Social and Economic Networks

Games with Strategic Substitutes

Xiang Sun

2019 Fall

Outline

1. A local public goods model
2. Equilibrium analysis
    - Specialized equilibrium and distributed equilibrium
    - Existence of equilibrium
3. Stable equilibrium
4. Extension
    - Imperfect substitutability
    - Convex costs
    - Heterogeneous agents

Reference

- Jackson, Chapter 9
- Yann Bramoullé and Rachel Kranton, Public goods in networks.
- Yann Bramoullé, Rachel Kranton, and Martin D'Amours, Strategic interaction and networks.

Strategic substitutes

- In games of strategic substitutes: an increase in other players' actions leads to relatively lower payoffs to higher actions of a given player.
- Example: Public good provision.
- How does the social or geographic structure affect the level and pattern of public good provision?
- Do people exert effort themselves or rely on others?

Effort

- The set of agents is $N = \{1, 2, \dots, n\}$.
- Let $x_i \in [0, +\infty)$ denote agent i's level of effort.
- The individual marginal cost of effort is constant and equal to c.
- Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ denote an effort profile of all agents.

Network

Agents are arranged in a network g.

- The adjacency matrix is $\boldsymbol{G}$, where the (i, j)-entry g_{ij} is:

$$g_{ij} = \begin{cases} 1, & \text{if agent } j \text{ is linked to agent } i, \\ 0, & \text{otherwise.} \end{cases}$$

- We assume $g_{ij} = g_{ji}$ and $g_{ii} = 1$.
- Once i and j are linked, each of them can get benefits directly from the other's effort.
- Let N_i denote the set of i's neighbors:

$$N_i = \{j \neq i \mid g_{ij} = 1\}.$$

Benefit

- Two assumptions:
  - An agent's effort is a substitute of the efforts of her (direct) neighbors, but not of individuals further away in the network.
  - A neighbor's effort is a perfect substitute with one's own.
- Each agent receives benefits from own and neighbors' effort according to a (twice differentiable) strictly concave benefit function $b(\cdot)$, where $b(0) = 0$, $b' > 0$, $b'' < 0$.
- With the assumptions above, an individual i has benefits

$$b\Big(x_i + \sum_{j \in N_i} x_j\Big).$$

Payoff

- An agent i's payoff from profile $\boldsymbol{x}$ in network g is

$$u_i(\boldsymbol{x}, g) = b\Big(x_i + \sum_{j \in N_i} x_j\Big) - cx_i.$$

- We have to assume

$$b'(+\infty) < c < b'(0).$$

Game

- Given a network structure g, agents simultaneously choose effort levels.
- For an effort profile $\boldsymbol{x}$, each agent i earns payoffs $u_i(\boldsymbol{x}, g)$.

Best reply

- First order condition:

$$b'\Big(x_i + \sum_{j \in N_i} x_j\Big) - c \leq 0$$

  with equality for interior points.

- Let x^* denote the effort level at which, to an individual agent, the marginal benefit equals its marginal cost: $b'(x^*) = c$.

* The existence of such an x^* is guaranteed by the assumption $c < b'(0)$.

- Given $\boldsymbol{x}$, i's best reply is

$$f_i(\boldsymbol{x}) = \max\Big\{x^* - \sum_{j \in N_i} x_j, 0\Big\}.$$

Nash equilibrium

- A profile $\boldsymbol{x}$ is a Nash equilibrium if and only if for every agent i
  - either $\sum_{j \in N_i} x_j \geq x^*$ and $x_i = 0$,
  - or $\sum_{j \in N_i} x_j \leq x^*$ and $x_i = x^* - \sum_{j \in N_i} x_j$.
- Agents want to exert effort as long as their total benefits are less than $b(x^*)$:
  - if the benefits they acquire from their neighbors are more than $b(x^*)$, they exert no effort.
  - if the benefits are less than $b(x^*)$, they exert effort up to the point where their benefits equal $b(x^*)$.

