Game Theory

Two-sided matching

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Marriage problem

A marriage problem (婚姻问题) is a triple $\Gamma = \langle M, W, \succ \rangle$, where

- M is a finite set of men,
- W is a finite set of women,
- $\succ = (\succ_i)_{i \in M \cup W}$ is a list of preferences. Here
  - $\succ_m$ denotes the strict preference of man m over $W \cup \{m\}$,
  - $\succ_w$ denotes the strict preference of woman w over $M \cup \{w\}$,

Matching

A matching (匹配) is a outcome, and is defined by a function $\mu\colon M \cup W \to M \cup W$ such that

- for all $m \in M$, if $\mu(m) \neq m$ then $\mu(m) \in W$,
- for all $w \in W$, if $\mu(w) \neq w$ then $\mu(w) \in M$,
- for all $m \in M$ and $w \in W$, $\mu(m) = w$ if and only if $\mu(w) = m$ (*i.e.*, a matching is mutual: you are matched with me if and only if I am matched with you).

We refer to $\mu(i)$ as the mate of i, and $\mu(i) = i$ means that agent i remains single under the matching μ.

Stable matching

- Suppose men and women are to be matched by a centralized mechanism, what properties should the matching satisfy?
- At least they should not have incentives to divorce.

Stable matching

A matching μ is stable (稳定) if it is

- individually rational: for each $i \in M \cup W$, $\mu(i) \succ_i i$.
- unblocked: there does not exist any pair (m, w) such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.

Example

There are three men and three women, with the following preferences:

m_1	m_2	m_3	w_1	w_2	w_3
w_2	w_1	w_1	m_1	m_3	m_1
w_1	w_3	w_2	m_3	m_1	m_3
w_3	w_2	w_3	m_2	m_2	m_2

All possible matchings are individually rational, since all pairs (m, w) are mutually acceptable.
The matching μ given below is unstable, since (m_1, w_2) is a blocking pair.

$$\mu = \begin{bmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{bmatrix} \qquad \mu' = \begin{bmatrix} w_1 & w_2 & w_3 \\ m_1 & m_3 & m_2 \end{bmatrix}.$$

The matching μ' is stable.

Deferred acceptance algorithm

Do stable matchings exist? If yes, how to find them?

For any marriage problem, the man-proposing DA (Gale-Shapley, 1962) operates as follows:

Step 1:

1. Each man m proposes to his first choice (if he has any acceptable choices).
2. Each woman rejects any offer except the best acceptable proposal and "holds" the most-preferred acceptable proposal (if any). Note that she does not accept him yet, but keeps him on a string to allow for the possibility that someone better may come along later.

Deferred acceptance algorithm (Cont.)

Step k:

1. Any man who was rejected at Step $(k-1)$ makes a new proposal to his most-preferred acceptable potential mate who has not yet rejected him (If no acceptable choices remain, he makes no proposal).
2. Each woman receiving proposals chooses her most-preferred acceptable proposal from the group consisting of the new proposers and the man on her string, if any. She rejects all the rest and again keeps the best-preferred in suspense.

End: The algorithm terminates when there are no more rejections. Each woman is matched with the man she has been holding in the last step. Any woman who has not been holding an offer or any man who was rejected by all acceptable women remains single.

Example

There are three men and two women, with the following preferences:

i	j	k	a	b
b	a	a	i	k
a		b	j	i
			k	

The procedure of DA is

Step	1	2	3	End
a	$j, \cancel{k}$	j	$\cancel{j}, i$	i
b	i	$\cancel{i}, k$	k	k
$\emptyset$	k	i	j	j

and the resulting matching is

$$\mu = \begin{bmatrix} i & j & k \\ a & \emptyset & b \end{bmatrix}.$$

Theorem on stability

Observation: As the algorithm proceeds, the tentative partners of a man is weakening, and the tentative partners of a woman is improving.

Theorem on stability

The men-proposing deferred acceptance algorithm gives a stable matching for each marriage problem.

Proof.

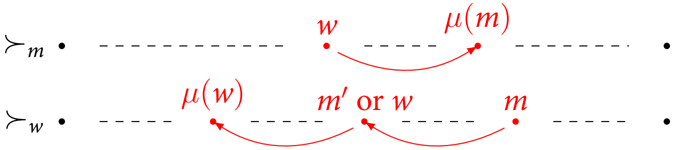
1. It suffices to show that the matching μ determined by the men-proposing deferred acceptance algorithm is not blocked by any pair (m, w).
2. Suppose that there is a pair (m, w), such that $m \neq \mu(w)$ and $w \succ_m \mu(m)$.

