Social and Economic Networks

Games with Strategic Complements

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Outline

1. A linear quadratic model
2. Equilibrium and Katz-Bonacich centrality
3. Key player and intercentrality
4. Key link

Reference

- Jackson, Chapter 9
- Chapter 11, *The Oxford Handbook of the Economics of Networks* (Edited by Yann Bramoullé, Andrea Galeotti, and Brian Rogers), Oxford University Press, 2016.
- Coralio Ballester, Antoni Calvó-Armengol, and Yves Zenou, Who's Who in Networks. Wanted: The Key Player.
- Coralio Ballester, Antoni Calvó-Armengol, and Yves Zenou, Delinquent Networks.

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Network

Consider a game where $N = \{1, 2, \ldots, n\}$ is a finite set of agents in network g.

- $g_{ij} = 1$ if agent i is connected to agent j, and $g_{ij} = 0$ otherwise.
- Links are taken to be reciprocal, so that $g_{ij} = g_{ji}$.
- By convention, $g_{ii} = 0$.
- We denote by $\boldsymbol{G}$ the $n \times n$ adjacency matrix with entry g_{ij}, which keeps track of all direct connections.

Utility

- Each agent i decides how much effort to exert, denoted by $x_i \in \mathbb{R}_+$.
- The utility of each agent i providing effort x_i in network g is given by:

$$u_i(x_i, x_{-i}, g) = \underbrace{\alpha_i x_i - \frac{1}{2} x_i^2}_{\text{individual part}} + \underbrace{\delta \sum_{j=1}^{n} g_{ij} x_i x_j}_{\text{local network effect}} .$$

- $\alpha_i \geq 0$: intrinsic marginal utility.

* It represents the exogenous heterogeneity of agent i that captures the observable characteristics of individual i (e.g., sex, race, age, parental education).

- $\delta > 0$ (sufficiently small) is the intensity of interactions.

Utility (Cont.)

$$u_i(x_i, x_{-i}, g) = \underbrace{\alpha_i x_i - \frac{1}{2}x_i^2}_{\text{individual part}} + \underbrace{\delta \sum_{j=1}^{n} g_{ij} x_i x_j}_{\text{local network effect}} .$$

- The second part, $\delta \sum_{j=1}^{n} g_{ij} x_i x_j$, corresponds to the local-aggregate effect of peers since each agent i is affected by the sum of efforts of the agents for which she has a direct connection.
- Strategic complementary:

$$\frac{\partial^2 u_i(\boldsymbol{x}, g)}{\partial x_i \partial x_j} = \delta g_{ij} \geq 0.$$

* Each player's relative payoff to taking an action is increasing in the set of neighbors who take this action.

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FOC

- Player i's utility

$$u_i(x_i, x_{-i}, g) = \alpha_i x_i - \frac{1}{2}x_i^2 + \delta \sum_{j=1}^{n} g_{ij} x_i x_j.$$

- FOC:

$$\frac{\partial u_i}{\partial x_i} = \alpha_i - x_i + \delta \sum_{j=1}^{n} g_{ij} x_j \leq 0$$

with equality whenever $x_i > 0$.

Heuristic analysis

- If an interior equilibrium exists, it solves

$$\alpha_i - x_i + \delta \sum_{j=1}^{n} g_{ij} x_j = 0 \text{ for each } i.$$

- We rewrite as:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \delta \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- That is, $\boldsymbol{\alpha} - \boldsymbol{x} + \delta \boldsymbol{G}\boldsymbol{x} = \boldsymbol{0}_n$ or $[\boldsymbol{I}_n - \delta \boldsymbol{G}]\boldsymbol{x} = \boldsymbol{\alpha}$.

Heuristic analysis (Cont.)

- We have

$$\boldsymbol{x} = [\boldsymbol{I}_n - \delta \boldsymbol{G}]^{-1} \boldsymbol{\alpha},$$

if the matrix $[\boldsymbol{I}_n - \delta \boldsymbol{G}]$ is invertible.

