

Game Theory

Auction design 4: Keyword auction

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- 1 Introduction
- 2 Generalized second price auction (GSP)
 - Payoff flows and quality score
 - Pricing
 - Equilibrium under GSP
 - Generalized English auction (GEA)
- 3 VCG for internet ads
 - GEA vs. VCG
 - VCG vs. GSP
- 4 Summary

Introduction

- In the 1990's, websites started quickly to generate revenue from advertising.
- The original method (mid 90's) was to sell ad space the same way it is sold in **magazine, billboards**, etc:
 - An advertiser rents/buy some space on a Web page (a banner).
 - The price is for a fixed number of displays (i.e., a fixed number of visitors).
- At the end of the 90's–early 00's, this advertising model proved being unadapted:
 - Advertisers can **target better viewers** (using cookies).
 - With search engines advertisers can know **users' interests** in **real time**.

Issues

- On the internet things can change very quickly, i.e., search engines sell a flow of perishable advertising service, and capacity can be wasted (no ad displayed for a particular query).
- Also, ad prices can change almost continuously.
- But price based on what?
 - **What Google wants:** for showing the add.
 - **What the advertiser wants:** the user performs a transaction on the advertiser's Web page.
 - **Solution:** the user clicked and is redirected: **pay-per-click** (PPC).
Measure of success:

$$\text{click-through rate} = \frac{\# \text{ users clicking on the ad}}{\# \text{ users "viewing" the ad}}.$$

The current model for ads in search engines

- Internet user enters a search term (a “query”).
- Gets a page with results:
 - First: **Sponsored links** (the ads).
 - Second: Most relevant links (organic search results).
- If the user clicks on a sponsored link:
 - 1 Sent to the advertiser's Web page.
 - 2 The advertiser pays the search engine for sending the user.
- The **position** of the sponsored link does matter:
higher displayed links are clicked **more often** than links displayed lower on the page.

Huge market

- For Google, about 90–95% of its revenue comes from ads (Facebook, Twitter, etc: similar ratio).
- Most expensive keywords (those are maximum prices):

keyword	Price per click
Insurance	\$54
Loans	\$45
Mortgage	\$45
Attorney	\$45
Credit	\$35

- Smallest price: ¢5.

Origins

First service to start an auction for displayed ads with PPC: **GoTo** in 1997 (renamed **Overture** in 2001, bought by **Yahoo!** in 2003), with a **first-price auction**.

- Fast, in real-time (bidders could adjust their bid at any moment).
- Very popular: Yahoo!, MSN used Overture.
- Problem: fast changing bids made the system **unstable**.

Issues with GoTo's auction

- Bidders ranked according to their bid.
- Each bidder pays its bid (1st price auction).

With a 1st price auction, there may not be an equilibrium, a situation where nobody wants to change its bid:

- 3 bidders (1, 2 and 3), with values per click of \$10, \$4, \$2.
- 2 spots:
1st spot is much more valuable than 2nd spot (much more clicks).
- If $b_1, b_2 > 2$, there's a bidding war between bidders 1 and 2, until we reach \$4.
- When $b_1 > 4$, bidder 2 sets $b_2 = 2.01$ (to prevent bidder 3 to enter), but then the war starts again.

Additional issue

If bidder 1 uses a (fast) robot to bid, while bidders 2 and 3 are (slow) humans then:

- Bids are slightly above \$2.00 for a long period of time.
- Sellers' revenue (i.e., search engine revenue) are “low” (even if values for bidder 1 and 2 are **much** higher).

A single bid for multiple items

Users' click rate depends on the rank of the ad in the page:

- Each position can be seen as a different item.
- So search engines should run **multiple items auctions**: several items sold at the same time.

But bidders are asked to submit a **single** bid. Why?

- A higher ad is clicked more: bidders have the **same preferences** over positions.
- In general, the **value per click does not depend on the position**: The probability that the user purchases does not depend on the rank of the ad.
- This does not mean that the probability of purchase is the same across advertisers!