Specialized equilibrium and distributed equilibrium

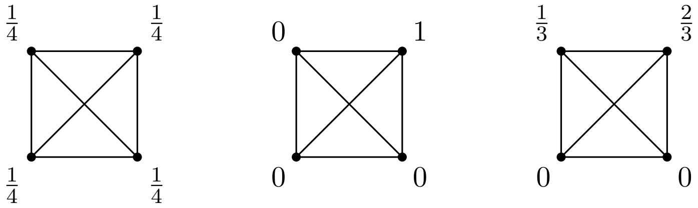
- We say a profile $\boldsymbol{x}$ is specialized when every agent either exerts the maximum amount of effort x^* or exerts no effort;

  * for all agents i, either $x_i = 0$ or $x_i = x^*$.
- We call an agent who exerts x^* a specialist.
- We say a profile $\boldsymbol{x}$ is distributed when every agent exerts some effort;

  * for all agents i, $0 < x_i < x^*$.
- Hybrid equilibria fall between these two extremes.

Example: Complete network

We set $x^* = 1$ for ease of exposition.



In the complete network, in any equilibrium, aggregate effort is x^*, and it can be split in any way among the agents.

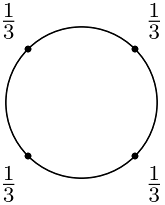
- effort could be equally distributed, so that each agent exerts $\frac{1}{4}x^*$,
- or one agent could be a specialist,
- or hybrid.

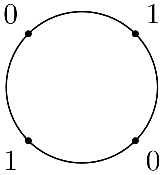
Example: Star

On the star, only specialized profiles are equilibria:

- either the center is a specialist,
- or the three agents at the periphery are specialists.

Example: Cycle

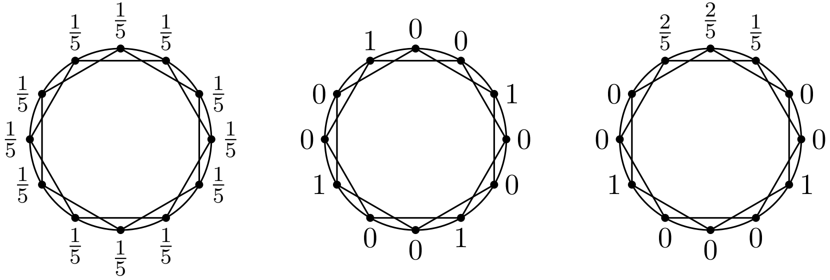




On the circle,

- effort can be distributed among the agents,
- or concentrated among specialists.

Example



- Effort could be equally distributed, so that each agent make some contribution.
- Agents can also be specialists, and other agents completely free ride.
- In hybrid equilibria, some agents specialize, some agents make smaller contributions, and other agents free ride.

Independent set

- An independent set (独立集) I of a network g is a set of agents such that no two agents who belong to I are linked; i.e., for every distinct i and j in I such that $g_{ij} = 0$.

* Each edge in the network has at most one endpoint in I.

- An independent set is maximal when it is not a proper subset of any other independent set.
- How to find a maximal independent set?

Intuition

- Given a maximal independent set I, every agent either belongs to I or is connected to an agent who belongs to I.
- We can partition the population into two disjoint sets of agents:
  - those who belong to maximal independent set I,
  - and those who are linked to an agent in I.
- Maximal independent sets are a natural notion in our context: in equilibrium no two specialists can be linked.

Result

Theorem

1. A specialized profile is a Nash equilibrium if and only if its set of specialists is a maximal independent set of the structure g.
2. Since for every g there exists a maximal independent set, there always exists a specialized Nash equilibrium.

The proof of "if" is easy.

Proof of “only if”

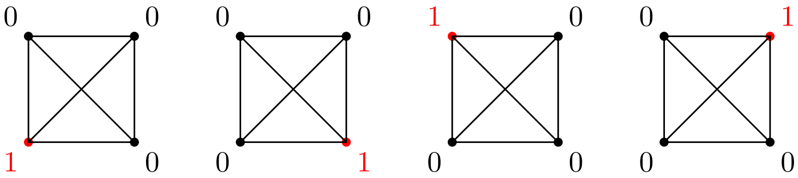
1. Consider a specialized equilibrium where I is the set of specialists.
2. For specialists, x^* is a best response if all their neighbors exert zero effort.