- Since $\boldsymbol{G}$ is a nonnegative symmetric matrix, all eigenvalues are real.
- Let $\lambda_1(\boldsymbol{G})$, $\lambda_2(\boldsymbol{G})$, ..., $\lambda_n(\boldsymbol{G})$ be eigenvalues of $\boldsymbol{G}$ such that

$$\lambda_1(\boldsymbol{G}) \geq \lambda_2(\boldsymbol{G}) \geq \cdots \geq \lambda_n(\boldsymbol{G}).$$

- Since each $g_{ii} = 0$, $\sum_{i=1}^{n} \lambda_i(\boldsymbol{G}) = \sum_{i=1}^{n} g_{ii} = 0$.

$\Rightarrow$ As long as $\boldsymbol{G}$ is not zero matrix, $\lambda_1(\boldsymbol{G}) > 0$.

- Clearly, if $\delta \lambda_1(\boldsymbol{G}) < 1$, then $[\boldsymbol{I}_n - \delta \boldsymbol{G}]$ is invertible.

Result

$[\boldsymbol{I}_n - \delta\boldsymbol{G}]^{-1}$ is well-define and nonnegative iff $\delta\lambda_1(\boldsymbol{G}) < 1$.

- $\boldsymbol{G}$ is symmetric, then it is diagonalizable:

$$\boldsymbol{P}\boldsymbol{G}\boldsymbol{P}^{-1} = \begin{pmatrix} \lambda_1(\boldsymbol{G}) & & & \\ & \lambda_2(\boldsymbol{G}) & & \\ & & \ddots & \\ & & & \lambda_n(\boldsymbol{G}) \end{pmatrix}$$

- Thus,

$$\begin{aligned} \boldsymbol{I}_n - \delta\boldsymbol{G} &= \boldsymbol{P}^{-1}[\boldsymbol{I}_n - \delta\boldsymbol{P}\boldsymbol{G}\boldsymbol{P}^{-1}]\boldsymbol{P} \\ &= \boldsymbol{P}^{-1}\begin{pmatrix} 1-\delta\lambda_1(\boldsymbol{G}) & & & \\ & 1-\delta\lambda_2(\boldsymbol{G}) & & \\ & & \ddots & \\ & & & 1-\delta\lambda_n(\boldsymbol{G}) \end{pmatrix}\boldsymbol{P} \end{aligned}$$

Equilibrium

Theorem

If $\delta\lambda_1(\boldsymbol{G}) < 1$ and each $\alpha_i > 0$, there is a unique pure-strategy Nash equilibrium $\boldsymbol{x}^*$, which is interior and given by

$$\boldsymbol{x}^* = [\boldsymbol{I}_n - \delta\boldsymbol{G}]^{-1}\boldsymbol{\alpha}.$$

Uniqueness

- The heuristic analysis has already established the existence and uniqueness of an interior equilibrium.
- We can show that a noninterior equilibrium fails to exist.
  - Let $\boldsymbol{y}$ be a noninterior equilibrium, where $y_i = 0$.
  - Then FOC implies $\alpha_i + \delta \sum_{j=1}^{n} g_{ij} y_j \leq 0$.
  - Since $\alpha_i > 0$, the inequality does not hold.
- If $\alpha_i = 0$ and $g_{ik} = 0$ for each k, then player i will choose 0 in an equilibrium (corner equilibrium).

Bonacich centrality

- Recall Bonacich centrality:

$$\boldsymbol{h}(g, a, b) = (\boldsymbol{I}_n - b\boldsymbol{G})^{-1} a\boldsymbol{G}\mathbf{1},$$

  where $a > 0$, $b > 0$, and b is sufficiently small such that the expression is well defined.
- $ad_i(g) = (a\boldsymbol{G}\mathbf{1})_i$ is the base value for i.
- b is the decay factor.
-

$$\boldsymbol{h}(g, a, b) = (\boldsymbol{I}_n - b\boldsymbol{G})^{-1} a\boldsymbol{G}\mathbf{1} = \left[\sum_{k=0}^{\infty} b^k \boldsymbol{G}^k\right] a\boldsymbol{G}\mathbf{1}.$$