1 Introduction

2 Generalized second price auction (GSP)

- Payoff flows and quality score
- Pricing
- Equilibrium under GSP
- Generalized English auction (GEA)

3 VCG for internet ads

- GEA vs. VCG
- VCG vs. GSP

4 Summary

Generalized Second Price Auction

- Each advertiser place a bid.
- Bids are ranked:
 - Highest bidder is shown first.
 - 2nd highest bidder is shown second.
 - 3rd highest bidder is shown third.
 - etc.
- The k -th highest bidder pays the bid (when the user clicks) of the $(k + 1)$ -th bidder.
- There is always at least 1 more bidder than the number of links awarded. If there are only 3 bidders then only 2 sponsored links are displayed.

1 Introduction

2 Generalized second price auction (GSP)

- Payoff flows and quality score
- Pricing
- Equilibrium under GSP
- Generalized English auction (GEA)

3 VCG for internet ads

- GEA vs. VCG
- VCG vs. GSP

4 Summary

Payoff flows

- The Click-Through-Rate. It gives a **probability** of being clicked.
- \Rightarrow Bidders (advertisers) need to think in terms of **expectation** of users consuming the good or, put differently, in terms of **flows of payoffs**.
- Easier to think in terms of **click frequency**: how many clicks per unit of time (hour, day, week).

Example

- An advertiser values the click at \$4.
- Click frequency of position $A = 200$ clicks/hour.
- Click frequency of position $B = 50$ clicks/hour.
- Price/click for $A = \$3$.
- Price/click for $B = \$1$.

Advertiser's preferred position depends on whether she consider the click frequency:

- Net payoff **per click** higher with position B :

$$\$4 - \$3 < \$4 - \$1.$$

- Net payoff **per hour** higher with position A :

$$200 \times (\$4 - \$3) > 50 \times (\$4 - \$1).$$

Quality score

- In 2005, Google modified the rules of the GSP auction.
- Today, everybody does the same.

The problem:

- Google would like to charge as much as possible.
- Key observation: click frequency depends on advertiser.
- But if click frequency is low \Rightarrow Google's revenue are low:

Revenue with Advertiser <i>A</i>		Revenue with Advertiser <i>B</i>
\$1/click	>	\$50/click
with 1000 clicks/hour		with 10 clicks/hour
= \$1000/hour		= \$500/hour

- Google needs to find a way to rank *A* above *B*.

Quality score (Cont.)

- For each advertiser, Google determines a **quality score** that depends on
 - CTR;
 - ad relevance;
 - user experience (UX);
 - other things (secret sauce).
- Example: Search for “under-wears” (not just before Feb 14!).

Visited site	User is	
	a man	a woman
Hanes	High score	Low score
Victoria Secret	Low score	High score
Craig's list	Very low score	Very low score

1 Introduction

2 Generalized second price auction (GSP)

- Payoff flows and quality score
- **Pricing**
- Equilibrium under GSP
- Generalized English auction (GEA)

3 VCG for internet ads

- GEA vs. VCG
- VCG vs. GSP

4 Summary

Pricing with quality scores

- For each bidder:

$$\text{Final score} = \text{Bid} \times \text{Quality score}.$$

- Advertisers are ranked according to the final score.
- Charge the k -th advertiser the **lowest price/click** p such that

$$p \times \text{quality score} > \text{final score of } (k + 1)\text{-th bidder}.$$

Pricing with quality scores (Cont.)

3 bidders (Pim, Pam, Poum), compete for 2 spots.

Bidder	Pim	Pam	Poum
Bids	\$6	\$4	\$2
Quality Score	2	4	1
Final score	12	16	2
Ranking	2nd	1st	3rd
Price/click	\$1.01	\$3.01	\$0

- Pam just needs a score higher than 12 to win against Pim.
 - If she bids \$3.01 then final score is 12.04.
 - So the price for Pam is \$3.01 per click.
- Pim just needs a score higher than 2 to win against Poum.
 - If he bids \$1.01 then final score is 2.02.
 - So the price for Poum is \$1.01 per click.
- Poum pays 0, ad not displayed.

1 Introduction

2 Generalized second price auction (GSP)

- Payoff flows and quality score
- Pricing
- **Equilibrium under GSP**
- Generalized English auction (GEA)

3 VCG for internet ads

- GEA vs. VCG
- VCG vs. GSP

4 Summary

Truthtelling

From now on, simple GSP **without** quality scores.

