Game Theory

Signaling games

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Introduction

- Signaling games (信号发送博弈) are a well-studied class of dynamic games of incomplete information.
- The concept of "signaling" (信号发送) refers to strategic models where informed agents (知情人) take some observable actions before uninformed agents (不知情的人) make their strategic decisions.
- Signaling games are a relatively simple setting in which to study
    - how players update beliefs based on observed actions (signals);
    - how players try to strategically reveal or conceal private information by their choice of actions.
- There are many applications of signaling games in economics (for example, Spence's job-market signaling model).

Timing

- A simple signaling game is a dynamic game of incomplete information involving two players: a Sender (S) and a Receiver (R).
- The timing of the game is as follows:
  1. Nature draws a type t_i for the Sender from a set of feasible types $T = \{t_1, \ldots, t_I\}$ according to a probability distribution $P(t_i)$, where $P(t_i) > 0$ for every i and $P(t_1) + \cdots + P(t_I) = 1$.
  2. The Sender observes t_i and then chooses a message m_j from a set of feasible messages $M = \{m_1, \ldots, m_J\}$.
  3. The Receiver observes m_j (but not t_i) and then chooses an action a_k from a set of feasible actions $A = \{a_1, \ldots, a_K\}$.
  4. Payoffs are given by $U_S(t_i, m_j, a_k)$ and $U_R(t_i, m_j, a_k)$.

Strategy

- Consider the following signaling game:

$$T = \{t_1, t_2\},\ A = \{a_1, a_2\},\ P(t_1) = p,\ M = \{m_1, m_2\}.$$

- The Sender has four pure strategies:

$$(m_1, m_1), (m_1, m_2), (m_2, m_1), (m_2, m_2).$$

- The strategy (m', m'') means the Sender of type t_1 chooses a message m' and type t_2 chooses a message m''.

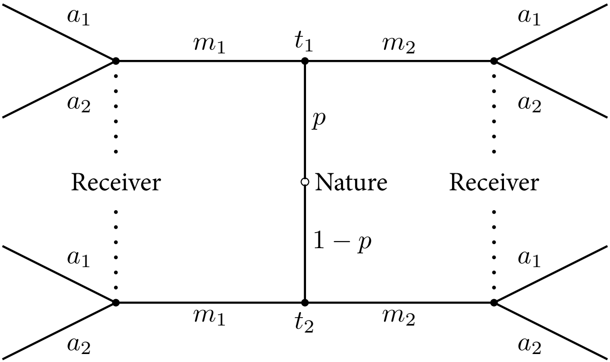
Strategy (Cont.)

- Similarly, the Receiver has four pure strategies:

$$(a_1, a_1), (a_1, a_2), (a_2, a_1), (a_2, a_2).$$

- The strategy (a', a'') means the Receiver plays a' if the Sender chooses m_1 and plays a'' if the Sender chooses m_2.
- We call Sender's strategies
  - (m_1, m_1), (m_2, m_2) to be pooling (混同) (because each type sends the same message);
  - (m_1, m_2), (m_2, m_1) to be separating (分离) (because each type sends a different message).

Illustration



1. Signaling games
   - Model
   - Perfect Bayesian equilibrium
   - Example
   - How to find PBE

2. Cheap talk games
   - Model
   - Informative communication

Signaling Requirement 1: Belief

We first translate the requirements for a perfect Bayesian equilibrium to the case of signaling games.

Signaling Requirement 1

After observing any message m_j from M, the Receiver must have a belief about which types could have sent m_j.

Denote this belief by the probability distribution $\mu(t_i|m_j)$, where $\mu(t_i|m_j) \geq 0$ for each $t_i \in T$, and $\sum_{t_i \in T} \mu(t_i|m_j) = 1$.

Signaling Requirement 2: Sequential rationality

Signaling Requirement 2R

For each $m_j \in M$, the Receiver's action $a^*(m_j)$ must maximize the Receiver's expected utility, given the belief $\mu(t_i|m_j)$ about which types could have sent m_j. That is, $a^*(m_j)$ solves

$$\max_{a_k \in A} \sum_{t_i \in T} \mu(t_i|m_j) U_R(t_i, m_j, a_k).$$

Signaling Requirement 2S

For each $t_i \in T$, the Sender's message $m^*(t_i)$ must maximize the Sender's utility, given the Receiver's strategy $a^*(m_j)$. That is, $m^*(t_i)$ solves

$$\max_{m_j \in M} U_S\big(t_i, m_j, a^*(m_j)\big).$$

Signaling Requirement 2: Sequential rationality (Cont.)