Katz-Bonacich centrality

- Katz-Bonacich centrality is defined as:

$$\boldsymbol{b}(g,\delta) = (\boldsymbol{I}_n - \delta \boldsymbol{G})^{-1}\mathbf{1}.$$

- It is obtained from the original Bonacich centrality by an affine transformation:

$$\boldsymbol{b}(g,\delta) = (\boldsymbol{I}_n - \delta \boldsymbol{G})^{-1}\mathbf{1} = \left[\sum_{k=0}^{\infty} \delta^k \boldsymbol{G}^k\right]\mathbf{1} = \mathbf{1} + \underbrace{\left[\sum_{k=0}^{\infty} \delta^k \boldsymbol{G}^k\right]\delta \boldsymbol{G}\mathbf{1}}_{\boldsymbol{h}(g,\delta,\delta)}.$$

- Weighted Katz-Bonacich centrality is defined as:

$$\boldsymbol{b}(g,\delta,\boldsymbol{\alpha}) = (\boldsymbol{I}_n - \delta \boldsymbol{G})^{-1}\boldsymbol{\alpha} = \sum_{k=0}^{\infty} \delta^k \boldsymbol{G}^k \boldsymbol{\alpha}.$$

Katz-Bonacich centrality (Cont.)

- Katz-Bonacich centrality:

$$\boldsymbol{b}(g, \delta) = (\boldsymbol{I}_n - \delta \mathbf{G})^{-1}\mathbf{1} = \sum_{k=0}^{\infty} \delta^k \mathbf{G}^k \mathbf{1}.$$

- The i-th component of $\boldsymbol{b}(g, \delta)$ is

$$\sum_{j=1}^{n} \sum_{k=0}^{\infty} \delta^k g_{ij}^k.$$

- Agent i's Katz-Bonacich centrality counts the total number of paths (not just the shortest paths) in g starting from i, weight by a decay factor δ that decreases with the length of these paths.
- $g_{ij}^k \geq 0$ measures the number of paths of length $k \geq 1$ in g from i to j.

Katz-Bonacich centrality (Cont.)

- Weighted Katz-Bonacich centrality:

$$\boldsymbol{b}(g,\delta,\boldsymbol{\alpha}) = (\boldsymbol{I}_n - \delta\mathbf{G})^{-1}\boldsymbol{\alpha} = \sum_{k=0}^{\infty} \delta^k \mathbf{G}^k \boldsymbol{\alpha}.$$

- The i-th component of $\boldsymbol{b}(g,\delta,\boldsymbol{\alpha})$ is

$$\sum_{j=1}^{n}\sum_{k=0}^{\infty} \delta^k g_{ij}^k \alpha_j.$$

- $\delta^k g_{ij}^k \alpha_j$ measures the number of paths of length $k \geq 1$ in g from i to j, weight by α_j relying on destinations and by a decay factor δ that decreases with the length of these paths.

Equilibrium

Theorem

If $\delta\lambda_1(\boldsymbol{G}) < 1$ and each $\alpha_i > 0$, there is a unique pure-strategy Nash equilibrium $\boldsymbol{x}^*$, which is interior and given by

$$\boldsymbol{x}^* = [\boldsymbol{I}_n - \delta\boldsymbol{G}]^{-1}\boldsymbol{\alpha} = \boldsymbol{b}(g, \delta, \boldsymbol{\alpha}).$$

- More central agents in the network will exert more effort.
- This is intuitively related to the equilibrium behavior, as the paths capture all possible feedbacks.

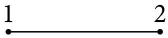
Equilibrium payoff

- For each agent i, the equilibrium utility is

$$u_i(\boldsymbol{x}^*, g) = \frac{1}{2}(x_i^*)^2 = \frac{1}{2}\big(\boldsymbol{b}(g, \delta, \boldsymbol{\alpha})\big)_i^2.$$

- The equilibrium utility of each agent is proportional to the square of her Katz-Bonacich centrality.

Example: K_2



- K_2: the complete graph with 2 nodes.
- The adjacency matrix is

$$\boldsymbol{G} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- By induction, for each $k \geq 1$,

$$\boldsymbol{G}^{2k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \boldsymbol{G}^{2k+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Example: K_2 (Cont.)

