Game Theory

Auction design 4: Keyword auction

Xiang Sun

2020 Fall

1 Introduction

2 Generalized Second Price Auction (GSP)
- Equilibrium under GSP
- Generalized English Auction (GEA)

3 VCG for internet ads

4 Summary

Introduction

- In the 1990's websites started quickly to generate revenue from advertising.
- The original method (mid 90's) was to sell ad space the same way it is sold in magazine, billboards, etc:
    - An advertiser rents/buy some space on a Web page (a banner).
    - The price is for a fixed number of displays (i.e., a fixed number of visitors).
- At the end of the 90's–early 00's this advertising model proved being unadapted:
    - Advertisers can target better viewers (using cookies).
    - With search engines advertisers can know users' interests in real time.

Issues

- On the internet things can change very quickly, i.e., search engines sell a flow of perishable advertising service, and capacity can be wasted (no ad displayed for a particular query).
- Also, ad prices can change almost continuously.
- But price based on what?
  - What Google wants: for showing the add.
  - What the advertiser wants: if the user performs a transaction on the advertiser's Web page.
  - Solution: the user clicked and is redirected: pay-per-click (PPC). Measure of success:

$$\text{click-through rate} = \frac{\sharp \text{ users clicking on the ad}}{\sharp \text{ users ''viewing'' the ad}}.$$

The current model for ads in search engines

- Internet user enters a search term (a "query").
- Gets a page with results:
  - First: Sponsored links (the ads).
  - Second: Most relevant links (organic search results).
- If the user clicks on a sponsored link:
  1. Sent to the advertiser's Web page.
  2. The advertiser pays the search engine for sending the user.
- The position of the sponsored link does matter: higher displayed links are clicked more often than links displayed lower on the page.

Huge market

- For Google, about 90–95% of its revenue comes from ads (Facebook, Twitter, etc: similar ratio).
- Most expensive keywords (those are maximum prices):

keyword	Price per click
Insurance	\$54
Loans	\$45
Mortgage	\$45
Attorney	\$45
Credit	\$35

- Smallest price: ¢5.

Origins

First service to start an auction for displayed ads with PPC: GoTo in 1997 (renamed Overture in 2001, bought by Yahoo! in 2003), with a first-price auction.

- Fast, in real-time (bidders could adjust their bid at any moment.
- Very popular: Yahoo!, MSN used Overture.
- Problem: fast changing bids made the system unstable.

Issues with GoTo's auction

- Bidders ranked according to their bid.
- Each bidder pays its bid (1st price auction).

With a 1st price auction there may not be an equilibrium, a situation where nobody wants to change its bid:

- 3 bidders (1, 2 and 3), with values per click of $\$10$, $\$4$, $\$2$.
- 1st spot much more valuable than 2nd spot (much more clicks).
- If $b_1, b_2 > 2$, there's a bidding war between bidders 1 and 2, until we reach $\$4$.
- When $b_1 > 4$, bidder 2 sets $b_2 = 2.01$ (to prevent bidder 3 to enter), but then the war starts again.

Additional issue

If bidder 1 uses a (fast) robot to bid, while bidders 2 and 3 are (slow) humans then:

- Bids are slightly above $\$2.00$ for a long period of time.
- Sellers' revenue (i.e., search engine revenue) are "low" (even if values for bidder 1 and 2 are much higher).

A single bid for multiple items

Users' click rate depends on the rank of the ad in the page:

- Each position can be seen as a different item.
- So search engines should run multiple items auctions: several items sold at the same time.

But bidders are asked to submit a single bid. Why?

- A higher ad is clicked more: bidders have the same preferences over positions.
- In general, the value per click does not depend on the position: The probability that the user purchases does not depend on the rank of the ad.
- This does not mean that the probability of purchase is the same across advertisers!

1 Introduction

2 Generalized Second Price Auction (GSP)
- Equilibrium under GSP
- Generalized English Auction (GEA)

3 VCG for internet ads

4 Summary

Generalized Second Price Auction

- Each advertiser place a bid.
- Bids are ranked:
  - Highest bidder is shown first.
  - 2nd highest bidder is shown second.
  - 3rd highest bidder is shown third.
  - etc.
- The k-th highest bidder pays the bid (when the user clicks) of the $(k+1)$-th bidder.
- There is always at least 1 more bidder than the number of links awarded. If there are only 3 bidders then only 2 sponsored links are displayed.

