Social and Economic Networks

Transportation Network

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Outline

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   - Equilibrium
   - Price of anarchy
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3. Congestion game
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Reference

- Easley and Kleinberg, Chapter 8
- Roughgarden, Chapter 11
- MIT Open Course Networks, Fall 2009, Lecture 12
- MIT Open Course Networks, Spring 2018, Lecture 15

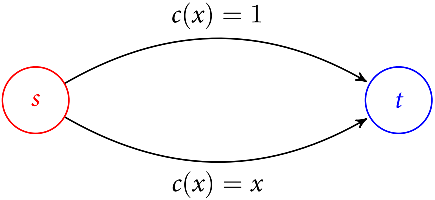
Transportation network

Traveling through a transportation network, or sending packets through the Internet, involves fundamentally game-theoretic reasoning:

- rather than simply choosing a route in isolation, individuals need to evaluate routes in the presence of the congestion resulting from the decisions made by themselves and everyone else.

Pigou's example

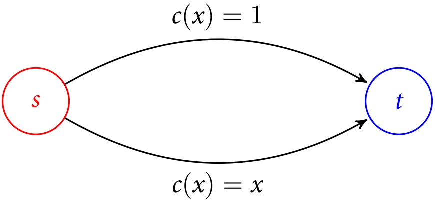
- Consider the simple network.



- The disjoint edges connect a source node s to a sink node t.
- Each edge is labeled with a cost function $c(\cdot)$, which describes the cost (i.e., travel time) incurred by users of the edge.

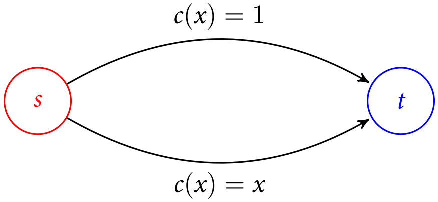
* It is a function of the amount of traffic routed on the edge.

Pigou's example (Cont.)



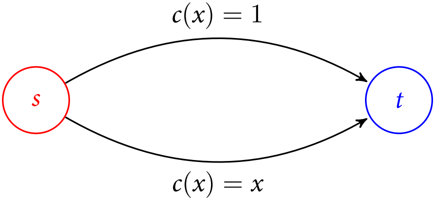
- The upper edge has the constant cost function $c(x) = 1$.
- ⋆ It represents a route that is relatively long but immune to congestion.
- The cost of the lower edge is governed by the function $c(x) = x$.
- ⋆ It increases as the edge gets more congested.
- The lower edge is cheaper than the upper edge iff less than one unit of traffic uses it.

Pigou's example (Cont.)



- Suppose that there is one unit of traffic, representing a large population of network users.
- Each user chooses independently between the two routes from s to t.
- Each user aims to minimize its cost.

Pigou's example: Equilibrium



- Selfish routing outcome: All traffic will use the lower edge.
- The lower route is never worse than the upper one, even when it is fully congested, and it is superior whenever some of the other users are foolish enough to take the upper route. (weakly dominant strategy)
- In this selfish routing outcome, all users incur one unit of cost.

Pigou's example: Social optimum

- Suppose we can control how the traffic is routed.
- Can we leverage this power to improve over the selfish routing outcome?
- We assign half of the traffic to each of the two routes.
    - The users forced onto the upper edge experience one unit of cost. They are not worse off.
    - The users forced onto the lower edge now enjoy lighter traffic conditions, and incur $\frac{1}{2}$ unit of cost.
- Therefore, we have lowered the cost of half of the users while making no one worse off.
- The average cost incurred by traffic has decreased from 1 to $\frac{3}{4}$.

Pigou's example: Price of anarchy

- The price of anarchy (无政府的代价):

$$\frac{\text{average cost of traffic in a outcome}}{\text{minimum-possible average cost}}.$$

- In Pigou's example, the price of anarchy is $\frac{4}{3}$.
- If the price of anarchy of a network is close to 1, then we conclude that the negative impact of selfish routing is relatively small.