- $[\boldsymbol{I} - \delta \boldsymbol{G}]^{-1}$ is well-defined when $\delta < 1$:

$$
\begin{aligned}
[\boldsymbol{I} - \delta \boldsymbol{G}]^{-1} &= \boldsymbol{I}_2 + \delta \boldsymbol{G} + \delta^2 \boldsymbol{G}^2 + \cdots \\
&= \frac{1}{1-\delta^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\delta}{1-\delta^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{1-\delta^2} \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}
\end{aligned}
$$

- Unique Nash equilibrium:

$$
\boldsymbol{x}^* = [\boldsymbol{I} - \delta \boldsymbol{G}]^{-1} \boldsymbol{\alpha} = \begin{pmatrix} \frac{\alpha_1 + \delta \alpha_2}{1-\delta^2} \\ \frac{\alpha_2 + \delta \alpha_1}{1-\delta^2} \end{pmatrix}.
$$

Example K_n

- The adjacency matrix is $\boldsymbol{G} = \mathbf{1}_{nn} - \boldsymbol{I}_n$.
- We can verify that

$$[\boldsymbol{I}_n - \delta \boldsymbol{G}]^{-1} = \frac{1}{1+\delta}\left[\boldsymbol{I}_n + \frac{\delta}{1-(n-1)\delta}\mathbf{1}_{nn}\right]$$

  and is well-defined when $\delta < \frac{1}{n-1}$.
- In equilibrium, for each i,

$$x_i^* = \frac{1}{1+\delta}\left[\alpha_i + \frac{\delta \sum_j \alpha_j}{1-(n-1)\delta}\right].$$

- Clearly, $x_i^* \geq x_j^*$ iff $\alpha_i \geq \alpha_j$.

Example: Regular network with degree d

- The adjacency matrix $\boldsymbol{G}$ satisfies:

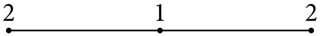
$$\boldsymbol{G}\mathbf{1}_n = d\mathbf{1}_n.$$

- Assume that $\alpha_i = 1$ for each i.
- The equilibrium is

$$\begin{aligned} \boldsymbol{x}^* &= [\boldsymbol{I}_n - \delta \boldsymbol{G}]^{-1}\mathbf{1}_n \\ &= \mathbf{1}_n + \delta \boldsymbol{G}\mathbf{1}_n + \delta^2 \boldsymbol{G}^2\mathbf{1}_n + \cdots \\ &= \mathbf{1}_n + \delta d\mathbf{1}_n + \delta^2 d^2\mathbf{1}_n + \cdots \\ &= \tfrac{1}{1-\delta d}\mathbf{1}_n, \end{aligned}$$

when $1 > \delta d$.

Example: $K_{1,2}$



- $K_{1,2}$: star network with center 1 and spokes 2 and 3.
- The adjacency matrix is

$$\boldsymbol{G} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

- By induction, for each $k \geq 1$,

$$\boldsymbol{G}^{2k} = \begin{pmatrix} 2^k & 0 & 0 \\ 0 & 2^{k-1} & 2^{k-1} \\ 0 & 2^{k-1} & 2^{k-1} \end{pmatrix} \text{ and } \boldsymbol{G}^{2k+1} = \begin{pmatrix} 0 & 2^k & 2^k \\ 2^k & 0 & 0 \\ 2^k & 0 & 0 \end{pmatrix}.$$

Example: $K_{1,2}$ (Cont.)

- $[\boldsymbol{I}_n - \delta \boldsymbol{G}]^{-1}$, well-defined when $\delta < \frac{1}{\sqrt{2}}$, is

$$
\begin{aligned}
[\boldsymbol{I}_n - \delta \boldsymbol{G}]^{-1} &= \boldsymbol{I}_n + \sum_{k=0}^{\infty} \delta^{2k+1} \boldsymbol{G}^{2k+1} + \sum_{k=1}^{\infty} \delta^{2k} \boldsymbol{G}^{2k} \\
&= \frac{1}{1-2\delta^2} \begin{pmatrix} 1 & \delta & \delta \\ \delta & 1-\delta^2 & \delta^2 \\ \delta & \delta^2 & 1-\delta^2 \end{pmatrix}.
\end{aligned}
$$