□

Theorem on stability (Cont.)

Proof.

3. Then m must have proposed to w at some step and subsequently been rejected in favor of someone (m' in the figure) that w likes better.



4. It is now clear that w must prefer her mate $\mu(w)$ to m and there is no instability.
5. Similar discussion applies to the pair (m, w) with $m \neq \mu(w)$ and $m \succ_w \mu(w)$.

$\square$

Theorem on optimality

Theorem on optimality

The matching determined by men-proposing DA algorithm is at least as good as any other stable matching for all men.

Proof.

Let us call a woman "achievable" for a particular man if there is a stable matching that sends him to her.

1. For contradiction, suppose that a man is rejected by an achievable woman in the DA procedure.
2. Consider the first step (say Step k) in which a man (call him m) is rejected by an achievable woman (call her w).

□

Theorem on optimality (Cont.)

Proof.

3. Then w keeps some other man m' at this step, so $m' \succ_w m$.
4. Let μ be a stable matching where $\mu(m) = w$.
5. Since this is the first step of DA where a man is rejected by an achievable woman, $w \succ_{m'} \mu(m')$. Otherwise,
    - Case 1: $\mu(m') \succ_{m'} w$, then m' is rejected by an achievable woman $\mu(m')$ before Step k.
    - Case 2: $\mu(m') = w = \mu(m)$, which leads to $m = m'$. Contradiction.
6. Thus, (m', w) blocks μ, contradicting the stability of μ.

$\square$

Theorem on optimality (Cont.)

- Intuitively, men may have different (individually) optimal matchings, since they have different preferences.
- However, restricting to the set of stable matchings, the stable matching resulting from men-proposing DA algorithm is optimal for every man.
- We refer to the outcome of the men-proposing deferred acceptance algorithm as the man-optimal stable matching.

Two-sidedness: Opposite interests

Likewise, we can define the woman-proposing DA, which produces the woman-optimal stable matching.

m_1	m_2	w_1	w_2
w_2	w_1	m_1	m_2
w_1	w_2	m_2	m_1

$$\mu_1 = \begin{bmatrix} m_1 & m_2 \\ w_2 & w_1 \end{bmatrix} \text{ and } \mu_2 = \begin{bmatrix} m_1 & m_2 \\ w_1 & w_2 \end{bmatrix}.$$

- μ_1 is man-optimally (woman-worst) stable.
- μ_2 is woman-optimally (man-worst) stable.

The man-optimal stable matching is not necessarily Pareto efficient for men (see previous example).

Rural hospital theorem

Rural hospital theorem (McVitie-Wilson, 1970)

The set of matched men (women) is the same across all stable matchings.

Proof.

1. Let $\bar{\mu}$ be the man-optimal stable matching and μ be any stable matching.
2. Then men prefer $\bar{\mu}$ to μ while women prefer μ to $\bar{\mu}$.

□

Rural hospital theorem (Cont.)

Proof.

3. Therefore, weakly more men is matched at $\bar{\mu}$ and weakly more women is matched at μ, i.e.,

$$\mu(M) \subseteq \bar{\mu}(M) \text{ and } \bar{\mu}(W) \subseteq \mu(W).$$

4. We also know that at each matching, the numbers of matched man and women are the same, i.e.,

$$\bar{\mu}(M) \xlongequal{\text{card}} \bar{\mu}(W) \text{ and } \mu(M) \xlongequal{\text{card}} \mu(W).$$

5. Hence, $|\bar{\mu}(M)| = |\mu(M)|$ and consequently, $\bar{\mu}(M) = \mu(M)$, where $\mu(M)$ is the set of matched men at μ.

□

Structure of the set of stable matchings

Let μ and μ' be two matchings. Define their joint $\mu \vee \mu'$ and meet $\mu \wedge \mu'$ by letting

- $\mu \vee \mu'(m) = \max_{\succ_m}\{\mu(m), \mu'(m)\}$ for all m,
  $\mu \vee \mu'(w) = \min_{\succ_w}\{\mu(w), \mu'(w)\}$ for all w,
- $\mu \wedge \mu'(m) = \min_{\succ_m}\{\mu(m), \mu'(m)\}$ for all m,
  $\mu \wedge \mu'(w) = \max_{\succ_w}\{\mu(w), \mu'(w)\}$ for all w.