- These two requirements imply that both the Receiver and the Sender act in an optimal way.
- Given the Sender's optimal strategy $m^*(t_i)$, i.e., m^* is a function from T into M, let $T_j = \{t_i \in T : m^*(t_i) = m_j\}$. T_j is the set of all types sending the message m_j.
- The information set corresponding to m_j is on the equilibrium path if $T_j \neq \emptyset$, and off the equilibrium path otherwise.

Signaling Requirement 3: Rational belief

Signaling Requirement 3

For each $m_j \in M$, if there exists $t_i \in T$ such that $m^*(t_i) = m_j$, i.e., $T_j \neq \emptyset$, then the Receiver's belief at the information set corresponding to m_j must follow from Bayes' rule and the Sender's strategy:

$$\mu(t_i|m_j) = \frac{P(t_i)}{\sum_{t \in T_j} P(t)}, \forall t_i \in T_j.$$

Perfect Bayesian Equilibria

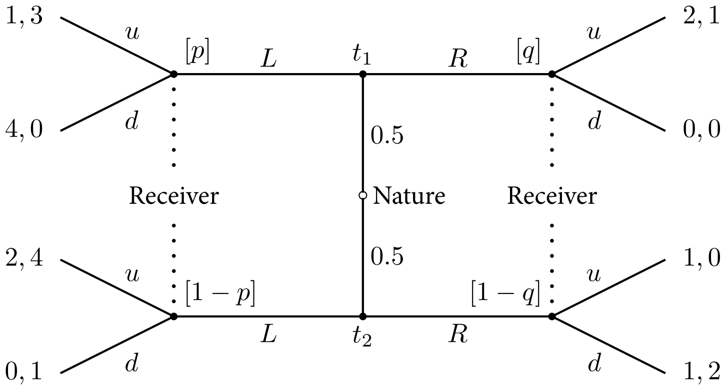
Definition

A pure-strategy perfect Bayesian equilibrium in a signaling game is a pair of strategies $m^*(t_i)$ and $a^*(m_j)$ and a belief $\mu(t_i|m_j)$ satisfying Signaling Requirements (1), (2R), (2S), and (3).

- A strategy for the Sender is a function from the type space T into the message space M.
- A strategy for the Receiver is a function from the message space M into the action space A.
- For a perfect Bayesian equilibrium of a signaling game, if the Sender's strategy is pooling (or separating), then we call the equilibrium pooling (or separating), respectively.

Example

Find all pure-strategy perfect Bayesian equilibria in the following signaling game.



The first (the second) number is the payoff to the Sender (the Receiver).

Formulation

- In this game,

$$T = \{t_1, t_2\}, P(t_1) = 0.5, M = \{L, R\}, A = \{u, d\}.$$

- The Sender's strategies are: (L, L), (L, R), (R, L) and (R, R), where (m', m'') means that type t_1 chooses m' and type t_2 chooses m''.
- The Receiver's strategies are: (u, u), (u, d), (d, u), and (d, d), where (a', a'') means that the Receiver plays a' following L and a'' following R.
- We analyze the possibility of the four Sender's strategies to constitute perfect Bayesian equilibria.

Case 1: PBE pooling on L

- Suppose the Sender adopts the strategy (L, L).
- By Signaling Requirement 3, we have $p = 1 - p = 0.5$. Given this belief (or any belief) of the Receiver, the Receiver's best response to message L is u, i.e., $a^*(L) = u$.
- For the message R, the Receiver's belief q cannot be determined by Sender's strategy, and thus we can choose any belief q. Furthermore, both $a^*(R) = u$ and $a^*(R) = d$ are possible for some q. Indeed $a^*(R) = u$ iff $q \geq \frac{2}{3}$; and $a^*(R) = d$ iff $q \leq \frac{2}{3}$.
- We only need to see if sending L is better than sending R for both types t_1 and t_2.