Example: $K_{p,q}$

- In a complete bipartite graph $K_{p,q}$, there are two disjoint groups P and Q in $K_{p,q}$ such that any node in P is connected to any node in Q.
- Let $p = |P|$ and $q = |Q|$. Thus, the network size satisfies $n = p + q$.
- The adjacency matrix is $\boldsymbol{G} = \begin{pmatrix} \mathbf{0}_{pp} & \mathbf{1}_{pq} \\ \mathbf{1}_{qp} & \mathbf{0}_{qq} \end{pmatrix}$.

Example: $K_{p,q}$ (Cont.)

- $[\boldsymbol{I}_n - \delta \boldsymbol{G}]^{-1}$ is

$$[\boldsymbol{I}_n - \delta \boldsymbol{G}]^{-1} = \begin{pmatrix} \boldsymbol{I}_p + \frac{\delta^2 q}{1-\delta^2 qp} \mathbf{1}_{pp} & \frac{\delta}{1-\delta^2 qp} \mathbf{1}_{pq} \\ \frac{\delta}{1-\delta^2 qp} \mathbf{1}_{qp} & \boldsymbol{I}_q + \frac{\delta^2 p}{1-\delta^2 qp} \mathbf{1}_{qq} \end{pmatrix}.$$

- For each i,

$$x_i^* = \alpha_i + \frac{\delta^2 q}{1 - \delta^2 qp} \sum_{s \in P} \alpha_s + \frac{\delta}{1 - \delta^2 qp} \sum_{t \in Q} \alpha_t.$$

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Key player problem

- For criminal behaviors, it seems relatively natural to consider games with strategic complementarities, since the higher are my friends' criminal efforts, the higher is my marginal utility of exerting criminal effort.
- One can apply the model that has been previously analyzed to study the criminal behaviors in an equilibrium.
- Question: Within a crime organization the police/government has the ability to remove one player, who should it be?
- That is, we want to find the player (criminal) such that removing him reduces total activity (crime) in a network the most.

Key player problem (Cont.)

- The key player problem is formulated as follows:

$$\arg\max_i \left[\sum_{k=1}^{n} b_k(g, \delta, \boldsymbol{\alpha}) - \sum_{k \neq i} b_k(g_{-i}, \delta, \boldsymbol{\alpha}_{-i}) \right].$$

- g_{-i} is the network when player i is removed, and $\boldsymbol{G}_{-i}$ is its adjacency matrix.
  - ⋆ $\boldsymbol{G}_{-i}$ is obtained from $\boldsymbol{G}$ be setting to 0 all of its i-th row and column coefficients.
- $\boldsymbol{\alpha}_{-i}$ is obtained from $\boldsymbol{\alpha}$ be setting to 0 of its i-th component.
- The first term is the sum of total activities in the original network.
- The second term is the resulting equilibrium total activity when i is removed.

Notation

- Let $\boldsymbol{M} = [\boldsymbol{I}_n - \delta \boldsymbol{G}]^{-1}$.
- $m_{ii}(g, \delta) = \sum_{k=0}^{\infty} \delta^k g_{ii}^k$:
  the number of all loops from i to i, weighted by δ^k.
- $m_{ij}(g, \delta) = \sum_{k=0}^{\infty} \delta^k g_{ij}^k$:
  the number of all the outer paths from i to $j \neq i$, weighted by δ^k.
- Since $\boldsymbol{b}(g, \delta) = [\boldsymbol{I}_n - \delta \boldsymbol{G}]^{-1} \mathbf{1}_n$, we have

$$b_i(g, \delta) = \sum_{j=1}^{n} m_{ij}(g, \delta) = m_{ii}(g, \delta) + \sum_{j \neq i} m_{ij}(g, \delta).$$

- Similarly, $\boldsymbol{b}(g, \delta, \boldsymbol{\alpha}) = [\boldsymbol{I}_n - \delta \boldsymbol{G}]^{-1} \boldsymbol{\alpha}$,

$$b_i(g, \delta, \boldsymbol{\alpha}) = \sum_{j=1}^{n} m_{ij}(g, \delta) \alpha_j.$$