Lattice Theorem (Conway)

If both μ and μ' are stable, then both $\mu \vee \mu'$ and $\mu \wedge \mu'$ are matchings and both are stable.

That is, the set of stable matchings is a lattice.

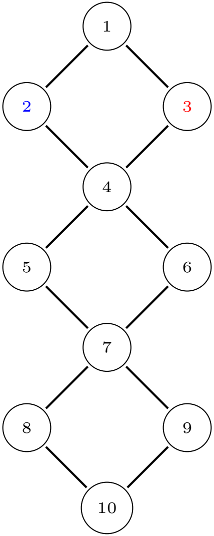
Example: Lattice structure

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
1	2	3	4	4	3	2	1
2	1	4	3	3	4	1	2
3	4	1	2	2	1	4	3
4	3	2	1	1	2	3	4

There are 10 stable matchings

No.	1	2	3	4	5	6	7	8	9	10
m_1	1	2	1	2	2	3	3	3	4	4
m_2	2	1	2	1	4	1	4	4	3	3
m_3	3	3	4	4	1	4	1	2	1	2
m_4	4	4	3	3	3	2	2	1	2	1

Example: Lattice structure (Cont.)



Incentives to misreport

A (centralized) mechanism φ maps each profile of agent's reported preferences $\succ = \big((\succ_m)_{m\in M}, (\succ_w)_{w\in W}\big)$ to a matching $\varphi(\succ)$.

Definition

φ is strategy-proof (抗策略操作) if it is a weakly dominant strategy for every agent to report his/her preference truthfully, i.e., for each $i \in M \cup W$, for each $\succ_{-i}$, and for each $\succ_i'$,

$$\varphi(\succ_i, \succ_{-i})(i) \succsim_i \varphi(\succ_i', \succ_{-i})(i).$$

Impossibility Theorem

Impossibility Theorem (Roth, 1982)

There is no stable mechanism that is strategy-proof.

Proof.

1. Revisit the example on the slide Two-sidedness.
2. If φ is a stable mechanism, then it has to select either μ_1 or μ_2 for the given preference profile.
3. If it selects μ_1 (the argument is similar if it selects μ_2), then w_1 will have incentive to misreport $\succ'_{w_1} : m_1, w_1$.
4. At $(\succ'_{w_1}, \succ_{-w_1})$, the only stable matching is μ_2, hence φ selects μ_2 and w_1 becomes better off.

$\square$

Remark

- The impossibility result is mainly due to the two-sidedness: when one side is happy, the other side is not and can gain from misreporting.
- As a direct consequence of the rural hospital theorem, under a stable mechanism, a man/woman can misreport to obtain any stable assignment.

Corollary

If φ is a stable mechanism, and μ is a stable matching at $(\succ_i, \succ_{-i})$, then there exists $\succ_i'$ such that $\varphi(\succ_i', \succ_{-i})(i) = \mu(i)$.

Proof of corollary

Proof of Corollary.

1. Let $\succ_i'$: $\mu(i), i$. That is, agent i misreport that only $\mu(i)$ is acceptable for him/her.
2. Note that μ is also stable under $(\succ_i', \succ_{-i})$.
3. Due to the rural hospital theorem, $|\mu(i)| = |\varphi(\succ_i', \succ_{-i})(i)|$.
4. That is, if i is matched at μ, he/she must also be matched at $\varphi(\succ_i', \succ_{-i})$.
5. Since only $\mu(i)$ is acceptable to i at $\succ_i'$ and φ is stable, $\varphi(\succ_i', \succ_{-i})(i) = \mu(i)$.

□

One-sided strategy-proofness

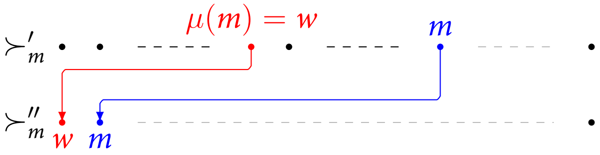
Theorem on one-sided strategy-proofness (Dubins and Freedam 1981, Roth 1982)

The man-proposing DA mechanism is strategy-proof for all men. Likewise, the woman-proposing DA is strategy-proof for all women.