Payoff flows

- The Click-Through-Rate. It gives a probability of being clicked.
- $\Rightarrow$ Bidders (advertisers) need to think in terms of expectation of users consuming the good or, put differently, in terms of flows of payoffs.
- Easier to think in terms of click frequency: how many clicks per unit of time (hour, day, week).

Example

- An advertiser values the click at $\$4$.
- Click frequency of position A = 200 clicks/hour.
- Click frequency of position B = 50 clicks/hour.
- Price/click for A = \$3.
- Price/click for B = \$1.

Advertiser's preferred position depends on whether she consider the click frequency:

- Net payoff per click higher with position B:

$$\$4 - \$3 < \$4 - \$1.$$

- Net payoff per hour higher with position A:

$$200 \times (\$4 - \$3) > 50 \times (\$4 - \$1).$$

Quality scores

- In 2005 Google modified the rules of the GSP auction.
- Today, everybody does the same.

The problem:

- Google would like to charge as much as possible.
- Key observation: click frequency depends on advertiser.
- But if click frequency is low $\Rightarrow$ Google's revenue are low:

Revenue with Advertiser A		Revenue with Advertiser B
\$1/click	$>$	\$50/click
with 1000 clicks/hour		with 10 clicks/hour
$=$ \$1000/hour		$=$ \$500/hour

- Google needs to find a way to rank A above B.

Quality score

- For each advertiser Google determines a quality score that depends on
  - CTR;
  - ad relevance;
  - user experience (UX);
  - other things (secret sauce).
- Example: Search for “under-wears” (not just before Feb 14!).

	User is	
Visited site	a man	a woman
Hanes	High score	Low score
Victoria Secret	Low score	High score
Craig’s list	Very low score	Very low score

Pricing with quality scores

- For each bidder:

$$\text{Final score} = \text{Bid} \times \text{quality score}.$$

- Advertisers are ranked according to the final score.
- Charge the k-th advertiser the lowest price/click p such that

$$p \times \text{quality score} \ > \ \text{final score of } (k+1)\text{-th bidder}.$$

Pricing with quality scores (Cont.)

3 bidders (Pim, Pam, Poum), compete for 2 spots.

Bidder	Pim	Pam	Poum
Bids	\$6	\$4	\$2
Quality Score	2	4	1
Final score	12	16	2
Ranking	2nd	1st	3rd
Price/click	\$1.01	\$3.01	\$0

- - Pam just needs a score higher than 12 to win against Pim.
  - If she bids \$3.01 then final score is 12.04.
  - So the price for Pam is \$3.01 per click.
- - Pim just needs a score higher than 2 to win against Poum.
  - If he bids \$1.01 then final score is 2.02.
  - So the price for Poum is \$1.01 per click.
- Poum pays 0, ad not displayed.

Truthtelling

From now on, simple GSP without quality scores.

Proposition

Truthtelling is not a dominant strategy under GSP (w/o quality scores).

- 3 bidders (A, B and C) compete for 2 spots. Values are $v_A = \$10$, $v_B = \$4$ and $v_C = \$2$.
- Click frequency: 200 clicks/hour for the 1st spot, 199 clicks/hour for the 2nd spot.
- If bidders bid truthfully, then:
    - 1st spot awarded to A, pays \$4/click (B's bid);
    - 2nd spot awarded to B, pays \$2/click (C's bid).
- Payoff A: $(\$10 - \$4) \times 200 = \$1,200$.
- If A bids instead \$3, she gets the 2nd position and payoff is $(\$10 - \$2) \times 199 = \$1,592$.

1 Introduction

2 Generalized Second Price Auction (GSP)
- Equilibrium under GSP
- Generalized English Auction (GEA)

3 VCG for internet ads

4 Summary

Equilibrium under GSP

- There is no dominant strategy under GSP.