$\Rightarrow$ This means that I is an independent set of the graph.

3. For a non-specialist i, exerting zero effort is a best response iff $\sum_{j \in N_i} x_j \geq x^*$ iff $|N_i \cap I| \geq 1$.

$\Rightarrow$ This means that all agents not in I are connected to at least one agent in $I \Rightarrow I$ is maximal.

Example: Complete network



- In a complete graph, an independent set can include at most one agent.
- For $n = 4$, there are four specialized equilibria, corresponding to each agent.

Example: Star

- On the star, there are two maximal independent sets: the agent at the center, and the three agents in the periphery.
- These two sets correspond to the two specialized equilibria (and only equilibria) for the star.

Example: Cycle

- In the circle, there are two maximal independent sets, each containing two agents on opposite sides of the circle.
- Again, these two sets correspond to the specialized equilibria for the circle.

1. A local public goods model
2. Equilibrium analysis
   - Specialized equilibrium and distributed equilibrium
   - Existence of equilibrium
3. Stable equilibrium
4. Extension
   - Imperfect substitutability
   - Convex costs
   - Heterogeneous agents

Stable equilibrium

- For any network, there are potentially many Nash equilibria—ranging from specialized to distributed.
- Stable equilibrium: robust for some small perturbations.

Nash tâtonnement

1. Given a profile $\boldsymbol{x}$, i's best response is denoted by

$$f_i(\boldsymbol{x}) = \max\Big\{x^* - \sum_{j \in N_i} x_j, 0\Big\}.$$

   * Define $\boldsymbol{f}$ as the collection of these individual best response:

$$\boldsymbol{f} = (f_1, f_2, \ldots, f_n).$$

2. If agents have a small admissible perturbation $\boldsymbol{\epsilon}$ on $\boldsymbol{x}$, we can consider agents' reactions:

$$f_i(\boldsymbol{x} + \boldsymbol{\epsilon}) \text{ or } \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{\epsilon}).$$

3. The successive reactions:

$$\boldsymbol{f}^n(\boldsymbol{x} + \boldsymbol{\epsilon}).$$

4. Question: Is $\lim_{n \to \infty} \boldsymbol{f}^n(\boldsymbol{x} + \boldsymbol{\epsilon}) = \boldsymbol{x}$ when $\boldsymbol{x}$ is an equilibrium?

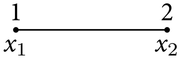
Stable equilibrium

Definition (Stable equilibrium)

An equilibrium $\boldsymbol{x}$ is stable if and only if there exists a positive number $\rho > 0$ such that for any vector $\boldsymbol{\epsilon}$ satisfying $|\epsilon_i| \leq \rho$ and $x_i + \epsilon_i \geq 0$ for each i, the sequence $\boldsymbol{x}^{[m]}$ converges to $\boldsymbol{x}$, where

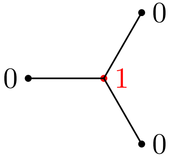
$$\boldsymbol{x}^{[0]} = \boldsymbol{x} + \boldsymbol{\epsilon},\ \boldsymbol{x}^{[1]} = \boldsymbol{f}(\boldsymbol{x}^{[0]}),\ \text{and } \boldsymbol{x}^{[m]} = \boldsymbol{f}(\boldsymbol{x}^{[m-1]}) = \boldsymbol{f}^m(\boldsymbol{x}^{[0]}).$$

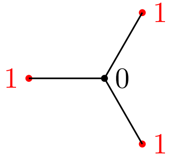
Example



- Consider an equilibrium (x_1, x_2).
- Clearly, $x_1 + x_2 = x^*$.
- Suppose that $x_2 > 0$.
- Consider a perturbation $(x_1 + \epsilon, x_2 - \epsilon)$.
- Then the best responses do not converge back to the original point:
$$\boldsymbol{x}^{[m]} = (x_1 + \epsilon, x_2 - \epsilon) \text{ for each } m \geq 1.$$
- No equilibrium is stable.

Example: Star





- The left one is not stable.
- The right one is stable.