Intercentrality

- We still assume that $\delta\lambda_1(\boldsymbol{G}) < 1$.
- The intercentrality (or key-player centrality) measure $c_i(g, \delta)$ is defined as follows:

$$c_i(g, \delta) = \frac{b_i(g, \delta, \boldsymbol{\alpha}) b_i(g, \delta, \mathbf{1}_n)}{m_{ii}(g, \delta)}.$$

Lemma

Lemma

Let $\boldsymbol{M}(g,\delta) = [\boldsymbol{I}_n - \delta \boldsymbol{G}]^{-1}$ be well-defined and nonnegative. Then

$$m_{ij}(g,\delta) m_{ik}(g,\delta) = m_{ii}[m_{jk}(g,\delta) - m_{jk}(g_{-i},\delta)]$$

for all $k \neq i \neq j$.

- Let $g^s_{j(i^c)k}$ denote the number of s-length paths from j to k that do not pass i.
- Let $g^s_{j(i)k}$ denote the number of s-length paths from j to k that do pass i.
- Since $\boldsymbol{G}$ is symmetric, $m_{jk}(g,\delta) = m_{kj}(g,\delta)$.

Proof of Lemma

$$
\begin{aligned}
& m_{ii}(g,\delta)\left[m_{jk}(g,\delta) - m_{jk}(g_{-i},\delta)\right] \\
={}& \sum_{p=1}^{\infty} \delta^p \sum_{\substack{r+s=p \\ r\geq 0, s\geq 1}} g_{ii}^{r}\left[g_{jk}^{s} - g_{j(i^c)k}^{s}\right] = \sum_{p=1}^{\infty} \delta^p \sum_{\substack{r+s=p \\ r\geq 0, s\geq 2}} g_{ii}^{r} g_{j(i)k}^{s} \\
={}& \sum_{p=1}^{\infty} \delta^p \sum_{\substack{r'+s'=p \\ r'\geq 1, s'\geq 1}} g_{ji}^{r'} g_{ik}^{s'} = m_{ji}(g,\delta) m_{ik}(g,\delta).
\end{aligned}
$$

Intercentrality (Cont.)

$$
\begin{aligned}
c_i(g,\delta) &= \frac{b_i(g,\delta,\boldsymbol{\alpha})b_i(g,\delta,\mathbf{1}_n)}{m_{ii}(g,\delta)} \\
&= b_i(g,\delta,\boldsymbol{\alpha}) + b_i(g,\delta,\boldsymbol{\alpha}) \sum_{j\neq i} \frac{m_{ij}(g,\delta)}{m_{ii}(g,\delta)} \\
&= b_i(g,\delta,\boldsymbol{\alpha}) + \left[\sum_{k=1}^{n} m_{ik}(g,\delta)\alpha_k\right]\left[\sum_{j\neq i} \frac{m_{ij}(g,\delta)}{m_{ii}(g,\delta)}\right] \\
&= b_i(g,\delta,\boldsymbol{\alpha}) + \sum_{j\neq i}\sum_{k=1}^{n} \frac{m_{ij}(g,\delta)m_{ik}(g,\delta)}{m_{ii}(g,\delta)}\alpha_k
\end{aligned}
$$

Intercentrality (Cont.)

$$
\begin{aligned}
c_i(g, \delta) &= b_i(g, \delta, \boldsymbol{\alpha}) + \sum_{j \neq i} \sum_{k=1}^{n} \frac{m_{ij}(g, \delta) m_{ik}(g, \delta)}{m_{ii}(g, \delta)} \alpha_k \\
&= b_i(g, \delta, \boldsymbol{\alpha}) + \sum_{j \neq i} \sum_{k=1}^{n} \left[m_{jk}(g, \delta) - m_{jk}(g_{-i}) \right] \alpha_k \\
&= b_i(g, \delta, \boldsymbol{\alpha}) + \sum_{j \neq i} \left[b_j(g, \delta, \boldsymbol{\alpha}) - b_j(g_{-i}, \delta, \boldsymbol{\alpha}_{-i}) \right]
\end{aligned}
$$

Player i's intercentrality is the sum of

- i's Katz-Bonacich centrality and
- i's contribution to the Katz-Bonacich centrality of every other player $j \neq i$.