Network

- We have a directed network $N = (V, E)$, with node set V, edge set E, and a single origin-destination pair.
- $\mathcal{P}$ denotes the set of paths between origin and destination.
- Each link $e \in E$ has a latency function $L_e(\cdot)$.
- * The latency function captures congestion effects.
- Let us assume for simplicity that $L_e(\cdot)$ is nonnegative, differentiable, and nondecreasing.
- We normalize total traffic to 1 and in the context of the game theoretic formulation here, $I = [0, 1]$.
- We also assume that all traffic is homogeneous. Each user wishes to minimize delay.

Network (Cont.)

- A flow or traffic pattern (distribution of strategy profile) a nonnegative vector
$$\boldsymbol{x} = (x_p)_{p \in \mathcal{P}},$$
such that $\sum_{p \in \mathcal{P}} x_p = 1$.
- x_p denotes the flow on path $p \in \mathcal{P}$.

* the proportion of users who choose the path p.

Network

- Suppose the flow is $\boldsymbol{x} = (x_p)_{p \in \mathcal{P}}$.
- The flow of each link $e \in E$ is

$$x_e = \sum_{p \ni e} x_p.$$

- Here the notation $p \ni e$ denotes the paths p that traverse link $e \in E$.
- The total delay (latency) cost of a routing pattern $\boldsymbol{x}$ is:

$$L(\boldsymbol{x}) = \sum_{e \in E} x_e L_e(x_e),$$

that is, it is the sum of latencies $L_e(x_e)$ for each link $e \in E$ multiplied by the flow over this link, x_e, summed over all links E.

Socially optimal routing

Socially optimal routing, defined as the flow minimizing aggregate delay, is given by $\boldsymbol{x}^S$ that is a solution to the following problem

$$\begin{aligned}
\underset{\boldsymbol{x}=(x_p)_{p\in\mathcal{P}}}{\text{minimize}} \quad & L(\boldsymbol{x}) = \sum_{e\in E} x_e L_e(x_e) \\
\text{subject to} \quad & \sum_{p\ni e} x_p = x_e \text{ for each } e \in E, \\
& \sum_{p\in\mathcal{P}} x_p = 1, x_p \geq 0 \text{ for each } p \in \mathcal{P}.
\end{aligned}$$

Equilibrium

- A flow $\boldsymbol{x}^E = (x_p^E)_{p\in\mathcal{P}}$ is an equilibrium (Nash equilibrium, or wardrop equilibrium) if for each path p with $x_p^E > 0$, there does not exist a path $p' \neq p$ such that

$$\sum_{e\in p} L_e(x_e^E) < \sum_{e\in p'} L_e(x_e^E).$$

* That is, for each user, their routing choice is optimal.

- Let $\boldsymbol{x}^E$ be an equilibrium, then it must be such that
  1. for all p and p' in $\mathcal{P}$ with $x_p, x_{p'} > 0$,

$$\sum_{e\in p} L_e(x_e^E) = \sum_{e\in p'} L_e(x_e^E).$$

  2. for any p and p' in $\mathcal{P}$ with $x_p > 0$ and $x_{p'} = 0$,

$$\sum_{e\in p} L_e(x_e^E) \leq \sum_{e\in p'} L_e(x_e^E).$$

Equilibrium: Characterization

Theorem

A flow $\boldsymbol{x}^E$ is an equilibrium iff it is a solution to

$$\begin{aligned}
\underset{\boldsymbol{x}=(x_p)_{p\in\mathcal{P}}}{\text{minimize}} \quad & \sum_{e\in E}\int_0^{x_e} L_e(z)\,\mathrm{d}z \\
\text{subject to} \quad & \sum_{p\ni e} x_p = x_e \text{ for each } e\in E, \\
& \sum_{p\in\mathcal{P}} x_p = 1,\, x_p \geq 0 \text{ for each } p\in\mathcal{P}.
\end{aligned}$$

If each L_i is strictly increasing, then $\boldsymbol{x}^E$ is unique.