  $\Rightarrow$ we need to look at the equilibria.
- But bids change very frequently
- We then have an (infinitely) repeated game, where:
  - Advertisers initially only know their value (but not the valuations of others).
  - Gradually learn the value of others.
- Problem: analysis of repeated game with incomplete information can become very complex and difficult.

Simplifying assumptions

- Easier way to analyze the outcome of the auction:
  If bids stabilize after some time, what bids can we observe?
- Simplifying assumptions:
  - Valuations are commonly known: after some time, advertisers learn all the relevant information.
  - Bids can change at any time: stable bids must be best-responses to each other.
    $\Rightarrow$ stable bids must form a Nash equilibrium of the simultaneous-move, one shot game of complete information.
  - But it's not enough, we want to capture the dynamic aspect. We'll use "simple strategies."

Simple strategy

A simple strategy is to try to force out the bidder who is just above me.

- Suppose I bid b and I have the k-th position.
- Opponent bids b' and has the $(k-1)$-th position.
- If I raise slightly my bid (i.e., ranking not affected) then:
  - my payoff doesn't change
  - but my opponent's payoff is affected (it decreases).
- Opponent can retaliate and bids slightly lower so that we swap our ranks.
- If I'm better off after such retaliation, I will decide to increase my bid.
- If I'm worse off after such retaliation, I don't change my bid.
- $\Rightarrow$ If bids are "stable," no bidder want to exchange his/her position with the bidder just above him/her.

Long run equilibrium

Notation:

- $\alpha_i = \sharp$ clicks frequency for position i.
- $v_j =$ valuation per click for advertiser j.
- $g(i) =$ identity of the bidder in position i.
- $p_i =$ payment per period for advertiser in position i.

Definition

An equilibrium of the simultaneous-move game induced by GSP is locally envy-free if a player cannot improve her payoff by exchanging bids with the bidder one position above her:

$$\underbrace{\alpha_i v_{g(i)} - p_i}_{\text{Payoff of bidder ranked } \sharp i} \geq \underbrace{\alpha_{i-1} v_{g(i)} - p_{i-1}}_{\text{Payoff of bidder ranked } \sharp i-1},$$

where $p_{i-1} = \alpha_{i-1} \times b_i$.

Nash equilibrium vs envy-freeness

Nash equilibrium of the one-shot bidding game and envy-freeness are different conditions.

- Nash equilibrium: If advertiser h deviates and takes the position of bidder h' then:
  - h pays the same price as h' if h' is at a lower position (payment depends on the bidder below h').
  - h pays a different price than h' if h' is at a higher position: The price h pays after deviating is the bid of h'. Before h' was paying the bid of the advertiser below her.
- Envy-freeness:

  Advertiser h want to take the position of h' and pay the same price h' was paying.

Long run equilibrium

So the long run equilibria of the GSP auctions are given two conditions:

- Nash equilibrium of the simultaneous-move game (one shot, not repeated).
- Local envy-freeness condition.

Such equilibria may be difficult to characterize. But we'll see that they are in fact equivalent to stable assignments (which are easier to describe).

Stable assignments

Consider a market between the positions and the advertisers.

- Let Mr. i is in charge of the i-th position.
- Mr. i's objective is to maximize profit: the price charged to the advertiser.
- If the i-th position is assigned to advertiser A:
    - Mr. i's net payoff = p_i.
    - Advertiser's net payoff = $\alpha_i \times v_A - p_i$.
    - Sum of payoffs = $p_i + (\alpha_i \times v_A - p_i) = \alpha_i \times v_A$.
    - $\Rightarrow$ Any "deal" between Mr. i and advertiser A consist of a division of $\alpha_i \times v_A$.

Stable assignments (Cont.)

Definition

An assignment μ together with p the prices is stable if there does not exist an advertiser i and a position k such that

- $\mu(i) \neq k$ (k not assigned to i).
- $\alpha_k \times v_i >$ (i's net payoff under μ) + (k's net payoff under μ).
    - The right-hand side is the minimum amount needed for i and k to be better off.
    - The left-hand side is the available amount if assigned together.

