Advanced Microeconomics I: Lecture Notes 1

1 we consider an entire economy in which consumers and firms interact through markets. There are two principal goals:

- To formally introduce and study two key concepts, the notions of Pareto optimality/efficiency and competitive/Walrasian equilibrium;
- To develop a somewhat special but analytically tractable context for the study of market equilibrium, the partial equilibrium model.

1 Pareto optimality and competitive equilibria

2 We introduce and discuss the concepts of Pareto optimality/efficiency and competitive/Walrasian equilibrium in a general setting.

3 Consider an economy consisting of I consumers, J firms, and L goods.

- Consumer i's preferences over consumption bundles $x_i = (x_{1i}, x_{2i}, \ldots, x_{Li})$ in his consumption set $X_i \subseteq \mathbb{R}^L$ are represented by the utility function u_i.
- The initial endowment of each good ℓ is denoted by $w_\ell \geq 0$.
- Each firm j has available to it the production possibilities summarized by the production set $Y_j \subseteq \mathbb{R}^L$. An element of Y_j is a production vector $y_j = (y_{1j}, y_{2j}, \ldots, y_{Lj}) \in \mathbb{R}^L$.
- If $(y_1, y_2, \ldots, y_J) \in (\mathbb{R}^L)^J$ are the production vectors of the J firms, the total/net amount of good ℓ available to the economy is $w_\ell + \sum_{j=1}^J y_{\ell j}$.

4 An economic allocation $(x_1, x_2, \ldots, x_I, y_1, y_2, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer i and a production vector $y_j \in Y_j$ for each firm j.

The allocation $(x_1, x_2, \ldots, x_I, y_1, y_2, \ldots, y_J)$ is feasible if

$$\sum_{i=1}^{I} x_{\ell i} \leq w_\ell + \sum_{j=1}^{J} y_{\ell j} \text{ for each good } \ell.$$

5 A feasible allocation $(x_1, x_2, \ldots, x_I, y_1, y_2, \ldots, y_J)$ is Pareto optimal/efficient if there is no other feasible allocation $(x'_1, x'_2, \ldots, x'_I, y'_1, y'_2, \ldots, y'_J)$ such that $u_i(x'_i) \geq u_i(x_i)$ for all consumers i and $u_{i_0}(x'_{i_0}) > u_{i_0}(x_{i_0})$ for some consumer i_0.

6 An allocation that is Pareto optimal uses society's initial resources and technological possibilities efficiently in the sense that there is no alternative way to organize the production and distribution of goods that makes some consumer better off without making some other consumer worse off.

7 Pareto optimality does not insure that an allocation is in any sense equitable. For example, using all of society's resources and technological capabilities to make a single consumer as well off as possible, subject to all other consumers receiving a subsistence level of utility, results in an allocation that is Pareto optimal but not in one that is very desirable on distributional grounds.

8 Pareto optimality serves as an important minimal test for the desirability of an allocation—it does, at the very least, say that there is no waste in the allocation of resources in society.

9 For further analysis, we assume that society's initial endowment and technological possibilities are owned by consumers.

- Consumer i initially owns $w_{\ell i}$ of good ℓ, where $\sum_i w_{\ell i} = w_\ell$. We denote consumer i's vector of endowments by $w_i = (w_{1i}, w_{2i}, \dots, w_{Li})$.
- Consumer i owns a share θ_{ij} of firm j (where $\sum_i \theta_{ij} = 1$), giving him a claim to fraction θ_{ij} of firm j's profits.

10 In a competitive economy, a market exists for each of the L goods, and all consumers and producers act as price takers.

The idea of price-taking assumption is that if consumers and producers are small relative to the size of the market, they will regard market prices as unaffected by their own actions.

Denote the price vector for goods by $p = (p_1, p_2, \dots, p_L)$.

11 The allocation $(x_1^*, x_2^*, \dots, x_I^*, y_1^*, y_2^*, \dots, y_J^*)$ and price vector $p^* \in \mathbb{R}^L$ constitute a competitive/Walrasian equilibrium if the following conditions are satisfied:

(i) Profit maximization: For each firm j, y_j^* solves

$$\max_{y_j \in Y_j} p^* \cdot y_j.$$

(ii) Utility maximization: For each consumer i, x_i^* solves

$$\begin{aligned} \underset{x_i \in X_i}{\text{maximize}} \quad & u_i(x_i) \\ \text{subject to} \quad & p^* \cdot x_i \le p^* \cdot w_i + \sum_{j=1}^{J} \theta_{ij} p^* \cdot y_j^*. \end{aligned}$$

(iii) Market clearing: For each good ℓ,

$$\sum_{i=1}^{I} x_{\ell i}^* = w_\ell + \sum_{j=1}^{J} y_{\ell j}^*.$$

12 Conditions (i) and (ii) reflect the underlying assumption, common to nearly all economic models, that agents seek to do as well as they can for themselves.