Distributed equilibrium is not stable

1. Consider an equilibrium where everyone exerts some effort.
2. Decrease the effort of an individual i by a small amount.
3. His neighbor(s) will adjust by increasing their own efforts.
4. This increase can lead i to reduce his effort even more.
5. In this case, the initial equilibrium is not stable.

Stable specialized equilibrium

- This process does not work in specialized equilibria when every agent j who exerts no effort is linked to two specialists.
- If we reduce the effort of these specialists, agent j will not adjust.
- He has access to two sources of information, and a small reduction will not lead him to increase his own effort.

Maximal independent set of higher order

- Given a network g, define a maximal independent set of order r as a maximal independent set I such that any individual not in I is connected to at least r individuals in I.
- The case $r = 1$ simply corresponds to maximal independent sets.
- While every graph contains maximal independent sets, not all graphs contain maximal independent sets of higher order.
  * For example, in the complete graph, there is no maximal independent set of order $r = 2$.
- Clearly, stable profiles correspond to maximal independent sets of order 2.

Result

Theorem (Stable equilibrium)

1. For any social structure g, an equilibrium is stable if and only if it is specialized and every non-specialist is connected to (at least) two specialists.
2. Hence, there exists a stable equilibrium in a graph g if and only if it has a maximal independent set of order 2.

Lemma

Lemma

If $\boldsymbol{x} \leq \boldsymbol{x}'$, then $f \circ f(\boldsymbol{x}) \leq f \circ f(\boldsymbol{x}')$.

Proof.

1. Suppose that $x_i \leq x_i'$ for all i.
2. Then $x^* - \sum_{j \in N_i} x_j \geq x^* - \sum_{j \in N_i} x_j'$.
3. Hence, $\max\left\{x^* - \sum_{j \in N_i} x_j, 0\right\} \geq \max\left\{x^* - \sum_{j \in N_i} x_j', 0\right\}$.
4. That is, $f(\boldsymbol{x}) \geq f(\boldsymbol{x}')$.
5. Applying f again to this inequality yields the result.

□

Step 1

Goal: A non-specialized equilibrium is not stable.

1. Let $J = \{j \mid 0 < x_j < x^*\}$.
2. Let $\rho > 0$ be a small number and define a perturbation $\boldsymbol{\epsilon}$ as follows:
$$\epsilon_j = \begin{cases} \rho, & \text{if } j \in J, \\ 0, & \text{if } j \notin J. \end{cases}$$
3. We wish to show that $\boldsymbol{f} \circ \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{\epsilon}) \geq \boldsymbol{x} + \boldsymbol{\epsilon}$.
4. Case 1: i is such that $x_i = 0$.
    - Then $i \notin J$ and $\epsilon_i = 0$, and hence $x_i + \epsilon_i = 0$.
    - Clearly, $f_i(\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{\epsilon})) \geq 0 = x_i + \epsilon_i$.

Step 1 (Cont.)

5. Case 2: i is such that $x_i = x^*$.
   - Then $i \notin J$, $\epsilon_i = 0$, and $x_i + \epsilon_i = x^*$.
   - Thus, for each $j \in N_i$, $f_j(\boldsymbol{x} + \boldsymbol{\epsilon}) = 0$.
   - Therefore, given $f_j(\boldsymbol{x} + \boldsymbol{\epsilon}) = 0$, i's best response is

$$f_i(\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{\epsilon})) = x^* = x_i + \epsilon_i.$$

Step 1 (Cont.)

6. Case 3: Suppose that $i \in J$.
   - For each $j \in N_i$, $x_j < x^*$; otherwise, i will deviate to 0. Also, for each $j \in N_i$ and $\ell \in N_j$, $x_\ell < x^*$.
   - If ρ is small enough, we have $\sum_{j \in N_k} \epsilon_j \leq |N_k| \cdot \rho \leq x_k$ for each $k \in J$.
   - Thus, $f_i(\boldsymbol{x} + \boldsymbol{\epsilon}) = x_i - \sum_{j \in N_i} \epsilon_j$.
   - For each $j \in N_i$ with $x_j = 0$, we have $\epsilon_j = 0$ and $f_j(\boldsymbol{x} + \boldsymbol{\epsilon}) = 0$.
   - For each $j \in N_i \cap J$, we have $\epsilon_j = \rho$ and $f_j(\boldsymbol{x} + \boldsymbol{\epsilon}) = x_j - \sum_{\ell \in N_j} \epsilon_\ell$.
   - Therefore, $f_i(f(\boldsymbol{x} + \boldsymbol{\epsilon})) = x_i + \sum_{j \in N_i \cap J} \sum_{\ell \in N_j} \epsilon_\ell$.
   - Since i has at least one neighbor in J, $\sum_{j \in N_i \cap J} \sum_{\ell \in N_j} \epsilon_\ell \geq \rho$, and hence

