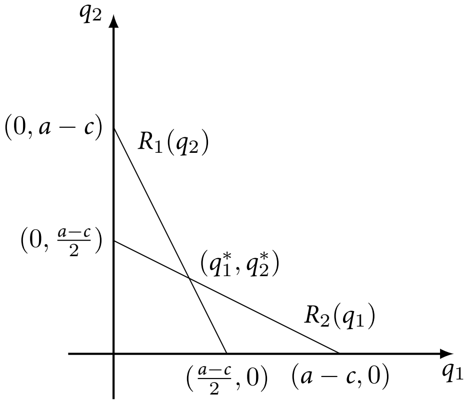
- We can obtain (q_1^*, q_2^*) by simultaneously solving

$$\begin{aligned} q_1^* &= \tfrac{1}{2}(a - q_2^* - c), \\ q_2^* &= \tfrac{1}{2}(a - q_1^* - c). \end{aligned}$$

- The unique Nash equilibrium is $(q_1^*, q_2^*) = \left(\frac{1}{3}(a-c), \frac{1}{3}(a-c)\right)$.

Cournot Model of Duopoly (Cont.)

Alternatively, we can solve for the Nash equilibrium graphically, i.e., (q_1^*, q_2^*) can be determined by the intersection of the two best response curves.



Bertrand Model of Duopoly

- Suppose two firms produce differentiated products and compete in prices.
- The demand for firm i is

$$q_i(p_i, p_j) = a - p_i + bp_j,$$

  where $b > 0$, which suggests that the two products are substitutes.
- Firms' marginal cost is again assumed to be c, where $0 < c < a$.
- Question: What is the Nash equilibrium?

Bertrand Model of Duopoly (Cont.)

- The strategy space of firm i is $S_i = [0, \infty)$ and any $p_i \in S_i$ is a strategy.
- The profit of firm i is

$$\pi_i(p_i, p_j) = (a - p_i + bp_j)(p_i - c).$$

- The pair of prices (p_i^*, p_j^*) is a Nash equilibrium if p_i^* solves

$$\max_{0 \leq p_i < \infty} (a - p_i + bp_j^*)(p_i - c),$$

which leads to

$$p_i^* = \frac{1}{2}(a + bp_j^* + c).$$

Bertrand Model of Duopoly (Cont.)

- The Nash equilibrium is determined by

$$\begin{aligned} p_1^* &= \tfrac{1}{2}(a + bp_2^* + c), \\ p_2^* &= \tfrac{1}{2}(a + bp_1^* + c). \end{aligned}$$

- The unique Nash equilibrium is $(p_1^*, p_2^*) = \left(\frac{a+c}{2-b}, \frac{a+c}{2-b}\right)$.
- The problem only makes sense if $b < 2$.

Motivating Example: Matching Pennies

Two players each has a penny and must choose whether to display it with heads or tails facing up. If the two pennies match (i.e., both are heads up or both are tails up), then player 2 wins player 1's penny; if the pennies do not match then 1 wins 2's penny.

		Player 2	
		Heads	Tails
Player 1	Heads	$-1, 1$	$1, -1$
	Tails	$1, -1$	$-1, 1$

Motivating Example: Matching Pennies (Cont.)

- In the Matching Pennies example, there is no Nash equilibrium by our previous definition.
- In such games, each player wants to outguess others, so that there is uncertainty regarding to the strategies chosen by the players.
- We need to introduce a broader definition of the strategies to incorporate such uncertainty by allowing players to randomize among their choices $\rightarrow$ mixed strategies.

Mixed Strategies

Definition

In a normal-form game $G = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$, suppose $S_i = \{s_{i1}, \ldots, s_{iK_i}\}$. Each strategy $s_{ik} \in S_i$ is a pure strategy for player i. A mixed strategy for player i is a probability distribution $p_i = (p_{i1}, \ldots, p_{iK_i})$, for $k = 1, \ldots, K_i$, where $p_{i1} + \cdots + p_{iK_i} = 1$ and $p_{ik} \geq 0$.

