

Game Theory

Social choice

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2020 Fall

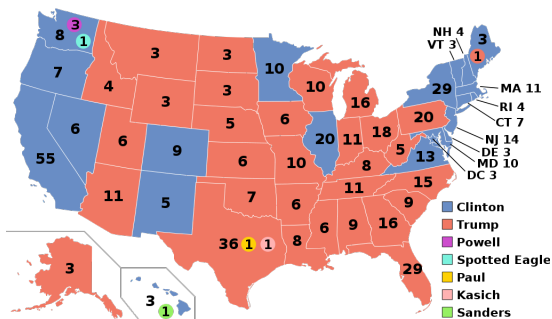
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Voting

- Voting seems very simple.
- But sometimes things go wrong.
 - In 2000, the US presidential election came down to Florida.
 - George Bush won by 537 votes.
 - But Ralph Nader got 97,421 votes. Twice as many Nader voters would have chosen Gore over Bush.

Voting

2016 US presidential election:



	Donald Trump	Hillary Clinton
Electoral vote	304	227
Popular vote	62,984,828	65,853,514
Percentage	46.1%	48.2%

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Setting

Our setting now:

- a set of **outcomes** or **alternatives**,
- agents have **preferences** over them,
- the ‘goal’: a **social choice function**: a mapping from profiles of preferences to a particular **outcome**.
 - Which such functions have desirable properties?

Preferences

- Given is a finite set of **outcomes** or **alternatives** O .
- Agents have (strict) **preferences**, \succ , over the outcomes: linear orders (or total orders).
- Linear orders \mathcal{L} : binary relations \succ that are total and transitive:
 - total: for every pair of outcomes $a \neq b$ either $a \succ b$ or $b \succ a$ (but not both: so it is complete and antisymmetric).
 - transitive: $a \succ b$ and $b \succ c$ implies $a \succ c$.
- Weak preferences \mathcal{L}_w : binary relations \succsim that are complete and transitive:
 - complete: for every a and b either $a \succsim b$ or $b \succsim a$ (both indicates indifference).
 - transitive: $a \succsim b$ and $b \succsim c$ implies $a \succsim c$.

Formal model

Given is a set of agents $N = \{1, 2, \dots, n\}$, a finite set of outcomes (or alternatives, or candidates) O , and the set of preferences over outcomes, \mathcal{L} .

Definition (Social choice function)

A social choice function is a function $C: \mathcal{L}^n \rightarrow O$.

Definition (Social welfare function)

A social welfare function is a function $C: \mathcal{L}^n \rightarrow \mathcal{L}$.

Voting schemes: Scoring rules

- Plurality (多数决)
 - pick the outcome which is most-preferred by the most people.
- Cumulative voting (累积投票制)
 - distribute e.g., 5 votes each.
 - possible to vote for the same outcome multiple times.
- Approval voting (认可投票制)
 - vote for as many outcomes as you “like”.

Voting schemes based on ranking

- Plurality with elimination (“instant runoff”, “transferable voting”
帶消除的多数决)
 - if some outcome has a majority, it is the winner.
 - otherwise, the outcome with the fewest votes is eliminated (may need some tie-breaking procedure).
 - repeat until there is a winner.
- Borda Rule, Borda Count (波达计数法)
 - assign each outcome a number.
 - the most preferred outcome gets a score of $n - 1$, the next most preferred gets $n - 2$, down to the n -th outcome which gets 0.
 - sum scores for each outcome, and choose one with highest score.
- Successive elimination (连续淘汰制)
 - in advance, decide an ordering of alternatives.
 - everyone votes for the first or second, and the loser is eliminated.
 - then vote for winner vs third alternative, and loser is eliminated.
 - continue until the last alternative is considered.

Condorcet consistency

- If there is a candidate or outcome that is preferred to every other candidate in pairwise majority-rule comparisons, that candidate should be chosen.
- There is not always a Condorcet winner.
- Sometimes, there is a cycle where A defeats B , B defeats C , and C defeats A , known as a Condorcet cycle.

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Condorcet example

499 agents: $A \succ B \succ C$

3 agents: $B \succ C \succ A$

498 agents: $C \succ B \succ A$

- What is the Condorcet winner? B
- What would win under plurality voting? A
- What would win under plurality with elimination? C

Sensitivity to losing candidate

35 agents: $A \succ C \succ B$

33 agents: $B \succ A \succ C$

32 agents: $C \succ B \succ A$

- What candidate wins under plurality voting? A
- What candidate wins under Borda voting? A
- Now consider dropping C . Now what happens under both Borda and plurality? B wins.