- Condition (i) states that each firm must choose a production plan that maximizes its profits, taking as given the equilibrium price vector of its outputs and inputs.
- Condition (ii) requires that each consumer chooses a consumption bundle that maximizes his utility given the budget constraint imposed by the equilibrium price vector and his wealth.

13 Condition (iii) requires that, at the equilibrium prices, the desired consumption and production levels identified in Conditions (i) and (ii) are in fact mutually compatible—the aggregate supply of each commodity equals the aggregate demand for it.

14 If the allocation $(x_1^*, x_2^*, \dots, x_I^*, y_1^*, y_2^*, \dots, y_J^*)$ and price vector $p^* \gg 0$ constitute a competitive equilibrium, then so do the allocation $(x_1^*, x_2^*, \dots, x_I^*, y_1^*, y_2^*, \dots, y_J^*)$ and price vector $\alpha p^* = (\alpha p_1^*, \alpha p_2^*, \dots, \alpha p_L^*)$ for any scalar $\alpha > 0$. As a result, we can normalize prices without loss of generality—we always normalize by setting one good's price equal to 1.

15 Lemma: If the allocation $(x_1, x_2, \dots, x_I, y_1, y_2, \dots, y_J)$ and price vector $p \gg 0$ satisfy the market clearing condition for all goods $\ell \neq k$, and if every consumer's budget constraint is satisfied with equality, so that $p \cdot x_i = p \cdot w_i + \sum_j \theta_{ij} p \cdot y_j$ for all i, then the market for good k also clears.

Proof. (1) Since $p \cdot x_i = p \cdot w_i + \sum_j \theta_{ij} p \cdot y_j$ for all i, we have

$$\sum_i p \cdot x_i = \sum_i p \cdot w_i + \sum_i \sum_j \theta_{ij} p \cdot y_j.$$

(2) It implies that

$$\sum_i \sum_\ell p_\ell x_{\ell i} = \sum_i \sum_\ell p_\ell w_{\ell i} + \sum_i \sum_j \theta_{ij} \sum_\ell p_\ell y_{\ell j}.$$

(3) Rearrange:

$$\sum_{\ell \neq k} \sum_i p_\ell x_{\ell i} + \sum_i p_k x_{ki} = \sum_{\ell \neq k} \sum_i p_\ell w_{\ell i} + \sum_i p_k w_{ki} + \sum_{\ell \neq k} \sum_i \sum_j \theta_{ij} p_\ell y_{\ell j} + \sum_i \sum_j \theta_{ij} p_k y_{kj}.$$

(4) Rearrange again:

$$\sum_{\ell \neq k} p_\ell \underbrace{\left[\sum_i x_{\ell i} - \overbrace{\sum_i w_{\ell i}}^{=w_\ell} - \overbrace{\sum_i \sum_j \theta_{ij} y_{\ell j}}^{=\sum_j y_{\ell j}} \right]}_{=0} = -p_k \left[\sum_i x_{ki} - \overbrace{\sum_i w_{ki}}^{=w_k} - \overbrace{\sum_i \sum_j \theta_{ij} y_{kj}}^{=\sum_j y_{kj}} \right].$$

(5) Since $p_k > 0$, we have that $\sum_i x_{ki} = w_k + \sum_j y_{kj}$.

□

2 Partial equilibrium analysis

16 Partial equilibrium (局部均衡) analysis envisions the market for one good that constitutes a small part of the overall economy. The small size of the market facilitates two important simplifications for the analysis of market equilibrium:

- when the expenditure on the good under study is a small portion of a consumer's total expenditure, only a small fraction of any additional dollar of wealth will be spent on this good; consequently, we can expect the wealth effects for it to be small.
- with dispersed substitution effects, the small size of the market under study should lead the prices of other goods to be approximately unaffected by changes in this market.