- Note that there are only K_i pure strategies for player i, but infinitely many mixed strategies.
- Any pure strategy s_{ik} is a special case of mixed strategies, i.e., $p_{ik} = 1$ and $p_{ij} = 0$ for all $j \neq k$.

Mixed Strategies: Example

- In the Matching Pennies example, $S_i = \{\text{Heads, Tails}\}$.
- Each player has two pure strategies: Heads or Tails.
- A mixed strategy for a player is a probability distribution $(p, 1-p)$, where p is the probability that the player chooses Heads, while $1-p$ is the probability that the player chooses Tails.
- $(\frac{1}{2}, \frac{1}{2})$ means playing Heads and Tails with an equal probability; $(\frac{1}{3}, \frac{2}{3})$ means playing Heads with a probability of $\frac{1}{3}$ and Tails with a probability of $\frac{2}{3}$.
- The mixed strategy $(1, 0)$ is simply a pure strategy of playing Heads.

Mixed Strategy Nash Equilibrium

- How to extend the definition of Nash equilibrium to include mixed strategies?
- Consider the case with two players.
- Suppose
$$S_1 = \{s_{11}, s_{12}, \ldots, s_{1J}\},$$
and
$$S_2 = \{s_{21}, s_{22}, \ldots, s_{2K}\}.$$
- Each $s_{1j} \in S_1$ is a pure strategy for player 1, and each $s_{2k} \in S_2$ is a pure strategy for player 2.

Expected payoff

- If player 1 thinks that player 2 will play a mixed strategy $p_2 = (p_{21}, \ldots, p_{2K})$, then player 1's expected payoff of playing a pure strategy s_{1j} is

$$v_1(s_{1j}, p_2) = \sum_{k=1}^{K} p_{2k} u_1(s_{1j}, s_{2k}).$$

- Player 1's expected payoff of playing a mixed strategy $p_1 = (p_{11}, \ldots, p_{1J})$ is

$$\begin{aligned} v_1(p_1, p_2) &= \sum_{j=1}^{J} p_{1j} \sum_{k=1}^{K} p_{2k} u_1(s_{1j}, s_{2k}) \\ &= \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} p_{2k} u_1(s_{1j}, s_{2k}). \end{aligned}$$

Mixed best response

- A mixed strategy $p_1 = (p_{11}, \ldots, p_{1J})$ is a best response to p_2 if
$$v_1(p_1, p_2) \geq v_1(p_1', p_2),$$
for all p_1' over S_1.
- Similarly, if player 2 thinks player 1 will play a mixed strategy $p_1 = (p_{11}, \ldots, p_{1J})$, then player 2's expected payoff of playing a mixed strategy $p_2 = (p_{21}, \ldots, p_{2K})$ is
$$\begin{aligned} v_2(p_1, p_2) &= \sum_{K=1}^{K} p_{2k} \sum_{j=1}^{J} p_{1j} u_2(s_{1j}, s_{2k}) \\ &= \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} p_{2k} u_2(s_{1j}, s_{2k}). \end{aligned}$$

Mixed Strategy Nash Equilibrium

Definition

In a two-player normal-form game $G = \langle S_1, S_2; u_1, u_2 \rangle$, the mixed strategies (p_1^*, p_2^*) are a Nash equilibrium if each player's mixed strategy is a best response to the other player's mixed strategy:

$$v_1(p_1^*, p_2^*) \geq v_1(p_1, p_2^*) \text{ for every } p_1 \text{ over } S_1,$$

and

$$v_2(p_1^*, p_2^*) \geq v_2(p_1^*, p_2) \text{ for every } p_2 \text{ over } S_2.$$

- How to find mixed-strategy Nash equilibria?

Find a Mixed Strategy Nash Equilibrium

- We consider the case with two players, each having two pure strategies.
- Let $p_1 = (r, 1-r)$ be a mixed strategy for player 1 and $p_2 = (q, 1-q)$ be a mixed strategy for player 2.
- Player 1's expected payoff of playing p_1, given player 2's strategy p_2, is
$$v_1(p_1, p_2) = rv_1(s_{11}, p_2) + (1-r)v_1(s_{12}, p_2).$$
- For each p_2 (or q), we need to compute r, denoted by $r^*(q)$, such that p_1 is a best response to p_2.