Case 1: PBE pooling on L (Cont.)

- If $a^*(R) = u$, i.e., (u, u) is the Receiver's strategy, then for type t_1, the Sender's payoff is 1 if L is sent and 2 if R is sent. Hence, sending L is not optimal.
- If $a^*(R) = d$, i.e., (u, d) is the Receiver's strategy, then for type t_1, the Sender's payoff is 1 if L is sent and 0 if R is sent, choosing L is optimal; for type t_2, choosing L is also optimal given $2 > 1$.
- Thus, (L, L) is the Sender's best response to the Receiver's strategy (u, d).
- Moreover, (u, d) is also the Receiver's best response to the Sender's strategy (L, L) if $q \leq \frac{2}{3}$.
- Therefore, $[(L, L), (u, d); p = \frac{1}{2}, q \leq \frac{2}{3}]$ is a pooling equilibrium.

Case 1: PBE pooling on L (Cont.)

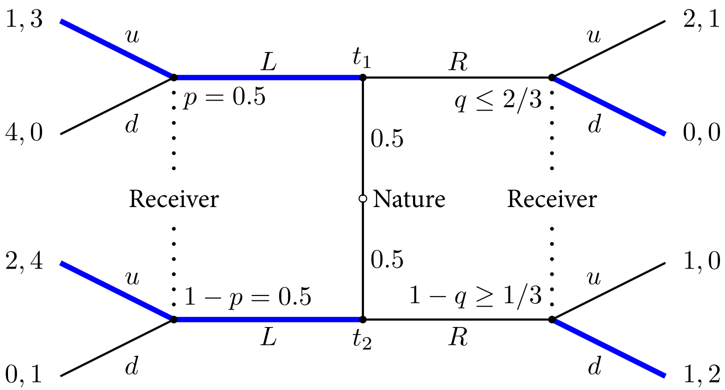


Figure: Pooling equilibrium: $[(L, L), (u, d); p = 0.5, q \leq \frac{2}{3}]$

Case 2: PBE pooling on R

- Suppose the Sender adopts the strategy (R, R).
- Then Signaling Requirement 3 implies that $q = 1 - q = \frac{1}{2}$. Given this belief, the Receiver's best response is to R is d, i.e., $a^*(R) = d$, since $\frac{1}{2} < 1$.
- For the message L, we can choose any belief p. But we know for any p, the Receiver's best response to L is u, i.e., $a^*(L) = u$.
- Given the Receiver's strategy (u, d), for type t_1, the Sender's payoff is 0 if R is sent and 1 if L is sent, and thus R is not optimal.
- Therefore, there is no equilibrium in which the Sender plays (R, R).

Case 3: Separation with t_1 playing L

- Suppose the Sender adopts the separating strategy (L, R).
- Then, Signaling Requirement 3 implies $p = 1$ and $q = 0$. For these beliefs, we must have $a^*(L) = u$, and $a^*(R) = d$.
- Given the Receiver's strategy (u, d), for type t_2, the Sender's payoff is 4 if L is sent and 2 if R is sent. Hence R is not optimal.
- Therefore, there is no equilibrium in which the Sender plays (L, R).

Case 4: Separation with t_1 playing R

- Suppose the Sender adopts the separating strategy (R, L).
- Then, Signaling Requirement 3 implies $p = 0$ and $q = 1$. For these beliefs, we have $a^*(L) = u$ and $a^*(R) = u$.
- Given the Receiver's strategy (u, u), for type t_1, the Sender's payoff is 1 if L is sent and 2 if R is sent. Hence R is optimal.
- For the Sender type t_2, the payoff is 2 if L is sent and 1 if R is sent. Hence L is also optimal.
- Therefore, $[(R, L), (u, u); p = 0, q = 1]$ is a separating perfect Bayesian equilibrium.

Case 4: Separation with t_1 playing R (Cont.)