Sensitivity to agenda setter

35 agents: $A \succ C \succ B$

33 agents: $B \succ A \succ C$

32 agents: $C \succ B \succ A$

- Who wins pairwise elimination, with the ordering A, B, C ? C
- Who wins with the ordering A, C, B ? B
- Who wins with the ordering B, C, A ? A

Another pairwise elimination problem

1 agent: $B \succ D \succ C \succ A$

1 agent: $A \succ B \succ D \succ C$

1 agent: $C \succ A \succ B \succ D$

- Who wins under pairwise elimination with the ordering A, B, C, D ?
- What is the problem with this?
 - all of the agents prefer B to D —the selected candidate is Pareto-dominated!

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Notation

- N is the set of **agents**.
- O is a finite set of **outcomes** with $|O| \geq 3$.
- \mathcal{L} is the set of all possible **strict preference orderings** over O .
 - for ease of exposition we switch to strict orderings
 - we will end up showing that desirable SWFs cannot be found even if preferences are restricted to strict orderings
- $(\succ_i)_{i \in N}$ is an element of the set \mathcal{L}^n (a **preference ordering for every agent**; the input to our social welfare function)
- \succ_W (or simply \succ) is the **preference ordering selected by the social welfare function W** .

PE and IIA

Definition

W is **Pareto efficient** if for any $o_1, o_2 \in O$, for each $i \in N$, $o_1 \succ_i o_2$ implies that $o_1 \succ o_2$.

When all agents agree on the ordering of two outcomes, the social welfare function must select that ordering.

Definition

W is **independent of irrelevant alternatives (IIA)** if, for any $o_1, o_2 \in O$ and any two preference profiles $(\succ'_i)_{i \in N}, (\succ''_i)_{i \in N} \in \mathcal{L}^n$, “for each i , $o_1 \succ'_i o_2$ if and only if $o_1 \succ''_i o_2$ ” implies that “ $o_1 \succ' o_2$ if and only if $o_1 \succ'' o_2$ ”.

The selected ordering between two outcomes should depend only on the relative orderings they are given by the agents.

Nondictatorship

Definition

W does not have a **dictator** if there is no agent i such that for any o_1 and o_2 , $o_1 \succ_i o_2$ implies $o_1 \succ o_2$.

- There does not exist a single agent whose preferences always determine the social ordering.
- We say that W is dictatorial if it fails to satisfy this property.

Theorem (Arrow, 1951)

Any social welfare function W that is Pareto efficient and independent of irrelevant alternatives is dictatorial.

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Social choice functions

- Maybe Arrow's theorem held because we required a whole preference ordering.
- Idea: **Social choice functions** might be easier to find.
- We'll need to **redefine our criteria** for the social choice function setting; PE and IIA discussed the ordering.

Criteria

- A social choice function C is **weakly Pareto efficient** if it never selects an outcome o_2 when there exists another outcome o_1 such that for each i , $o_1 \succ_i o_2$.
 - A dominated outcome can't be chosen.
- C is **monotonic** if, for any $o \in O$ and any preference profile $(\succ_i) \in \mathcal{L}^n$ with $C((\succ_i)_i) = o$, then for any other preference profile (\succ'_i) with the property that for each $i \in N$, for each $o' \in O$, $o \succ'_i o'$ if $o \succ_i o'$, it must be that $C((\succ'_i)_i) = o$.
 - An outcome o must remain the winner whenever the support for it is increased in a preference profile under which o was already winning.
- C is **dictatorial** if there exists an agent j such that C always selects the top choice in j 's preference ordering.

Muller and Satterthwaite

Theorem (Muller and Satterthwaite, 1977)

Any social choice function that is weakly Pareto efficient and monotonic is dictatorial.

- Perhaps contrary to intuition, social choice functions are **no simpler** than social welfare functions after all.
- The proof repeatedly “probes” a social choice function to determine the relative social ordering between given pairs of outcomes.
- Because the function must be defined for all inputs, we can use this technique to construct a full social welfare ordering.

Plurality

- Plurality satisfies weak PE and ND, so it must not be monotonic.
- Consider the following preferences:

3 agents: $a \succ b \succ c$

2 agents: $b \succ c \succ a$

2 agents: $c \succ b \succ a$

Plurality chooses a .

- Increase support for a by moving c to the bottom:

3 agents: $a \succ b \succ c$

2 agents: $b \succ c \succ a$

2 agents: $b \succ a \succ c$

Now plurality chooses b .

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Single-peaked preferences

- Sometimes voters' preferences have nicer properties.
- Prominent case: candidates can be ordered from left to right.
- Voters: have a most-preferred candidate and then candidates who are more extreme are less-preferred.