Find a Mixed Strategy Nash Equilibrium (Cont.)

$r^*(q)$ is the set of solutions to $\max_r v_1(p_1, p_2)$:

$$r^*(q) = \begin{cases} 1, & \text{if } v_1(s_{11}, p_2) > v_1(s_{12}, p_2); \\ [0,1], & \text{if } v_1(s_{11}, p_2) = v_2(s_{12}, p_2); \\ 0, & \text{if } v_1(s_{11}, p_2) < v_2(s_{12}, p_2). \end{cases}$$

Find a Mixed Strategy Nash Equilibrium (Cont.)

- Similarly, player 2's expected payoff is

$$v_2(p_1, p_2) = qv_2(p_1, s_{21}) + (1-q)v_2(p_1, s_{22}).$$

- Given p_1, the best response for player 2 is denoted by $q^*(r)$, which is the set of solutions to $\max_q v_2(p_1, p_2)$:

$$q^*(r) = \begin{cases} 1, & \text{if } v_2(p_1, s_{21}) > v_2(p_1, s_{22}); \\ [0,1], & \text{if } v_2(p_1, s_{21}) = v_2(p_1, s_{22}); \\ 0, & \text{if } v_2(p_1, s_{21}) < v_2(p_1, s_{22}). \end{cases}$$

Find a Mixed Strategy Nash Equilibrium (Cont.)

- A mixed strategy Nash equilibrium is an intersection of the two best-response correspondences $r^*(q)$ and $q^*(r)$.
- If (r^*, q^*) is a mixed strategy Nash equilibrium, then

$$r^* = r^*(q^*),\ q^* = q^*(r^*).$$

Matching Pennies

- Find a Nash equilibrium in the game of Matching Pennies.

		Player 2	
		Heads	Tails
Player 1	Heads	$-1, 1$	$1, -1$
	Tails	$1, -1$	$-1, 1$

- Let $p_1 = (r, 1-r)$ be a mixed strategy for player 1, where r is the probability player 1 chooses Heads.
- Similarly, let $p_2 = (q, 1-q)$ be a mixed strategy for player 2, where q is the probability player 2 chooses Heads.
- What is $r^*(q)$ and $q^*(r)$?

Matching Pennies (Cont.)

- For player 1,

$$v_1(s_{11}, p_2) = q \cdot (-1) + (1-q) \cdot 1 = 1 - 2q,$$
$$v_1(s_{12}, p_2) = q \cdot 1 + (1-q) \cdot (-1) = -1 + 2q.$$

- Player 1 chooses Heads (i.e., $r^*(q) = 1$) if and only if

$$1 - 2q > -1 + 2q \Leftrightarrow 0 \le q < \tfrac{1}{2}.$$

- We have

$$r^*(q) = \begin{cases} 1, & \text{if } 0 \le q < \frac{1}{2}; \\ [0,1], & \text{if } q = \frac{1}{2}; \\ 0, & \text{if } \frac{1}{2} < q \le 1. \end{cases}$$

Matching Pennies (Cont.)

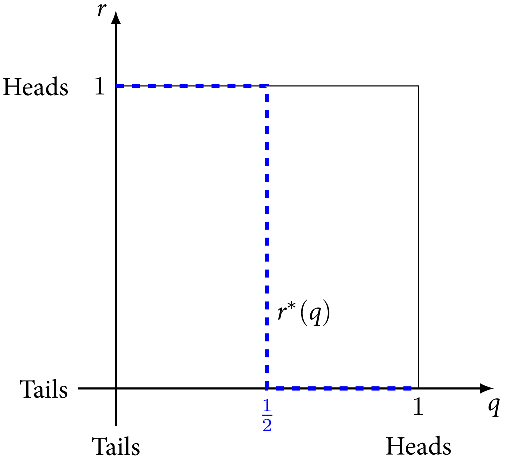


Figure: Best response correspondence for player 1: $r^*(q)$

Matching Pennies (Cont.)