- Stable assignments are known to be a fundamental property in real-life markets (that involve assignments).
- Markets that do not produce stable assignments tend to perform poorly or collapse.

Example

position	advertiser	price	click frequency	valuation
1st	i	6	200	15
2nd	j	5	150	13

- Value to be shared between 1st and i = $200 \times 15 = 3,000$
- 1st position' payoff = $6 \times 200 = 1,200$.
- Value to be shared between 2nd and j = $150 \times 13 = 1,950$
- j's payoff = $150 \times (13 - 5) = 1,200$.

But j and 1st together can generate $13 \times 200 = 2,600$! For instance, with a price of 6.5 they get

- 1st: $200 \times 6.5 = 1,300 > 1,200$
- j: $200 \times (13 - 6.5) = 1,300 > 1,200$.

Equilbrium vs stable assignments

Long run equilibria (of the GSP) and stable assignments obey to two different logic:

- Equilibria:

  Only the bidders deviate. If a bidder deviates and takes a new position she does not need to ask that position's permission.

  - $\Rightarrow$ Positions are mere objects, they don't have the right to an opinion.
  - $\Rightarrow$ A bidder deviates if she is better off. It does not matter if the position gets lower revenue.

- Stability:

  If a bidder wants a different position, that will be possible only if the position agrees.

  - It's not sufficient that the advertiser is better off at the new position.
  - The new position also needs to be better off.

Equilbrium vs stable assignments (Cont.)

Proposition

The outcome of any locally envy-free equilibrium in the GSP auction is a stable assignment.
Furthermore, if there are more advertisers than positions then any stable assignment is the outcome of a locally envy-free equilibrium.

Proof: envy-freeness $\Rightarrow$ stability

1. Suppose we have an envy-free equilibrium and that $p_1, \ldots, p_n$ are that payments received by positions $1, \ldots, n$.
2. For notational simplicity, assume that in equilibrium advertiser h at position h (for all $h \geq 1$).
3. Take an advertiser assigned to position k, looking at position h.

   We want to show that:

   $$\text{Nash equilibrium} + \text{Envy-free} \Rightarrow \text{Stable assignment.}$$
4. How do we prove that an assignment is stable?

Claim

Claim: If for any h and k the equilibrium is such that

$$\alpha_k v_k - p_k \geq \alpha_h v_k - p_h, \tag{1}$$

then it corresponds to a stable assignment.

Proof.

1. p_h = Position h's payoff.

   $\alpha_h v_k$ = Size of the surplus to be shared between position h and advertiser k (if matched together).
2. $\alpha_h v_k - p_h$ is advertiser's k maximal payoff she can hope if position h agrees to taker her (instead of advertiser h):
3. But Eq. (1) implies

   k's payoff at position $k >$ max. payoff can hope with pos. h.
4. $\Rightarrow$ there is no way that h and k can be better off together. So the assignment is stable.

Proof strategy

1. We start with an equilibrium (Nash equilibrium + locally envy-freeness).
2. We take any advertiser k assigned to position k. At the equilibrium k does not want to change the bid and obtain position h:

   k would get a lower payoff if she changes her bid to get position h:

$$\underbrace{\alpha_k v_k - p_k}_{k\text{'s equilibrium payoff}} \geq \underbrace{\alpha_h v_k - \hat{p}_h}_{k\text{'s payoff if deviates}} . \tag{2}$$

   (can do that bidding b such that $b_{h+1} < b < b_h$)
3. We'll show that (2) gives the stable assignment condition (1).

Case 1: advertiser k & position $h > k$

- We assumed that the equilibrium of repeated game is also a Nash equilibrium of the one-shot game, so we have

$$\underbrace{\alpha_k v_k - p_k}_{k\text{'s equilibrium payoff}} \geq \underbrace{\alpha_h v_k - \hat{p}_h}_{k\text{'s payoff if deviates}} .$$

- But payment to position h depends on bid of advertiser $h+1$. Since advertiser h does not change her bid (only advertiser h),

$$\underbrace{\hat{p}_h}_{\text{what } k \text{ would pay if deviates}} = \underbrace{p_h}_{\text{what } h \text{ was paying}} .$$

- So we can rewrite we get:

$$\alpha_k v_k - p_k > \alpha_h v_k - p_h.$$

- That's the stability condition (1), what we wanted.