- Intuition: Men are not punished when applying to preferred women.
- Stronger result: DA is the unique stable and one-sided strategy-proof mechanism. (Alcalde and Barberà 1994)

One-sided strategy-proofness: Proof

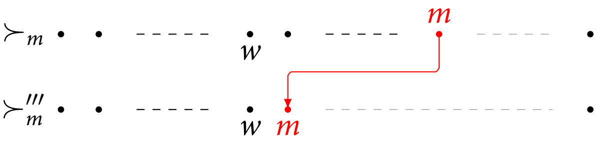
1. In the marriage problem $\langle M, W, \succ \rangle$, suppose that man m misreports $\succ'_m$. Let $\mathrm{DA}^M[\succ'_m, \succ_{-m}] = \mu$. It is sufficient to show that by truthfully reporting $\succ_m$, m will be weakly better off.
2. Case 1: If $\mu(m) = m$ or $m \succ_m \mu(m)$, nothing needs to be proved.
3. Case 2: Suppose that $\mu(m) = w$.
4. Suppose m reports $\succ''_m$: w, m, *i.e.*, only w is acceptable to him.



   1. At $(\succ''_m, \succ_{-m})$, μ is still stable due to less desires.
   2. Since m is matched to w under μ, rural hospital theorem implies that m being unmatched will be unstable at $(\succ''_m, \succ_{-m})$.

One-sided strategy-proofness: Proof (Cont.)

5. Consider $\succ_m''' : \ \ldots, w, m$, which is obtained by truncating the true preference from w.



   1. m being unmatched will also be unstable at $(\succ_m''', \succ_{-m})$: If a matching making m single is stable under $(\succ_m''', \succ_{-m})$, then it is also stable under $(\succ_m'', \succ_{-m})$.
   2. Therefore, under $\text{DA}^M[\succ_m''', \succ_{-m}]$, m is matched to some woman weakly better than w.
   3. As the DA procedure is the same under $(\succ_m''', \succ_{-m})$ and $(\succ_m, \succ_{-m})$, m will be weakly better off by truthfully reporting $\succ_m$.

1. Marriage problem
2. Stable matching and deferred acceptance algorithm
3. Incentive
4. Practice
   - New York City high school match
   - Boston public school match
   - Chinese college admissions

Background

- Over 90,000 students enter high schools each year.
- The old NYC system was decentralized:
  - Each student can submit a list of at most 5 schools.
  - Each school obtains the list of students who listed it, and independently make offers.
  - There were waiting lists (run by mail), and 3 rounds of move waiting lists.
- Problems with the old system:
  - The system left 30,000 children unassigned to any of their choices and they are administratively assigned.
  - Strategic behavior by schools: school principals were concealing capacities.

Two-sided market

- In New York City, schools behave strategically.
- Deputy Chancellor of Schools (NYT 19 November 2004):

  > *Before you might have had a situation where a school was going to take 100 new children for 9th grade, they might have declared only 40 seats and then placed the other 60 children outside the process.*

- Unlike Boston, the market seems to be really two-sided, i.e., we should treat both students and schools are strategic players.

Student-proposing DA

Since NYC is a two-sided matching market, the student-proposing DA is the big winner:

- DA implements a stable matching (probably more important for NYC than for Boston.)
- DA is strategy-proof for students: it is a dominant strategy for every student to report true preferences.
- There is no stable mechanism that is strategy-proof for schools.
- When the market is large, it is almost strategy-proof for schools to report true preferences: Recall there are 90000 students and over 500 public high schools in New York City.

Change to student-proposing DA

NYC Department of Education changed the mechanism to the student-proposing DA, except for some details:

- Students can rank only 12 schools.
- Seats in a few schools, called specialized high schools (such as Stuyvesant and Bronx High School of Science), is assigned in an earlier round, separately from the rest.
- Some top students are granted to get into a school when they rank the school as their first choices.
- All unmatched students in the main round will be assigned in the supplementary round, where the random serial dictatorship is used.

Change to student-proposing DA (Cont.)

- These features come from historical constraints and could not be changed.
- This make it technically incorrect to use standard results in two-sided matching, but they seem to be small enough a problem (it may be interesting to study if this is true and why or why not.)

Effect of changes in the mechanism

- Over 70,000 students were matched to one of their choice schools: an increase of more than 20,000 students compared to the previous year match.
- An additional 7,600 students matched to a school of their choice in the third round.
- 3,000 students did not receive any school they chose, a decrease from 30,000 who did not receive a choice school in the previous year.