17 Because of the fixity of other prices, we are justified in treating the expenditure on these other goods as a single composite commodity, called the numeraire (一般等价物).

18 Consider a two-good quasilinear model.

- There are two commodities: the numeraire and good ℓ. Let m_i and x_i denote consumer i's consumption of the numeraire and good ℓ, respectively.
- We let each consumer's consumption set be $\mathbb{R} \times \mathbb{R}_+$. We assume for convenience that consumption of the numeraire m can take negative values—this is to avoid dealing with boundary problems.
- Each consumer i has a utility function that takes the quasilinear form

  $$u_i(m_i, x_i) = m_i + \phi_i(x_i).$$

  We assume that ϕ_i is bounded above and twice differentiable, with $\phi_i'(x_i) > 0$ and $\phi_i''(x_i) < 0$ for all $x_i \geq 0$. We normalize $\phi_i(0) = 0$.

  The quasilinearity implies that the wealth effect on good ℓ is 0.[1]
- We normalize the price of the numeraire to equal 1, and we let p denote the price of good ℓ.
- Each firm j is able to produce good ℓ from good m. The amount of the numeraire required by firm j to produce $q_j \geq 0$ units of good ℓ is given by the cost function $c_j(q_j)$.
- Letting z_j denote firm j's use of good m as an input, its production set is therefore

  $$Y_j = \{(-z_j, q_j) \mid q_j \geq 0, z_j \geq c_j(q_j)\}.$$

  We assume that c_j is twice differentiable, with $c_j'(q_j) > 0$ and $c_j''(q_j) \geq 0$ at all $q_j \geq 0$.
- For simplicity, we assume that there is no initial endowment of good ℓ—all amounts consumed must be produced by the firms. Consumer i's initial endowment of the numeraire is the scalar $w_{mi} > 0$, and we let $w_m = \sum_i w_{mi}$.

19 Given the price p^* for good ℓ, firm j's equilibrium output level q_j^* must solve

$$\max_{q_j \geq 0} p^* q_j - c_j(q_j),$$

which has the necessary and sufficient first order condition

$$p^* \leq c_j'(q_j^*) \text{ with equality if } q_j^* > 0.$$

Note that the second order condition is satisfied: $-c_j''(q_j) < 0$.

20 Consumer i's equilibrium consumption vector (m_i^*, x_i^*) must solve

$$\begin{aligned} \underset{m_i \in \mathbb{R}, x_i \in \mathbb{R}_+}{\text{maximize}} \quad & m_i + \phi_i(x_i) \\ \text{subject to} \quad & m_i + p^* x_i \leq w_{mi} + \sum_{j=1}^{J} \theta_{ij} [p^* q_j^* - c_j(q_j^*)]. \end{aligned}$$

[1] Consider the problem

$$\begin{aligned} \underset{m_i, x_i}{\text{maximize}} \quad & u_i(m_i, x_i) = m_i + \phi_i(x_i) \\ \text{subject to} \quad & m_i + p x_i = w_i. \end{aligned}$$

The solution x_i^*, satisfying $\phi_i'(x_i^*) = \lambda p^*$, only depends on p^*.

In any solution to this problem, the budget constraint holds with equality; otherwise, consumer i can increase m_i to get better.

Substituting for m_i from the constraint, we can rewrite consumer i's problem

$$\max_{x_i \in \mathbb{R}_+} \phi_i(x_i) - p^* x_i + \left[w_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*)) \right]$$

Since the second order condition is satisfied ($\phi_i''(x_i) < 0$), the necessary and sufficient first order condition is

$$\phi_i'(x_i^*) \leq p^* \text{ with equality if } x_i^* > 0.$$

21 It is convenient to adopt the convention of identifying an equilibrium allocation by $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ with the understanding that consumer i's equilibrium consumption of the numeraire is $m_i^* = w_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*)) - p^* x_i^*$ and that firm j's equilibrium usage of the numeraire as an input is $z_j^* = c_j(q_j^*)$.

22 It is clear that $p^* > 0$ in any competitive equilibrium; otherwise consumers would demand an infinite amount of good ℓ ($\phi_i' > 0$).

By Lemma, we need only to check that the market for good ℓ clears—no need to check for the numeraire.