Case 2: advertiser k & position $k-1$

- Let's rewrite the stability condition (1) replacing h by $k-1$ to see what we need to show:

$$\alpha_k v_k - p_k \geq \alpha_{k-1} v_k - p_{k-1}. \tag{3}$$

  So we need to show that the equilibrium conditions are such that (3) holds true.
- But that's the condition of envy-freeness! The definition of long run equilibrium requires envy-freeness. So we're done.

Now let's do the same for positions $k-2, k-3, \ldots, 1$.

Case 3: advertiser k & position $m < k-1$

Claim

Equilibrium $\Rightarrow$ Assortative match (bidders ranked by their valuations), i.e., $v_k \geq v_{k+1}$ for all k.

Proof.

1. Nash Equilibrium condition: nobody wants to more one position down:
$$\alpha_k v_k - p_k \geq \alpha_{k+1} v_k - p_{k+1}. \tag{4}$$
2. Envy-freeness: nobody wants to more one position up:
$$\alpha_{k+1} v_{k+1} - p_{k+1} \geq \alpha_k v_{k+1} - p_k. \tag{5}$$

□

Proof of Claim

Proof.

3. Add (4) and (5) and we get

$$\alpha_k v_k + \alpha_{k+1} v_{k+1} \geq \alpha_{k+1} v_k + \alpha_k v_{k+1}$$

iff

$$v_k(\alpha_k - \alpha_{k+1}) \geq v_{k+1}(\alpha_k - \alpha_{k+1}).$$

4. Since $\alpha_k > \alpha_{k+1}$ (higher position = more clicks), we have

$$v_k \geq v_{k+1},$$

claim is proved!

□

Case 3: advertiser k & position $m < k - 1$ (Cont.)

Now, suppose advertiser k and position $m < k - 1$ want to rematch. Since equilibrium locally envy-free:

$$\begin{aligned}
\alpha_k v_k - p_k &\geq \alpha_{k-1} v_k - p_{k-1}, \\
\alpha_{k-1} v_{k-1} - p_{k-1} &\geq \alpha_{k-2} v_{k-1} - p_{k-2}, \\
\alpha_{k-2} v_{k-2} - p_{k-2} &\geq \alpha_{k-3} v_{k-2} - p_{k-3}, \\
&\vdots \\
\alpha_{m+2} v_{m+2} - p_{m+2} &\geq \alpha_{m+1} v_{m+2} - p_{m+1}, \\
\alpha_{m+1} v_{m+1} - p_{m+1} &\geq \alpha_m v_{m+1} - p_m.
\end{aligned}$$

Case 3: advertiser k & position $m < k - 1$ (Cont.)

Observe that for any $h > 1$,

$$\underbrace{\alpha_h v_h - p_h \geq \alpha_{h-1} v_h - p_{h-1}}_{\text{envy-freeness condition}} \quad \Rightarrow \quad v_h \leq \frac{p_{h-1} - p_h}{\alpha_{h-1} - \alpha_h}.$$

Since $\alpha_h < \alpha_{h-1}$ for any h, and since and $v_k < v_h$ for any $h < k$, we can replace v_h by v_k in the second inequality and we get for any $h = 2, \ldots, k - 1$,

$$v_k \leq \frac{p_{h-1} - p_h}{\alpha_{h-1} - \alpha_h} \text{ and } \alpha_h v_k - p_h \geq \alpha_{h-1} v_k - p_{h-1}.$$

So $j = m, \ldots, k - 1$, from

$$\alpha_j v_j - p_j \geq \alpha_{j-1} v_j - p_{j-1},$$

one can obtain

$$\alpha_j v_k - p_j \geq \alpha_{j-1} v_k - p_{j-1}.$$

Case 3: advertiser k & position $m < k - 1$ (Cont.)