     $f_i(f(\boldsymbol{x} + \boldsymbol{\epsilon})) \geq x_i + \epsilon_i$.

Step 1 (Cont.)

7. Therefore, $f \circ f(x + \epsilon) \geq x + \epsilon$.
8. By applying the lemma we can see that for any finite number m, $f^{2m}(x + \epsilon) \geq x + \epsilon$, which is strictly greater than x.
9. Therefore, the sequence of best-responses never converges back to x and the equilibrium is not stable.

Step 2

Goal: A specialized equilibrium such that a non-specialist is connected to a unique specialist is not stable.

1. Consider a specialized equilibrium $\boldsymbol{x}$ such that i is a non-specialist who is connected to a unique specialist j.
2. Let $\rho > 0$ be a small number and define a perturbation ϵ as follows:
$$\epsilon_\ell = \begin{cases} \rho, & \text{if } \ell = i, \\ 0, & \text{if } \ell \neq i. \end{cases}$$

Step 2 (Cont.)

3. Then, clearly, $x_j^{[1]} = x^* - \rho$ and $x_\ell^{[1]} = x_\ell$ for each $\ell \neq j$.
4. Next, $x_\ell^{[2]} = x_\ell$ for any ℓ except for neighbors of j whose only specialist neighbor is j.

   These agents, which include i, all play $x_\ell^{[2]} = \rho$.
5. This means that $\boldsymbol{f} \circ \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{\epsilon}) \geq \boldsymbol{x} + \boldsymbol{\epsilon}$ and we can apply the same argument as above, hence this equilibrium is not stable.

Step 3

Goal: Specialized equilibria in which every non-specialist is connected to (at least) two specialists are stable.

1. Take $\boldsymbol{x}$ such an equilibrium, let I be the set of specialists in $\boldsymbol{x}$.
2. Let $\delta = \frac{1}{n^2}x^*$.
3. Consider a perturbation $\boldsymbol{\epsilon}$ such that for each $i \in I$, $|\epsilon_i| < \rho$ and $x_i + \epsilon_i \geq 0$.
4. For each $i \notin I$,
   - $\sum_{j \in N_i} x_j^{[0]} = \sum_{j \in N_i} (x_j + \epsilon_j) = |N_i \cap I| x^* + \sum_{j \in N_i} \epsilon_j$.
   - Since $|\sum_{j \in N_i} \epsilon_j| < n\delta \leq x^*$ and $|N_i \cap I| \geq 2$, we have $\sum_{j \in N_i} x_j^{[0]} \geq x^*$.
   - Hence, $x_i^{[1]} = 0$.
   - * Despite the perturbation, non-specialists still do no effort.

Step 3 (Cont.)

5. For each $i \in I$,
   - $\sum_{j \in N_i} x_j^{[0]} = \sum_{j \in N_i} (x_j + \epsilon_j) = \sum_{j \in N_i} \epsilon_j$.
   - Hence, $x_i^{[1]} = x^* - \sum_{j \in N_i} \epsilon_j$.
6. For each $i \notin I$,
   - $\sum_{j \in N_i} x_j^{[1]} = |N_i \cap I| x^* - \sum_{j \in N_i \cap I} \sum_{\ell \in N_j} \epsilon_\ell$.
   - Since $|\sum_{j \in N_i \cap I} \sum_{\ell \in N_j} \epsilon_\ell| \leq n^2 \delta \leq x^*$ and $|N_i \cap I| \geq 2$, we have again $x_i^{[2]} = 0$.
7. For each $i \in I$, since $x_j^{[1]} = 0$ for each $j \in N_i$, we have $x_i^{[2]} = x^*$.
8. We showed that $\boldsymbol{f} \circ \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{\epsilon}) = \boldsymbol{x}$, hence $\boldsymbol{f}^m(\boldsymbol{x} + \boldsymbol{\epsilon}) = \boldsymbol{x}$ for all $m \geq 2$, hence the equilibrium is stable.