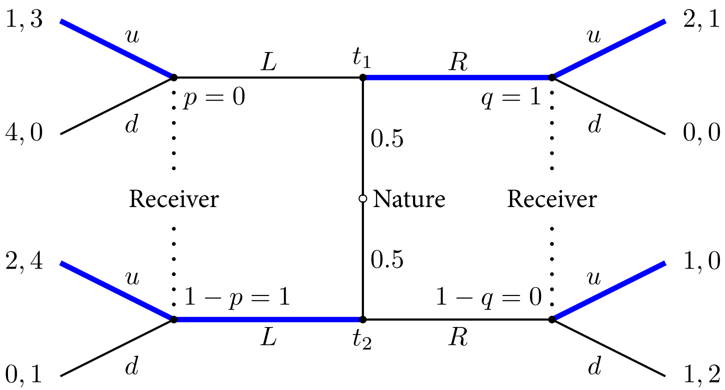


Figure: Separating equilibrium: $[(R, L), (u, u); p = 0, q = 1]$

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How to find PBE

How to find (pure-strategy) perfect Bayesian equilibria in signaling games:

1. Start with a strategy of the Sender (pooling or separating);
2. If possible, calculate the beliefs of the Receiver using Bayes' rules. Otherwise, choose arbitrary beliefs;
3. Given the beliefs, find out the best response of the Receiver;
4. Check whether the Sender's strategy is a best response to the Receiver's strategy.

How to find PBE (Cont.)

- Consider an alternative way to find perfect Bayesian equilibria.
- We first find Bayeisan Nash equilibria, and then check which equilibria are perfect Bayesian equilibria.
- Consider the following bi-matrix to represent the game:

		Receiver			
		(u,u)	(u,d)	(d,u)	(d,d)
Sender	(L,L)	$1,\underline{2},\underline{3.5}$	$\underline{1},\underline{2},\underline{3.5}$	$\underline{4},0,0.5$	$4,0,0.5$
	(L,R)	$1,1,1.5$	$\underline{1},1,\underline{2.5}$	$\underline{4},\underline{1},0$	$\underline{4},\underline{1},1$
	(R,L)	$\underline{2},\underline{2},\underline{2.5}$	$0,\underline{2},2$	$2,0,1$	$0,0,0.5$
	(R,R)	$\underline{2},1,0.5$	$0,1,\underline{1}$	$2,\underline{1},0.5$	$0,\underline{1},\underline{1}$

- Two (pure-strategy) Bayesian Nash equilibria: $\big((L,L),(u,d)\big)$ and $\big((R,L),(u,u)\big)$

How to find PBE (Cont.)

- To check whether they are perfect Bayesian equilibria, we only need to find beliefs, satisfying all four Signaling Requirements.
- For (L, L), Bayes' rule requires $p = \frac{1}{2}$ and there is no requirement for q. Given the belief, $a^*(L) = u$, and $a^*(R) = d$ iff $q \leq \frac{2}{3}$. Thus (u, d) is a best response to (L, L) iff $p = \frac{1}{2}$ and $q \leq \frac{2}{3}$.
- For (R, L), Bayes' rule requires $p = 0$ and $q = 1$. Given this belief, $a^*(L) = u$ and $a^*(R) = u$. Thus (u, u) is a best response to (R, L).
- Therefore, $[(L, L), (u, d); p = \frac{1}{2}, q \leq \frac{2}{3}]$ and $[(R, L), (u, u); p = 0, q = 1]$ are two perfect Bayesian equilibria.

Cheap talk games

- Cheap talk games (空谈博弈) are analogous to signaling games, but the Sender's messages are just talk, i.e., costless, non-binding, nonverifiable claims.
- Cheap talk cannot be informative in some cases (for example, Spence's job-market signaling model).
- There are situations where cheap talk can convey some information (although may not be fully precise), for example, Stein (1989), Matthews (1989), Austen-Smith (1990).
- In general, cheap talk can be informative under certain conditions.

Model

The timing of the simplest cheap talk game is identical to the timing of the simplest signaling game (only payoff functions differ):

1. Nature draws a type t_i for the Sender from a set of feasible types $T = \{t_1, \ldots, t_I\}$ according to a probability distribution $P(t_i)$, where $P(t_i) > 0$ for every i and $P(t_1) + \cdots + P(t_I) = 1$.
2. The Sender observes t_i and then chooses a message m_j from a set of feasible messages $M = \{m_1, \ldots, m_J\}$.
3. The Receiver observes m_j (but not t_i) and then chooses an action a_k from a set of feasible actions $A = \{a_1, \ldots, a_K\}$.
4. Payoffs are given by $U_S(t_i, a_k)$ and $U_R(t_i, a_k)$ (independent of m_j).