So we can rewrite the equations replacing v_j by v_k for $j = m, \ldots, k - 1$.

$$\begin{aligned}
\alpha_k v_k - p_k &\geq \alpha_{k-1} v_k - p_{k-1}, \\
\alpha_{k-1} v_k - p_{k-1} &\geq \alpha_{k-2} v_k - p_{k-2}, \\
\alpha_{k-2} v_k - p_{k-2} &\geq \alpha_{k-3} v_k - p_{k-3}, \\
&\vdots \\
\alpha_{m+2} v_k - p_{m+2} &\geq \alpha_{m+1} v_k - p_{m+1}, \\
\alpha_{m+1} v_k - p_{m+1} &\geq \alpha_m v_k - p_m.
\end{aligned}$$

Adding all these inequalities yields

$$\alpha_k v_k - p_k \geq \alpha_m v_k - p_m.$$

We're done! Envy-free $\Rightarrow$ stable.

1 Introduction

2 Generalized Second Price Auction (GSP)

- Equilibrium under GSP
- Generalized English Auction (GEA)

3 VCG for internet ads

4 Summary

Generalized English Auction

- The Vickrey auction is a one-shot/simultaneous version of the English auction.
- Can we define a "generalized English auction" so that GSP would be its one-shot/simultaneous version?

Rules

- The is a clock showing the current price, which increases over time.
- Start: price $= 0$, all advertisers are in the auction.
- Advertiser can drop at any time. Their bid is the price on the clock at the time they drop out.
- Auction over when the next-to-last advertiser drops out.

Outcome

- Last advertiser ranked 1st, all other ranked according to the time they dropped out (the latest, the higher).
- Each advertiser pays the bid of the advertiser ranked just below him/her.

Generalized English Auction

Proposition

There is a unique (perfect Bayesian) equilibrium of the generalized English auction, where an advertiser with valuation v drops out at the price

$$p^* = v - \frac{\alpha_i}{\alpha_{i-1}}(v - b_{i+1}),$$

where $i = \sharp$ advertiser remaining (including him/her).

These prices imply that the payoffs in the generalized English auction are the same as in the VCG auction.

Intuition

- Suppose there are i bidders remaining (including me), and the next highest bid is b_{i+1}.
- The next bidder who drops out will pay b_{i+1}. If I'm the next to drop out my payoff is

$$\alpha_i \times (v - b_{i+1}).$$

- If I wait a bidder to drop out (at price p) and drop out just after, my payoff is

$$\alpha_{i-1} \times (v - p).$$

Intuition (Cont.)

We have $\alpha_{i-1} > \alpha_i$, so if $p = b_{i+1}$ then

$$\alpha_i \times (v - b_{i+1}) < \alpha_{i-1} \times (v - p).$$

- But the right hand side decreases in p.
- So there will be a price p^* such that waiting and not waiting gives the same payoff:

$$\alpha_i \times (v - b_{i+1}) = \alpha_{i-1} \times (v - p^*)$$

iff

$$p^* = v - \frac{\alpha_i}{\alpha_{i-1}}(v - b_{i+1}).$$

1. Introduction
2. Generalized Second Price Auction (GSP)
   - Equilibrium under GSP
   - Generalized English Auction (GEA)
3. VCG for internet ads
4. Summary

Comparing GEA and VCG

Proposition

Equilibrium payoffs of GEA = payoffs of VCG.

$\Rightarrow$ GEA = sequential version of VCG.

Proof.

- Suppose $v_A > v_B > v_C > v_D$, 3 links at most.
  - 4 bidders $\Rightarrow$ 3 links,
  - 3 bidders $\Rightarrow$ 2 links.

$\square$

Proof

- VCG $\Rightarrow$ bidders truthful $\Rightarrow$ assortative assignment.
  If assignment not assortative then social value is not maximized.
- Take bidder B.
- Social welfare of others if bidder B present:

$$\alpha_1 v_A + \alpha_3 v_C.$$

- Social welfare if bidder B NOT present:

$$\alpha_1 v_A + \alpha_2 v_C.$$

- Price for B:

$$\alpha_1 v_A + \alpha_2 v_C - \alpha_1 v_A - \alpha_3 v_C = v_C(\alpha_2 - \alpha_3).$$

Proof (Cont.)