Boston mechanism

Round 1: For each school, a priority ordering is exogenously determined. (In case of Boston, priorities depend on home address, whether the student has a sibling already attending a school, and a lottery number to break ties.)

Round 2: Each student submits a preference ranking of the schools.

Round 3: The final round is the student assignment based on preferences and priorities:

- **Step 1:** In Step 1 only the top choices of the students are considered. For each school, consider the students who have listed it as their top choice and assign seats of the school to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her top choice.

Boston mechanism (Cont.)

Round 3: The final round is the student assignment based on preferences and priorities:

Step 1:

Step k: Consider the remaining students. In Step k only the k-th choices of these students are considered. For each school still with available seats, consider the students who have listed it as their k-th choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her k-th choice.

End: The algorithm terminates when no more students are assigned. At each step, every assignment is final.

Background

- Students entering grades K, 6, and 9 submit preferences over schools.
- Students have priorities at schools set by the school system:
  1. Students who already attend the school,
  2. Students who live in a walk zone and have their siblings already attending the school,
  3. Students whose siblings are already attending the school,
  4. Students who live in a walk zone,
  5. All other students.
- Priorities are weak, i.e., there are many students in each priority class: This is going to be important but for now let's ignore the issue.

Example

Consider the following problem:

i	j	k	a	b
b	a	a	i	k
a		b	j	i
			k	

The procedure of the Boston mechanism is

Step	1	End
a	$j, \cancel{k}$	j
b	i	i
$\emptyset$	k	k

Example (Cont.)

- Student i is on the list of school b, and students j and k are on the list of schools a where j has higher priority.
- So i is assigned to b, j is assigned to a, and k remains unmatched.
- The resulting matching is

$$\mu = \begin{bmatrix} i & j & k \\ b & a & \emptyset \end{bmatrix}.$$

Problems

- The Boston mechanism is not necessarily stable.
  The matching μ is blocked by the pair (k, b).
- The Boston mechanism is not strategy-proofness.
  If k misreports her preference as $P'_k:\ b, a, \emptyset$ instead, the Boston mechanism produces the following matching

$$\mu' = \begin{bmatrix} i & j & k \\ \emptyset & a & b \end{bmatrix},$$

and student k benefits from submitting a false preference.

Problems (Cont.)

- A student (for example, k) who ranks a school (b) as her second choice loses her priority to students (i) who rank it as their first choice.
- Thus, it is risky for the student to use her first choice at a highly sought-after school if she has relatively low priority there. If she does not receive her first choice, she might drop far down list.
- Besides, the Boston mechanism gives students incentive to misreport their preferences by improving the ranking of schools in their choice lists for which they have high priority.
- There is experimental evidence on preference manipulation under Boston mechanism.

Worries in Boston mechanism is real

St. Petersburg Times (14 September 2003):

Make a realistic, informed selection on the school you list as your first choice. It's the cleanest shot you will get at a school, but if you aim too high you might miss. Here's why: If the random computer selection rejects your first choice, your chances of getting your second choice school are greatly diminished. That's because you then fall in line behind everyone who wanted your second choice school as their first choice. You can fall even farther back in line as you get bumped down to your third, fourth and fifth choices.

The 2004–2005 BPS School Guide:

For a better choice of your 'first choice' school . . . consider choosing less popular schools.

Worries in Boston mechanism is real (Cont.)

Advice from the West Zone Parents Group[1] meeting (27 October 2003)

> *One school choice strategy is to find a school you like that is undersubscribed and put it as a top choice, or, find a school that you like that is popular and put it as a first choice and find a school that is less popular for a "safe" second choice.*

[1] This group is a well-informed group of approximately 180 members who meet regularly prior to admissions time to discuss Boston school choice for elementary school, recommends two types of strategies to its members.

Changes

- Abdulkadiroğlu et al. found that of the 15135 students, 19% (2910) listed two over-demanded schools as their top two choices, and about 27% (782) of these ended up unassigned.
- For Boston Public School system, the Boston mechanism was replaced by DA in 2006.

Chinese college admissions

- To alleviate the problem of high-scoring students not being accepted by any universities, the parallel mechanism was proposed by Zhenyi Wu (吴振一).
- A Chinese parallel mechanism was first implemented in Hunan tier 0 college admissions in 2001.
- From 2001 to 2012, variants of the mechanism have been adopted by 28 provinces to replace Boston mechanisms.