Proof

- Rewrite the minimization problem as

$$\begin{aligned} \underset{\boldsymbol{x}=(x_p)_{p\in\mathcal{P}}}{\text{minimize}} \quad & \sum_{e\in E}\int_0^{\sum_{p\ni e} x_p} L_e(z)\,\mathrm{d}z \\ \text{subject to} \quad & \sum_{p\in\mathcal{P}} x_p = 1, x_p \geq 0 \text{ for each } p \in \mathcal{P}. \end{aligned}$$

- Since each L_e is nondecreasing, this is a convex program. Therefore, first-order conditions are necessary and sufficient.
- Lagrangian:

$$\sum_{e\in E}\int_0^{\sum_{p\ni e} x_p} L_e(z)\,\mathrm{d}z - \lambda\Big(\sum_{p\in\mathcal{P}} x_p - 1\Big) - \sum_{p\in\mathcal{P}} \mu_p x_p.$$

Proof (Cont.)

- First-order condition on x_p is

$$\sum_{e \in p} L_e(x_e^E) = \sum_{e \in p} L_e\left(\sum_{p \ni e} x_p^E\right) = \lambda + \mu_p.$$

  Complementary slackness: $\mu_p \geq 0$ with equality whenever $x_p^E > 0$.
- If $x_p^E > 0$, FOC implies

$$\sum_{e \in p} L_e(x_e^E) = \sum_{e \in p} L_e\left(\sum_{p \ni e} x_p^E\right) = \lambda.$$

- That is, the multiplier λ will be equal to the lowest cost path, which then implies the result that for all $p, p' \in \mathcal{P}$ with $x_p^E, x_{p'}^E > 0$,

$$\sum_{e \in p} L_e(x_e^E) = \lambda = \sum_{e \in p'} L_e(x_e^E).$$

Proof (Cont.)

- If $x_p^E = 0$, FOC implies

$$\sum_{e \in p} L_e(x_e^E) = \sum_{e \in p} L_e\left(\sum_{p \ni e} x_p^E\right) = \lambda + \mu_p \geq \lambda.$$

- That is, for paths with $x_p^E = 0$, the cost can be higher.
- Finally, if each L_e is strictly increasing, then the set of equalities $\sum_{e \in p'} L_e(x_e^E) = \sum_{e \in p} L_e(x_e^E)$ admits a unique solution.

Proof: Intuition

- Suppose there is a traffic graph with x_e people driving along edge e.
- Let the energy of the edge e be

$$E(e) = \int_0^{x_e} L_e(z)\,\mathrm{d}z.$$

  If $x_e = 0$, let $E(e) = 0$.
- Let the total energy of the traffic network be the sum of the energies of every edge.
- Take a choice of routes that minimizes the total energy. Such a choice must exist because there are finitely many choices of routes. That will be an equilibrium.

Proof: Intuition (Cont.)

- Assume, for contradiction, this is not the case.
- Then, there is at least some drivers who can switch the route and improve the travel time.
- Suppose the original route is $p = (e_0, e_1, \ldots, e_n)$ while the new route is $p' = (e'_0, e'_1, \ldots, e'_m)$.
- Then $\sum_{i=0}^{n} L_{e_i}(x_{e_i}) > \sum_{i=0}^{m} L_{e'_i}(x_{e'_i})$.
- Let E be total energy of the traffic graph, and consider what happens when the route p is removed.
  - The energy of each edge e_i will be reduced by $\int_{x_e-\epsilon}^{x_e} L_e(z)\,\mathrm{d}z$.
  - So E will be reduced by $\sum_{i=0}^{n} \int_{x_{e_i}-\epsilon}^{x_{e_i}} L_{e_i}(z)\,\mathrm{d}z$.

Proof: Intuition (Cont.)