- For player 2,

$$v_2(p_1, s_{21}) = r \cdot 1 + (1 - r) \cdot (-1) = -1 + 2r,$$
$$v_2(p_1, s_{22}) = r \cdot (-1) + (1 - r) \cdot 1 = 1 - 2r.$$

- Player 2 chooses Heads (i.e., $q^*(r) = 1$) if and only if

$$-1 + 2r > 1 - 2r \Leftrightarrow \tfrac{1}{2} < r \leq 1.$$

- We have

$$q^*(r) = \begin{cases} 1, & \text{if } \frac{1}{2} < r \leq 1; \\ [0, 1], & \text{if } r = \frac{1}{2}; \\ 0, & \text{if } 0 \leq r < \frac{1}{2}. \end{cases}$$

Matching Pennies (Cont.)

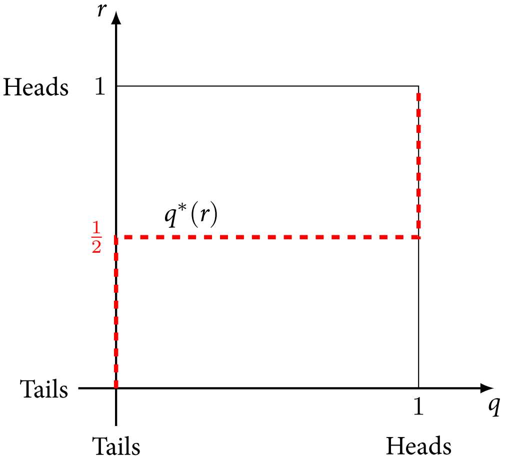


Figure: Best response correspondence for player 2: $q^*(r)$

Matching Pennies (Cont.)

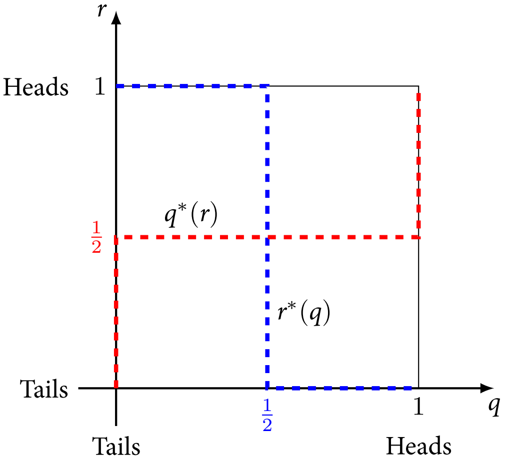


Figure: Mixed strategy Nash equilibrium in Matching Pennies

Matching Pennies (Cont.)

- The graphs of best response correspondences $r^*(q)$ and $q^*(r)$ intersect only once at the point where $q = \frac{1}{2}$ and $r = \frac{1}{2}$.
- $p_1^* = (\frac{1}{2}, \frac{1}{2})$ and $p_2^* = (\frac{1}{2}, \frac{1}{2})$ are the only Nash equilibrium in mixed strategies!

Battle of the Sexes

- Consider the example Battle of the Sexes.
- Let $(r, 1-r)$ be a mixed strategy in which Husband chooses Opera with probability r, and $(q, 1-q)$ be a mixed strategy in which Wife chooses Opera with probability q.
- There are three Nash equilibria: $(r=0, q=0)$, $(r=1, q=1)$ and $(r=\frac{1}{3}, q=\frac{2}{3})$.

Battle of the Sexes (Cont.)