Preliminary

- The key feature of the cheap talk game is that the message has no direct effect on the payoffs of the Sender and the Receiver.
- The message can only be informative by changing the Receiver's belief about the Sender's type.
- Since anything can be said (i.e., M can be a very large set), it is typically assumed that $M = T$.
- The definition of perfect Bayesian equilibrium in a cheap talk game is identical to that in a signaling game.
- One key difference between these two games is that there always exists a pooling equilibrium in a cheap talk game.

Pooling equilibrium

- The following is a pooling equilibrium:

$$m^*(t_i) = t^*,\ \mu(t_i \mid m_j) = P(t_i),\ a^*(m_j) = a^*$$

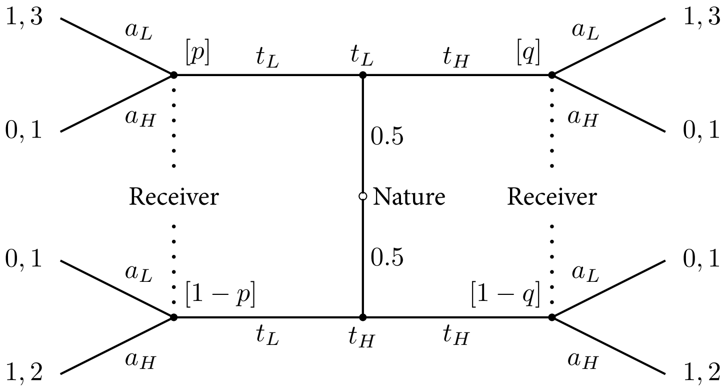
for all $t_i \in T$ and $m_j \in M$, where t^* is any message, and a^* solves

$$\max_{a_k \in A} \sum_{t_i \in T} P(t_i) U_R(t_i, a_k).$$

- In this pooling equilibrium, the Sender of all types sends the same message t^*, while the Receiver keeps the prior belief of all messages and takes an action optimally according to the belief.
- We call it a babbling equilibrium, which is not informative.
- An interesting question is whether there exists any non-pooling equilibrium in which communication can be effective.

Example

Find all pure-strategy perfect Bayesian equilibria of the following game.



Example (Cont.)

- Note that the above signaling game is indeed a cheap talk game, since neither the Sender's payoff nor the Receiver's payoff depends on the signals.
- Clearly, there are two pooling equilibria:

$$[(t_L, t_L), (a_L, a_L); p = \tfrac{1}{2}, q \geq \tfrac{1}{3}],$$

and

$$[(t_H, t_H), (a_L, a_L); p \geq \tfrac{1}{3}, q = \tfrac{1}{2}].$$

- There also exists a separating equilibrium:

$$[(t_L, t_H), (a_L, a_H); p = 1, q = 0].$$

Cheap talk game with two types and two actions

- Consider a two-type, two-action example:

$$T = \{t_L, t_H\},\ P(t_L) = p,\ A = \{a_L, a_H\},\ M = T.$$

- We use the following matrix to represent the payoffs: the first (second) number is the payoff to the Sender (Receiver).

	t_L	t_H
a_L	$x, 1$	$y, 0$
a_H	$z, 0$	$w, 1$

It is independent of messages.

- Note that the above matrix differs from the normal-form representation of the game.

Cheap talk game with two types and two actions (Cont.)

- Consider the following separating equilibrium:
  - the Sender's strategy: $[m^*(t_L) = t_L, m^*(t_H) = t_H]$;
  - the Receiver's beliefs: $\mu(t_L|t_L) = 1$ and $\mu(t_L|t_H) = 0$;
  - the Receiver's strategy: $[a^*(t_L) = a_L, a^*(t_H) = a_H]$.
- In the above equilibrium, each type of the Sender tells the truth.
- It can be shown that the separating equilibrium exists iff $x \geq z$ and $y \leq w$.
- In other words, the Sender's and the Receiver's interests perfectly align.
- In general, Crawford and Sobel (1982) have shown that more communication can occur through cheap talk when players' preferences are more closely aligned.