- If the new route $p' = (e'_0, e'_1, \ldots, e'_m)$ is then added, the total energy E will be increased by $\sum_{i=0}^{m} \int_{x_{e'_i}}^{x_{e'_i}+\epsilon} L_{e'_i}(z)\,\mathrm{d}z$.
- Because the new route is less costly than the original route, E must decrease relative to the original configuration, contradicting the assumption that the original set of routes minimized the total energy.
- Therefore, the choice of routes minimizing total energy is an equilibrium.

Equilibrium: Algorithm

Best response dynamics:

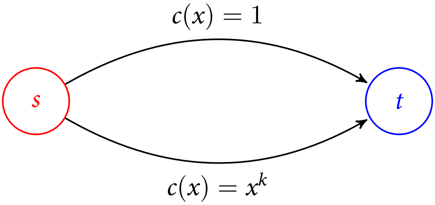
- Let $\boldsymbol{x}$ be some flow.
- If $\boldsymbol{x}$ is not at equilibrium:
  - compute the energy $E(\boldsymbol{x})$.
  - for each driver in $\boldsymbol{x}$:
    - for each alternate path p':
    - compute the energy e' of the flow when driver takes path p'.
    - if $e' < E(\boldsymbol{x})$:
      modify $\boldsymbol{x}$ so that driver takes path p'.
  - continue until no further improvement.

Price of anarchy

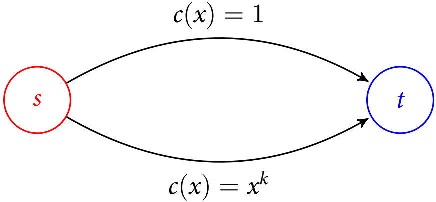
- $$\text{Price of anarchy} = \frac{\text{average cost of traffic in a outcome}}{\text{minimum-possible average cost}}.$$
- Price of anarchy could be close to ∞, i.e., very inefficient.
- Price of anarchy could be close to 1.

Equilibrium: Inefficiency

- In Pigou's example, the equilibrium fails to minimize total delay—hence it is inefficient.
- In fact, it can be arbitrarily inefficient.



Equilibrium: Inefficiency (Cont.)

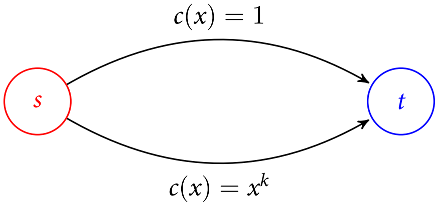


- In this example, socially optimal routing involves

$$\min_{x_1} x_1 L_1(x_1) + (1 - x_1) L_2(1 - x_1) = x_1 + (1 - x_1)^{k+1}.$$

- $x_1^S = 1 - (k+1)^{-\frac{1}{k}}$ and $x_2^S = (k+1)^{-\frac{1}{k}}$.
- Thus, $L(\boldsymbol{x}^S) = x_1^S L_1(x_1^S) + x_2^S L_2(x_2^S) = (k+1)^{-\frac{k+1}{k}} + 1 - (k+1)^{-\frac{1}{k}}$.

Equilibrium: Inefficiency (Cont.)



- The equilibrium again has $x_1^E = 0$ and $x_2^E = 1$.
- Thus, $L(\boldsymbol{x}^E) = 1$.

Price of anarchy

- Therefore, price of anarchy is

$$\frac{L(\boldsymbol{x}^E)}{L(\boldsymbol{x}^S)} = \frac{1}{(k+1)^{-\frac{k+1}{k}} + 1 - (k+1)^{-\frac{1}{k}}}.$$

- When $k \to \infty$, price of anarchy approaches ∞.
- Thus, the equilibrium can be arbitrarily inefficient relative to the social optimum.