B's price with GEA:

- Not difficult to show that C (D) drops out before B (D).
- Price of B is bid of C (the bidder who left just before).
- Bid of C depends on bid of D. Bid of $D = 0$.

$$bid_C = v_C - \frac{\alpha_3}{\alpha_2}(v_C - 0).$$

- To get the price per period for B we multiply by α_2:

$$\text{price per period for} B = \alpha_2\left(v_C - \frac{\alpha_3}{\alpha_2}v_C\right) = v_c(\alpha_2 - \alpha_3).$$

Comparing VCG and GSP

The envy-freeness condition written for advertiser $i+1$ is

$$v_{i+1}\alpha_{i+1} - p_{i+1} \geq \alpha_i v_{i+1} - p_i,$$

which can be re-written as

$$p_i \geq v_{i+1}(\alpha_i - \alpha_{i+1}) + p_{i+1}.$$

Left-hand side is thus a lower bound (per period) of the revenue of position i.

- Take 4 bidders, 3 slots, $v_A > v_B > v_C > v_D$.

$$p_1 \geq v_B(\alpha_1 - \alpha_2) + p_2 \tag{6}$$
$$p_2 \geq v_C(\alpha_2 - \alpha_3) + p_3 \tag{7}$$
$$p_3 \geq v_D\alpha_3. \tag{8}$$

- Add (6), (7) and (8) and we get

$$p_1 \geq v_B(\alpha_1 - \alpha_2) + v_C(\alpha_2 - \alpha_3) + v_D\alpha_3.$$

  Lower bound for revenue of position 1.

- Add (7) and (8) and we get

$$p_2 \geq v_C(\alpha_2 - \alpha_3) + v_D\alpha_3.$$

  Lower bound for revenue of position 2.

- (8) is the lower bound for revenue of position 3.

Now take the VCG auction.

- Payment for bidder A (to get 1st position):

$$\begin{aligned} p_1 = & \underbrace{v_B\alpha_1 + v_C\alpha_2 + v_D\alpha_3}_{\text{max social value when } A \text{ not here}} - \underbrace{(v_B\alpha_2 + v_C\alpha_3)}_{\text{social value of others when } A \text{ here}} \\ = & v_B(\alpha_1 - \alpha_2) + v_C(\alpha_2 - \alpha_3) + v_D\alpha_3. \end{aligned}$$

- Payment for bidder B (to get 2nd position):

$$p_2 = v_C(\alpha_2 - \alpha_3) + v_D\alpha_3.$$

- Payment for bidder C (to get 3rd position):

$$p_3 = v_D\alpha_3.$$

Comparing VCG and GSP

So, we have

$$\text{Revenue with GSP} \geq \text{Revenue with VCG}.$$

Why does Facebook uses VCG?

- On Facebook more uncertainty on CTR. Using VCG means advertisers spend more time figuring out the value of their valuations.

  Life simpler for advertisers.
- Revenue lower for Facebook $\Rightarrow$ revenue higher for advertisers.

  In the long run, advertisers may prefer Facebook.
- Few ads per page, computational issue disappears.

1. Introduction
2. Generalized Second Price Auction (GSP)
   - Equilibrium under GSP
   - Generalized English Auction (GEA)
3. VCG for internet ads
4. Summary

Take-away

- Most ads on the internet are allocated through an auction.
- Bidders have a valuation per click, but make decisions taking into account the click-through rate (click frequency).
- A popular format is the Generalized Second-Price auction:
  - Advertisers place one bid for a spot on the webpage.
  - Advertisers allocated spots so that.

$$\text{bid rank} = \text{CTR rank of spot on the page.}$$

  - Bidders pay per click the bid of the bidder rank just below them.
- Truthful bidding is not a dominant strategy with the GSP.

Take-away (Cont.)

- Long run equilibrium of the repeated GSP game yield stable assignments.
- Website's revenue are higher with the GSP than with the VCG.
- In practice websites multiply advertisers with a quality score: it maximizes the payoff flows for the website.
- Some websites (e.g., Facebook) use the VCG auction.