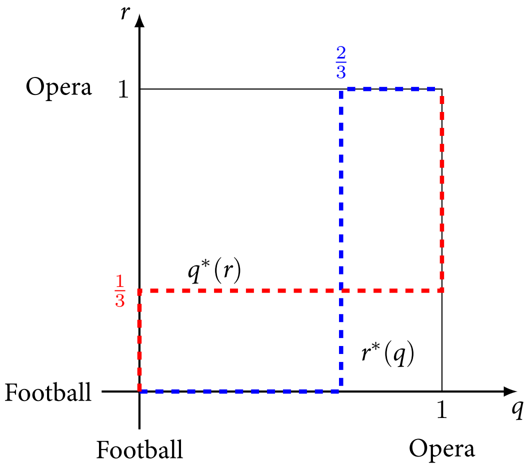


Figure: Nash equilibria in Battle of the Sexes

Mixed Strategy Nash Equilibrium

- What if there are more than two strategies for a player?
- We can first eliminate strictly dominated (pure) strategies.
- The following result is important:

Proposition

The pure strategies played with a positive probability in a mixed strategy Nash equilibrium survive IESDS.

Mixed Strategy Nash Equilibrium

- Example:

Player 2

Player 1		L	C	R
	U	$2, 3$	$1, 1$	$4, 2$
	M	$1, 1$	$3, 2$	$2, 0$
	D	$0, 5$	$0, 5$	$3, 4$

- Using IESDS, we can first eliminate D, and then R.
- The reduced game is

	L	C
U	$2, 3$	$1, 1$
M	$1, 1$	$3, 2$

which is identical to Battle of the Sexes.

Mixed Strategy Nash Equilibrium

- In general, let $p = (p_1, \ldots, p_n)$ be a mixed strategy profile, where $p_i = (p_{i1}, \ldots, p_{iK_i})$, for $i = 1, \ldots, n$.
- The expected payoff for player i is

$$v_i(p) = \sum_{j=1}^{K_i} p_{ij} v_i(p_1, \ldots, p_{i-1}, s_{ij}, p_{i+1}, \ldots, p_n).$$

- The mixed strategy p_i^* is a best response to $p_{-i} = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$ if

$$v_i(p_i^*, p_{-i}) \geq v_i(p_i, p_{-i})$$

for all probability distribution p_i over S_i.

Mixed Strategy Nash Equilibrium

Definition

In a normal-form game $G = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$, the mixed strategies $(p_1^*, \ldots, p_n^*)$ are a (mixed strategy) Nash equilibrium if each player's mixed strategy is a best response to the other players' mixed strategies in terms of expected payoff, i.e.,

$$v_i(p_i^*, p_{-i}^*) \geq v_i(p_i, p_{-i}^*)$$

for every p_i over S_i, and for all $i = 1, \ldots, n$.

Existence of Nash equilibrium

Theorem (Nash, 1950)

In the n-player normal-form game $G = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$, if n is finite and S_i is finite for every i, then there exists at least one Nash equilibrium, possibly involving mixed strategies.

- The conditions are sufficient but not necessary conditions for the existence of a Nash equilibrium.
- Recall that in both Cournot and Betrand competition models, Nash equilibrium exists but the strategy space is infinite.

Strictly Dominated Strategy and Best Response

- Before we know that if a (pure) strategy is a strictly dominated strategy, then it can never be a best response.
- But the reverse may not be true.
- Once we have considered mixed strategies, then the reverse is also true.

Proposition

A pure strategy is a strictly dominated strategy if and only if it is never a best response.

Strictly Dominated Strategy and Best Response

- A pure strategy can be strictly dominated by a mixed strategy, even if it is not strictly dominated by any pure strategy!
- Example:

Player 2

		L	R
Player 1	U	3, –	0, –
	M	0, –	3, –
	D	1, –	1, –

- D is not strictly dominated by either U or M.
- But D is strictly dominated by a strategy $(\frac{1}{2}, \frac{1}{2}, 0)$, i.e., playing U and M with a half probability.
- D is a strictly dominated strategy $\rightarrow$ D is never a best response.

Strictly Dominated Strategy and Best Response

- A pure strategy can be a best response to a mixed strategy, even if it is not a best response to any pure strategy!