Bound of price of anarchy

- Given x_e,
  - $E(e) = \int_0^{x_e} L_e(z)\,\mathrm{d}z$ is the energy of e.
  - let $L(e) = x_e \cdot L_e(x_e)$.
- Since L_e is increasing, $L(e) \geq E(e)$.
- Suppose each $L_e(x_e) = a_e x_e + b_e$ is a linear function.
- Then

$$
\begin{aligned}
L(e) = a_e x_e^2 + b_e x_e &\leq 2\left(\tfrac{a_e}{2} x_e^2 + b_e x_e\right) \\
&= 2 \int_0^{x_e} (a_e z + b_e)\,\mathrm{d}z = 2E(e).
\end{aligned}
$$

Bound of price of anarchy (Cont.)

- Suppose $\boldsymbol{x}$ is a flow,
  - let $E(\boldsymbol{x})$ denote the total energy,
  - let $L(\boldsymbol{x})$ denote the total cost.
- Then we have

$$\frac{1}{2}L(\boldsymbol{x}) \leq E(\boldsymbol{x}) \leq L(\boldsymbol{x}).$$

- Suppose we start with a socially optimal flow $\boldsymbol{x}^S$.
- The best response dynamics leads to an equilibrium flow $\boldsymbol{x}^E$.

Bound of price of anarchy (Cont.)

- In the process of best response dynamics, the cost could increase while the energy is decreasing.
- That is,
$$E(\boldsymbol{x}^E) \le E(\boldsymbol{x}^S).$$
- Since
$$L(\boldsymbol{x}^E) \le 2E(\boldsymbol{x}^E) \text{ and } E(\boldsymbol{x}^S) \le L(\boldsymbol{x}^S),$$
we have
$$L(\boldsymbol{x}^E) \le 2E(\boldsymbol{x}^E) \le 2E(\boldsymbol{x}^S) \le 2L(\boldsymbol{x}^S).$$
- The social cost of the equilibrium is at most twice the cost of the social optimum.

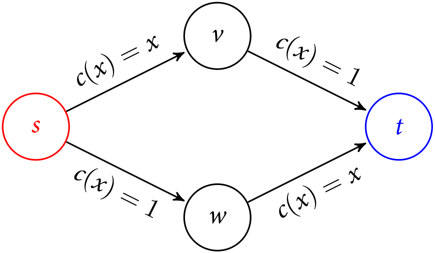
Bounds of price of anarchy

- The coolest statement that might be true is that highly nonlinear cost function are the only obstacle to a small price of anarchy.
- That is, every selfish routing network with not-too-nonlinear cost functions, no matter how complex, has price of anarchy close to 1.
- The worse-case price of anarchy:

Description	Typical representative	Price of anarchy
Linear	$ax + b$	$\frac{4}{3}$
Quadratic	$ax^2 + bx + c$	$\frac{3\sqrt{3}}{3\sqrt{3}-2} \approx 1.6$
Cubic	$ax^3 + bx^2 + cx + d$	$\frac{4\sqrt[3]{4}}{4\sqrt[3]{4}-3} \approx 1.9$
Quartic	$ax^4 + bx^3 + cx^2 + dx + e$	$\frac{5\sqrt[4]{5}}{5\sqrt[4]{5}-4} \approx 2.2$
Degree $\leq p$	$\sum_{i=0}^{p} a_i x^i$	$\frac{(p+1)\sqrt[p]{p+1}}{(p+1)\sqrt[p]{p+1}-p} \approx \frac{p}{\ln p}$

Braess's paradox

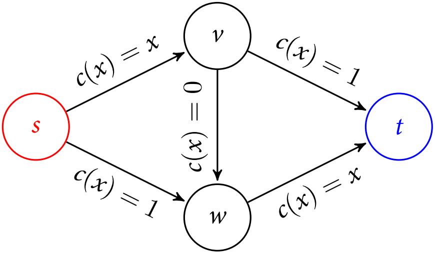
- Consider the four-node network.



- There are two disjoint routes from s to t, each with combined cost $1 + x$, where x is the amount of traffic that uses the route.
- The routes are therefore identical, and selfish traffic should split evenly between them.
- There is one unit of traffic.
- Then all users experience $\frac{3}{2}$ units of cost in the selfish outcome.