Remark

- Clearly, $\delta\lambda_1(\boldsymbol{G}_{-i}) \leq \delta\lambda_1(\boldsymbol{G}) < 1$:

$$\begin{aligned}\lambda_1(\boldsymbol{G}_{-i}) &= \max_{|\boldsymbol{x}|=1} \boldsymbol{x}^\intercal \cdot \boldsymbol{G}_{-i} \cdot \boldsymbol{x} \\ &= \max_{|\boldsymbol{x}_{-i}|=1} \boldsymbol{x}_{-i}^\intercal \cdot \boldsymbol{G}_{-i} \cdot \boldsymbol{x}_{-i} \\ &\leq \max_{|\boldsymbol{x}|=1} \boldsymbol{x}^\intercal \cdot \boldsymbol{G} \cdot \boldsymbol{x} = \lambda_1(\boldsymbol{G}).\end{aligned}$$

* It can be also proved by Cauchy interlacing theorem.

- Then $\boldsymbol{b}(g_{-i}, \delta, \boldsymbol{\alpha}_{-i}) = [\boldsymbol{I}_n - \delta\boldsymbol{G}_{-i}]^{-1}\boldsymbol{\alpha}_{-i}$ is well-defined.

$\Rightarrow$ $b_j(g_{-i}, \delta, \boldsymbol{\alpha}_{-i})$ is well-defined.

Result

Theorem

A player i^* solves the key-player problem iff i^* is an agent with the highest intercentrality in g.

If the police/government has the ability to remove one criminal, it should remove the one with highest intercentrality.

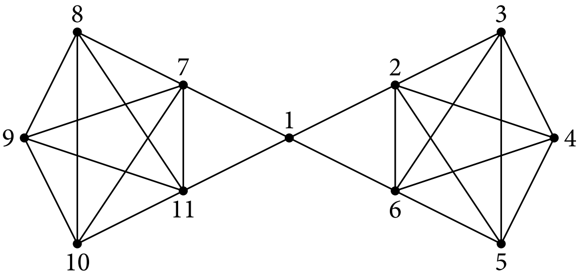
Proof.

$$c_i(g,\delta) = b_i(g,\delta,\boldsymbol{\alpha}) + \sum_{j\neq i}\left[b_j(g,\delta,\boldsymbol{\alpha}) - b_j(g_{-i},\delta,\boldsymbol{\alpha}_{-i})\right]$$

$$= \sum_{k=1}^{n} b_k(g,\delta,\boldsymbol{\alpha}) - \sum_{k\neq i} b_k(g_{-i},\delta,\boldsymbol{\alpha}_{-i}).$$

$\square$

Example



Player	$\delta = 0.1$ b_i	$\delta = 0.1$ c_i	$\delta = 0.2$ b_i	$\delta = 0.2$ c_i
1	1.75	2.92	8.33	41.67
2	1.88	3.28	9.17	40.33
3	1.72	2.79	7.78	32.67

Example (Cont.)

- Player 2 always displays the highest Katz-Bonacich centrality.
  - It has the highest number of direct connections.
  - Besides, it is directly connected to the bridge delinquent 1, which gives them access to a very wide and diversified span of indirect connections.
- For low values of δ, the direct effect on delinquency reduction prevails, and player 2 are the key player—with highest intercentrality.
- When δ is higher, though, the most active delinquents are no longer the key players. Now, indirect effects matter a lot, and eliminating delinquent 1 has the highest joint direct and indirect effect on aggregate delinquency reduction.

Remark

- In key-player policy, the planner perturbs the network by removing a delinquent and all other delinquents are allowed to change their effort after the removal, but the network is not "rewired," that is, individuals do not optimally change their relationships (links) with their friends.
- First, it would be extremely difficult to solve a network formation problem every time a player is removed.
- Second, in the context of a short-term policy and because friendship relationships take longer to adjust than the level of criminal activity.

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