1. A local public goods model
2. Equilibrium analysis
   - Specialized equilibrium and distributed equilibrium
   - Existence of equilibrium
3. Stable equilibrium
4. Extension
   - Imperfect substitutability
   - Convex costs
   - Heterogeneous agents

Imperfect substitutability

- An individual's own effort could be more beneficial to himself than to his neighbors.
- Let benefits equal

$$b\left(x_i + \delta \sum_{j \in N_i} x_j\right),$$

where $\delta \in (0, 1]$ measures the extent to which own efforts and neighbor's efforts are substitutes.

Equilibrium

- On any graph if δ is low enough, there exists a unique Nash equilibrium and in this equilibrium all agents do strictly positive effort.
  - * there is a unique equilibrium and it is a distributed equilibrium.
- Specialization disappears when δ is low enough.
  - * when efforts by one's neighbors have little value, individuals must exert some effort on their own.
- When agents have sufficiently many neighbors in a network and the network admits maximal independent sets of sufficiently high order, there exist specialized equilibria.

Result

Theorem

Suppose an agent's benefits are $b\Big(x_i + \delta \sum_{j \in N_i} x_j\Big)$ where $\delta \in (0, 1]$. Let s be the smallest integer larger than or equal to $\frac{1}{\delta}$. Then, a specialized profile is an equilibrium if and only if the set of specialists is a maximal independent set of order s of the network g.

Example

- On the circle when n is even, there are maximal independent sets of order 2. Hence, there exist specialized equilibria for $\delta \geq \frac{1}{2}$. The equilibria involve alternate agents exerting x^*.
- On the star, there is a maximal independent set of order $n-1$. The profile where all agents at the periphery are specialists and the center exerts no effort is an equilibrium for $\delta \geq \frac{1}{n-1}$.

Convex costs

- Suppose that effort costs $c(\cdot)$ are increasing and convex and that $c'(0) > b'(+\infty)$.
- Convex costs drive the outcome towards effort sharing.
- In complete networks, there is now a unique equilibrium where all individuals exert the same amount of effort.
- Specialization can still emerge in graphs that are not complete.

Result

Let effort level x^* still represent how much an isolated individual experiments. It now satisfies $b'(x^*) = c'(x^*)$.

Theorem

Suppose that $c(\cdot)$ is increasing and convex and that $c'(0) > b'(+\infty)$. Let s be the smallest integer such that $b'(sx^*) \leq c'(0)$. Then, a specialized profile is a Nash equilibrium if and only if the set of specialists is a maximal independent set of order s of the network g.

1. A local public goods model
2. Equilibrium analysis
   - Specialized equilibrium and distributed equilibrium
   - Existence of equilibrium
3. Stable equilibrium
4. Extension
   - Imperfect substitutability
   - Convex costs
   - Heterogeneous agents

Heterogeneous agents

- Suppose that agents can derive different benefits from the public good and have different costs of effort.
- Let the benefit functions and cost functions be $b\left(x_i + \sum_{j \in N_i} x_j\right)$ and $c_i x_i$.
- Equilibrium outcomes can be represented by an idiosyncratic threshold effort level x_i^* such that in equilibrium,

$$x_i = \begin{cases} 0, & \text{if } \sum_{j \in N_i} x_j \geq x_i^*, \\ x_i^* - \sum_{j \in N_i} x_j, & \text{otherwise.} \end{cases}$$

- Individuals with higher benefits or lower costs have higher thresholds.

Result

- On the complete network, there is a (generically) unique equilibrium where the individual with highest threshold exerts all the effort.
- Heterogeneity thus leads to specialization.

Theorem

All specialized Nash equilibria correspond to a maximal independent set of the network. There always exists a specialized equilibrium where the agent with highest threshold is a specialist.