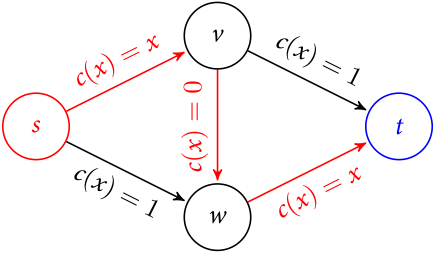
Braess's paradox (Cont.)

- To decrease the cost encountered by the traffic, we build a short, high-capacity edge connecting the midpoints of the two existing routes.



- How should selfish traffic react?

Braess's paradox (Cont.)



- As in Pigou's example, the cost of the new route $s \to v \to w \to t$ is never worse than that along the two original paths, and it is strictly less whenever some traffic fails to use it.
- All users will deviate to the new route.
- All of the traffic now experiences two units of cost.

Braess's paradox (Cont.)

- Braess's paradox shows that the intuitively helpful action off adding a new zero-cost edge can increase the cost experienced by all of the traffic.
- In Seoul, South Korea, a speeding up of traffic around the city was seen when a motorway was removed as part of the Cheonggyecheon (清溪川) restoration project.
- In Stuttgart, Germany, after investments into the road network in 1969, the traffic situation did not improve until a section of newly built road was closed for traffic again.
- In 1990 the temporary closing of 42nd Street in New York City for Earth Day reduced the amount of congestion in the area.

Braess's paradox (Cont.)

- In 2008 Youn, Gastner and Jeong demonstrated specific routes in Boston, New York City and London where that might actually occur and pointed out roads that could be closed to reduce predicted travel times.
- In 2009, New York experimented with closures of Broadway at Times Square and Herald Square, which resulted in improved traffic flow and permanent pedestrian plazas.

Potential games

- A finite game (or a finite-player game with infinite strategies) is a potential game if there exists a function $\Phi\colon S \to \mathbb{R}$ such that $\Phi(s_i, s_{-i})$ gives information about $u_i(s_i, s_{-i})$ for each i.
- If so, Φ is referred to as the potential function.
- The potential function has a natural analogy to "energy" in physical systems.
- It will be useful both for locating pure-strategy Nash equilibria.

Potentials

- A function $\Phi\colon S \to \mathbb{R}$ is called an ordinal potential function for the game if for each $i \in I$ and all $s_{-i} \in S_{-i}$, for all $x, z \in S_i$,

$$u_i(x, s_{-i}) - u_i(z, s_{-i}) \geq 0 \text{ iff } \Phi(x, s_{-i}) - \Phi(z, s_{-i}) \geq 0,$$

  and

$$u_i(x, s_{-i}) - u_i(z, s_{-i}) > 0 \text{ iff } \Phi(x, s_{-i}) - \Phi(z, s_{-i}) > 0.$$

- A function $\Phi\colon S \to \mathbb{R}$ is called an exact potential function for the game if for each $i \in I$ and all $s_{-i} \in S_{-i}$, for all $x, z \in S_i$,

$$u_i(x, s_{-i}) - u_i(z, s_{-i}) = \Phi(x, s_{-i}) - \Phi(z, s_{-i}).$$

Potential games

- A finite game G is called an ordinal (exact) potential game if it admits an ordinal (exact) potential.
- In what follows, we refer to ordinal potential games as potential games, and only add the “exact” qualifier when this is necessary.
- A finite-player game G with infinite strategy space is a potential game if it admits a continuous potential function.

Pure-strategy Nash equilibria in potential games

Theorem

Every potential game has at least one pure strategy Nash equilibrium.

- The global maximum of an ordinal potential function is a pure-strategy Nash equilibrium.
- Suppose that s^* corresponds to the global maximum.
- Then, for any $i \in I$, we have, by definition, $\Phi(s_i^*, s_{-i}^*) - \Phi(s, s_{-i}^*) \geq 0$ for all $s \in S_i$.
- Since Φ is a potential function,

$$u_i(s_i^*, s_{-i}^*) - u_i(s_i, s_{-i}^*) \geq 0 \text{ for all } s \in S_i.$$

- Thus, s^* is a pure-strategy Nash equilibrium.