23 The allocation $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ and the price p^* constitute a competitive equilibrium iff

$$p^* \leq c_j'(q_j^*) \text{ with equality if } q_j^* > 0, j = 1, \ldots, J, \tag{1}$$

$$\phi_i'(x_i^*) \leq p^* \text{ with equality if } x_i^* > 0, i = 1, \ldots, I, \tag{2}$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*. \tag{3}$$

24 At any interior solution,

- Condition (1) says that firm j's marginal benefit from selling an additional unit of good ℓ, p^*, exactly equals its marginal cost $c_j'(q_j^*)$.
- Condition (2) says that consumer i's marginal benefit from consuming an additional unit of good ℓ, $\phi_i'(x_i^*)$, exactly equals its marginal cost p^*.
- Condition (3) is the market-clearing equation.

25 If $\max_i \phi_i'(0) > \min_j c_j'(0)$, then the aggregate consumption and production of good ℓ must be strictly positive in a competitive equilibrium.

Otherwise, $x_i^* = 0$ and $c_j^* = 0$ for each i and j. Then $\phi_i'(0) \leq p^* \leq c_j'(0)$ for each i and j. Thus, $\max_i \phi_i'(0) \leq p^* \leq \min_j c_j'(0)$—contradiction.

In the following, we assume $\max_i \phi_i'(0) > \min_j c_j'(0)$.

26 We can derive the aggregate demand function for good ℓ from Condition (2).

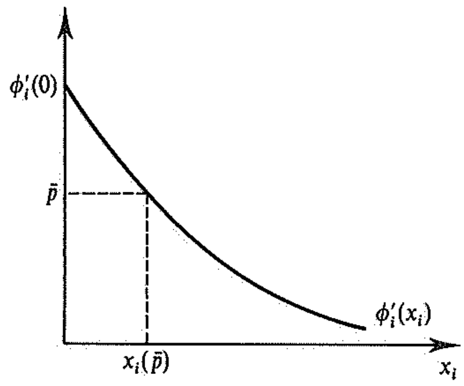
(1) Since $\phi_i'' < 0$, ϕ' is strictly decreasing function taking all values in the set $(0, \phi_i'(0)]$.

(2) For each possible level of $p > 0$, we can solve for a unique level of x_i, denoted $x_i(p)$, that satisfies Condition (2).

- If $p < \phi_i'(0)$, then $x_i(p) \neq 0$, and hence $\phi_i'\big(x_i(p)\big) = p$.

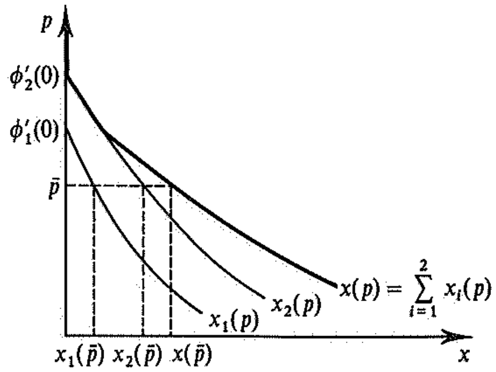
- If $p \geq \phi_i'(0)$, then $x_i(p) = 0$.

(3) Graphic illustration



27 The function x_i derived above is consumer i's Walrasian demand function for good ℓ. It is continuous and strictly decreasing at any $p < \phi_i'(0)$ (it is nonincreasing at all $p > 0$).

28 The aggregate demand function for good ℓ is the function $x(p) = \sum_i x_i(p)$, which is continuous and strictly decreasing at any $p < \max_i \phi_i'(0)$ (it is nonincreasing at all $p > 0$). Note that $x(p) = 0$ whenever $p \geq \max_i \phi_i'(0)$.



29 The aggregate supply function can be similarly derived from Condition (1). We consider a special case here: every c_j is strictly convex and $c_j'(q_j) \to \infty$ as $q_j \to \infty$.[2]

(1) For any $p > 0$, we can solve for a unique level of q_j that satisfies Condition (1).

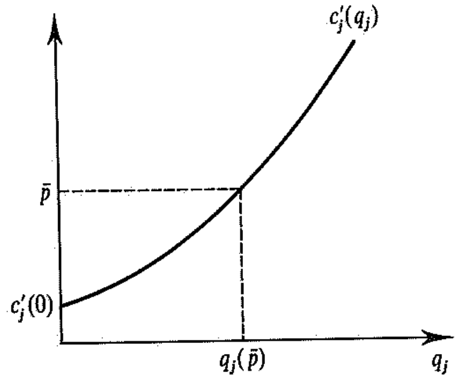
- If $p > c_j'(0)$, then $q_j(p) \neq 0$, and hence $p = c_j'\big(q_j(p)\big)$.
- If $p \leq c_j'(0)$, then $q_j(p) = 0$.