		Player 2	
		L	R
Player 1	U	$3, -$	$0, -$
	M	$0, -$	$3, -$
	D	$2, -$	$2, -$

- D is not a best response to L or R.
- D is a best response to a mixed strategy $(q, 1-q)$ chosen by player 2, if

$$2 \geq 3q \text{ and } 2 \geq 3(1-q),$$

i.e., $\frac{1}{3} \leq q \leq \frac{2}{3}$.
- D is not a "never best response" $\rightarrow$ D is not a strictly dominated strategy!

Summary

- We have considered simple static games of complete information.
- Two basic questions in game theory:
  1. How to describe a game $\rightarrow$ normal-form representation
  2. How to solve a game? IESDS or Nash equilibrium
- Mixed strategies: players' uncertainty about others' strategies
- Existence of equilibrium: Nash's Theorem

Question 1

In the following normal-form games, what strategies survive iterated elimination of strictly dominated strategies? What are the pure-strategy Nash equilibria?

	L	C	R
T	$2, 0$	$1, 1$	$4, 2$
M	$3, 4$	$1, 2$	$2, 3$
B	$1, 3$	$0, 2$	$3, 0$

	L	R
U	$1, 3$	$-2, 0$
M	$-2, 0$	$1, 3$
D	$0, 1$	$0, 1$

Question 2

There are three computer companies, each of which can choose to make large (L) or small (S) computers. The choice of company 1 is denoted by S_1 or L_1, and similarly, the choices of companies 2 and 3 are denoted S_i or L_i of $i = 2$ or 3. The following table shows the profit each company would receive according to the choices which the three companies could make. What is the outcome of IESDS and the Nash equilibria of the game?

	S_2S_3	S_2L_3	L_2S_3	L_2L_3
S_1	$-10, -15, 20$	$0, -10, 60$	$0, 10, 10$	$20, 5, 15$
L_1	$5, -5, 0$	$-5, 35, 15$	$-5, 0, 15$	$-20, 10, 10$

Question 3

Players 1 and 2 are bargaining over how to split one dollar. Both players simultaneously name shares they would like to have, s_1 and s_2, where $0 \leq s_1, s_2 \leq 1$. If $s_1^2 + s_2^2 \leq 1/2$, then the players receive the shares they named; if $s_1^2 + s_2^2 > 1/2$, then both players receive zero. What are the pure-strategy Nash equilibria of this game? Now we change the payoff rule as follows: If $s_1^2 + s_2^2 < 1/2$, then the players receive the shares they named; if $s_1^2 + s_2^2 \geq 1/2$, then both players receive zero. What are the pure-strategy Nash equilibria of this game?

Question 4

A two-person game is called a zero-sum game (also called a matrix game) if $u_1(s_1, s_2) + u_2(s_1, s_2) = 0$ for all $s_1 \in S_1$ and $s_2 \in S_2$. Show that (s_1^*, s_2^*) is a pure-strategy Nash equilibrium of a two-person zero-sum game if and only if

$$u_1(s_1, s_2^*) \leq u_1(s_1^*, s_2^*) \leq u_1(s_1^*, s_2), \quad \forall s_1 \in S_1,\ s_2 \in S_2.$$

Consider a two-person zero-sum game in strategic form with finitely many strategies for each player (not just two), and assume that player I has two particular pure strategies T and B and that player II has two pure strategies l and r so that both (T, l) and (B, r) are Nash equilibria of the game. Show that there are at least two further pure-strategy Nash equilibria.
Prove that, for each player, the payoffs for the given equilibria are equal.

Question 5

Consider the following two-person game.

		Player 2	
		X	Y
Player 1	A	$9, 9$	$0, 8$
	B	$8, 0$	$7, 7$

1. Suppose that Player 1 thinks that Player 2 will play her strategy X with probability y and her strategy Y with probability $1 - y$. For what value of y will Player 1 be indifferent between his two strategies?
2. If y is less than this value what strategy will Player 1 prefer? If y is greater than that value?
3. Graph the best responses of Player 1 to Player 2's mixed strategy.
4. Repeat this analysis with the roles of the players reversed.

Deadline: September 23, 2019 (Monday), 23:59.