Examples of ordinal potential games

- Cournot competition.
- Each of n firms chooses quantity $q_i \in (0, \infty)$.
- The payoff function for player i given by $u_i(q_i, q_{-i}) = q_i \cdot (P(Q) - c)$.
- We define the function

$$\Phi(q_1, \ldots, q_n) = (\prod_{i=1}^{n} q_i)(P(Q) - c).$$

- Note that for all i and all q_{-i}, for all $q_i, q_i' > 0$,

$$u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) > 0 \text{ iff } \Phi_i(q_i, q_{-i}) - \Phi_i(q_i', q_{-i}) > 0$$

- Φ is therefore an ordinal potential function for this game.

Examples of exact potential games

- Cournot competition (again).
- Suppose now that $P(Q) = a - bQ$ and costs $c_i(q_i)$ are arbitrary.
- We define the function

$$\Phi^*(q_1, \ldots, q_n) = a \sum_{i=1}^{n} q_i - b \sum_{i=1}^{n} q_i^2 - b \sum_{1 \leq i < j \leq n} q_i q_j - \sum_{i=1}^{n} c_i(q_i).$$

- It can be shown that for all i and all q_{-i}, for all $q_i, q_i' > 0$,

$$u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) = \Phi_i^*(q_i, q_{-i}) - \Phi_i^*(q_i', q_{-i}).$$

- Φ is an exact potential function for this game.

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Congestion model

- Congestion model: $C = \langle I, R, (S_i)_{i \in I}, (c^j)_{j \in R} \rangle$ where
- $I = \{1, 2, \ldots, n\}$ is the set of players.
- $R = \{1, 2, \ldots, m\}$ is the set of resources.
- $S_i \subseteq M$ is the set of resource combinations (e.g., links or common resources) that player i can take/use.
- A strategy for player i is $s_i \in S_i$, corresponding to the resources that this player is using.
- $c^j(k)$ is the benefit for the negative of the cost to each user who uses resource j if k users are using it.

Congestion game

- Define congestion game $\langle I, (S_i), (u_i) \rangle$ with utilities

$$u_i(s_i, s_{-i}) = \sum_{j \in s_i} c^j(k_j),$$

where k_j is the number of users of resource j under strategies s.

Theorem

Every congestion game is a potential game and thus has a pure-strategy Nash equilibrium.

Proof

- For each j define $\bar{k}_j^i$ as the usage of resource j excluding player i, i.e.,
$$\bar{k}_j^i = \sum_{i' \neq i} 1_{j \in s_{i'}},$$
where $1_{j \in s_{i'}}$ is the indicator for the event that $j \in s_{i'}$.
- When others are using the strategy profile s_{-i}, the utility difference of player i from two strategies s_i and s_i' is
$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) - \sum_{j \in s_{i'}} c^j(\bar{k}_j^i + 1).$$

Proof (Cont.)

- Now consider the function

$$\Phi(s) = \sum_{j \in \cup_{i' \in I} s_{i'}} \left[\sum_{k=1}^{k_j} c^j(k) \right].$$

- We can also write

$$\Phi(s_i, s_{-i}) = \sum_{j \in \cup_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k) \right] + \sum_{j \in s_i} c^j(\bar{k}_j^i + 1).$$

Proof (Cont.)

- Therefore,

$$\begin{aligned}\Phi(s_i, s_{-i}) - \Phi(s'_i, s_{-i}) &= \sum_{j \in \cup_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k) \right] + \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) \\ &\quad - \sum_{j \in \cup_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k) \right] - \sum_{j \in s'_i} c^j(\bar{k}_j^i + 1) \\ &= \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) - \sum_{j \in s'_i} c^j(\bar{k}_j^i + 1) \\ &= u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}).\end{aligned}$$