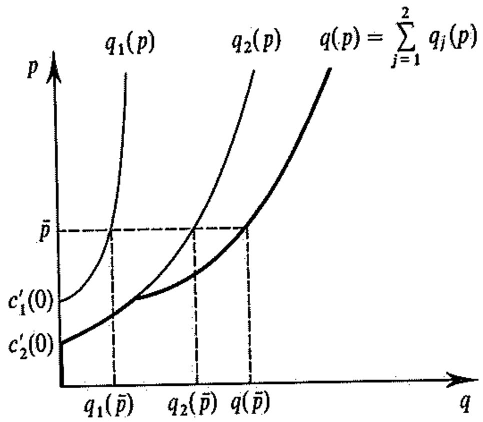
(2) Graphic illustration

30 The function q_j derived above is firm j's supply function for good ℓ. It is continuous and strictly increasing at any $p > c_j'(0)$ (it is nondecreasing at all $p > 0$).

31 The aggregate supply function for good ℓ is the function $q(p) = \sum_j q_j(p)$, which is continuous and strictly increasing at any $p > \min_j \phi_j'(0)$ (it is nondecreasing at all $p > 0$). Note that $q(p) = 0$ whenever $p \leq \min_j c_j'(0)$.

[2]For the general case, please see MWG page 320–321.





32 To find the equilibrium price of good ℓ, we need only find the price p^* at which aggregate demand equals aggregate supply, that is, $x(p^*) = q(p^*)$.

(1) We have already assumed that $\max_i \phi_i'(0) > \min_j c_j'(0)$.

(2) At any $p \geq \max_i \phi_i'(0)$, we have $x(p) = 0$ and $q(p) > 0$.

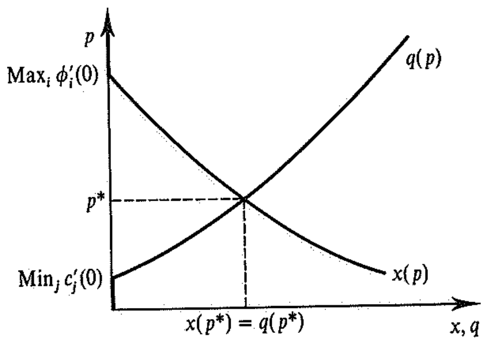
(3) At any $p \leq \min_j c_j'(0)$, we have $q(p) = 0$ and $x(p) > 0$.

(4) Thus, the equilibrium price p^* (if exists) should be in $\big(\min_j c_j'(0), \max_i \phi_i'(0)\big)$.

(5) The existence follows from the continuity of x and q.

(6) The equilibrium price p^* is unique.

(7) Graphic illustration



3 The fundamental welfare theorems

33 We would like to study the link between the set of Pareto optimal allocations and the set of competitive equilibria.

34 When consumers' preferences are quasilinear, the boundary of the economy's utility possibility set is linear and all points in this boundary are associated with consumption allocations that differ only in the distribution of the numeraire among consumers.

Step 1: We fix the consumption and production levels of good ℓ at $(\bar{x}_1, \ldots, \bar{x}_I, \bar{q}_1, \ldots, \bar{q}_J)$.

(1) With these production levels, the total amount of the numeraire available for distribution among consumers is $w_m - \sum_j c_j(\bar{q}_j)$.

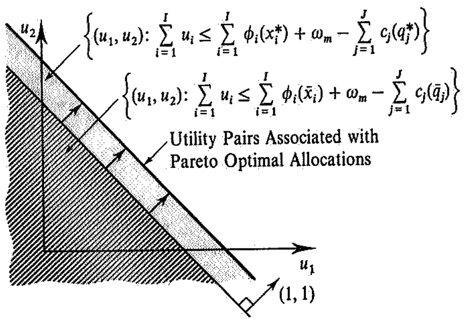
(2) The quasilinear form of the utility functions allows for an unlimited unit-for-unit transfer of utility across consumers through transfers of the numeraire.

(3) The set of utilities that can be attained for the I consumers by appropriately distributing the available amounts of the numeraire is given by

$$\left\{ (u_1, \ldots, u_I) \;\middle|\; \sum_{i=1}^{I} u_i \leq \sum_{i=1}^{I} \phi_i(\bar{x}_i) + w_m - \sum_{j=1}^{J} c_j(\bar{q}_j) \right\}.$$

(4) The boundary of this set is a hyperplane with normal vector $(1, \ldots, 1)$.

Step 2: By altering the consumption and production levels of good ℓ, we necessarily shift the boundary of this set in a parallel manner.



35 Every Pareto optimal allocation must involve the consumption and production levels of good ℓ, $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$, that extend this boundary as far out as possible.

These are called optimal consumption and production levels for good ℓ. Note that they are not uniquely determined; see MWG page 325 Footnote 15.

As long as these optimal consumption and production levels for good ℓ are determined, Pareto optimal allocations can differ only in the distribution of the numeraire among consumers.

36 The optimal consumption and production levels can be obtained by solving the following problem:

$$\begin{aligned} \underset{(x_i),(q_j)}{\text{maximize}} \quad & \sum_{i=1}^{I} \phi_i(x_i) + w_m - \sum_{j=1}^{J} c_j(q_j), \\ \text{subject to} \quad & \sum_{i=1}^{I} x_i = \sum_{j=1}^{J} q_j. \end{aligned}$$

37 The term $\sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$ is called Marshallian aggregate surplus. It can be thought of as the total utility generated from consumption of good ℓ less its cost of production.

38 It is clear that the second order condition is satisfied. The first order conditions yield necessary and sufficient conditions for optimal consumption and production levels.

Let μ be the multiplier on the constraint, the optimal values $(x_1^*, \dots, x_I^*, q_1^*, \dots, q_J^*)$ and the multiplier μ satisfy the following $I + J + 1$ conditions:

$$\mu \leq c_j'(q_j^*) \text{ with equality if } q_j^* > 0, j = 1, \dots, J, \tag{4}$$

$$\phi_i'(x_i^*) \leq \mu \text{ with equality if } x_i^* > 0, i = 1, \dots, I, \tag{5}$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*. \tag{6}$$

These conditions exactly parallel Conditions (1)–(3) with μ replacing p^*.

Since any competitive equilibrium allocation has consumption and production levels of good ℓ, $(x_1^*, \dots, x_I^*, q_1^*, \dots, q_J^*)$, that satisfies Conditions (1)–(3), it also satisfies the above three conditions by letting $\mu = p^*$. That is, any competitive equilibrium outcome in this model is Pareto optimal.

39 Theorem (The first fundamental theorem of welfare economies): If the price p^* and allocation $(x_1^*, \dots, x_I^*, q_1^*, \dots, q_J^*)$ constitute a competitive equilibrium, then this allocation is Pareto optimal.

40 This theorem is a formal expression of Adam Smith's "invisible hand" and is a result that holds with considerable generality.

On the other hand, in the models we establish the first fundamental welfare theorem, markets are "complete" in the sense that there is a market for every relevant commodity and all market participants act as price takers.

41 We have already known that good ℓ's equilibrium price p^*, its equilibrium consumption and production levels $(x_1^*, \dots, x_I^*, q_1^*, \dots, q_J^*)$, and firms' profits are unaffected by changes in consumers' wealth levels.

As a result, a transfer of one unit of the numeraire from consumer i to consumer i' will cause these consumers' equilibrium consumption of the numeraire to change by exactly the amount of the transfer and will cause no other changes.

Thus, by appropriately transferring endowments of the numeraire, the resulting competitive equilibrium allocation can be made to yield any utility vector in the boundary of the utility possibility set.

42 Theorem (The second fundamental theorem of welfare economies): For any Pareto optimal levels of utility $(u_1^*, \dots, u_I^*)$, there are transfers of the numeraire $(T_1, \dots, T_I)$ satisfying $\sum_i T_i = 0$, such that a competitive equilibrium reached from the endowments $(w_{m1} + T_1, \dots, w_{mI} + T_I)$ yields precisely the utilities $(u_1^*, \dots, u_I^*)$.

43 In more general competitive economies, a critical requirement, in addition to those needed for the first fundamental welfare theorem, turns out to be convexity of preferences and production sets.

4 Homework

- Reading: 10.A–C