Proposition

Truthtelling is not a dominant strategy under GSP (w/o quality scores).

- 3 bidders (A, B and C) compete for 2 spots.
- Values are $v_A = \$10$, $v_B = \$4$ and $v_C = \$2$.
- Click frequency: 200 clicks/hour for the 1st spot, 199 clicks/hour for the 2nd spot.
- If bidders bid truthfully, then:
 - 1st spot awarded to A, pays \$4/click (B's bid);
 - 2nd spot awarded to B, pays \$2/click (C's bid).
- Payoff A: $(\$10 - \$4) \times 200 = \$1,200$.
- If A bids instead \$3, she gets the 2nd position and payoff is $(\$10 - \$2) \times 199 = \$1,592$.

Equilibrium under GSP

- There is no dominant strategy under GSP.
⇒ we need to look at the equilibria.
- But bids change very frequently.
- We then have an (infinitely) repeated game, where:
 - Advertisers initially only know their value (but not the valuations of others).
 - Gradually learn the value of others.
- Problem: analysis of repeated game with incomplete information can become very complex and difficult.

Simplifying assumptions

- Easier way to analyze the outcome of the auction:
If bids stabilize after some time, what bids can we observe?
- Simplifying assumptions:
 - Valuations are commonly known: after some time, advertisers learn all the relevant information.
 - Bids can change at any time: stable bids must be best-responses to each other.
⇒ stable bids must form a Nash equilibrium of the simultaneous-move, one shot game of complete information.
 - But it's not enough, we want to capture the dynamic aspect. We'll use "simple strategies."

Simple strategy

A **simple strategy** is to try to force out the bidder who is just above me.

- Suppose I bid b and I have the k -th position.
- Opponent bids b' and has the $(k - 1)$ -th position.
- If I raise **slightly** my bid (i.e., ranking not affected) then:
 - my payoff doesn't change
 - but my opponent's payoff is affected (it decreases).
- Opponent can retaliate and bids slightly lower so that we swap our ranks.
- If I'm better off after such retaliation, I will decide to increase my bid.
- If I'm worse off after such retaliation, I don't change my bid.
- \Rightarrow If bids are "stable," no bidder want to exchange his/her position with the bidder just above him/her.

Long run equilibrium

Notation:

- $\alpha_i = \#$ clicks frequency for position i .
- $v_j =$ valuation per click for advertiser j .
- $g(i) =$ identity of the bidder in position i .
- $p_i =$ payment per period for advertiser in position i .

Definition

An **equilibrium** of the simultaneous-move game induced by GSP is **locally envy-free** if a player cannot improve her payoff by exchanging bids with the bidder one position above her:

$$\underbrace{\alpha_i v_{g(i)} - p_i}_{\text{Payoff of bidder ranked } \#i} \geq \underbrace{\alpha_{i-1} v_{g(i)} - p_{i-1}}_{\text{Payoff of bidder ranked } \#i-1},$$

where $p_{i-1} = \alpha_{i-1} \times b_i$.

Nash equilibrium vs. envy-freeness

Nash equilibrium of the one-shot bidding game and envy-freeness are **different** conditions.

- **Nash equilibrium:** If advertiser h deviates and takes the position of bidder h' then:

- h pays the **same price** as h' if h' is at a **lower position** (payment depends on the bidder below h').
- h pays a **different price** than h' if h' is at a **higher position**:
The price h pays after deviating is the bid of h' . Before h' was paying the bid of the advertiser below her.

- **Envy-freeness:**

Advertiser h want to take the position of h' and **pay the same price** h' was paying.

Long run equilibrium

So the long run equilibria of the GSP auctions are given two conditions:

- Nash equilibrium of the simultaneous-move game (one shot, not repeated).
- Local envy-freeness condition.

Such equilibria may be difficult to characterize. But we'll see that they are in fact equivalent to **stable assignments** (which are easier to describe).

Stable assignments

Consider a market between the positions and the advertisers.

- Let Mr. i is in charge of the i -th position.
- Mr. i 's objective is to maximize profit: the price charged to the advertiser.
- If the i -th position is assigned to advertiser A :
 - Mr. i 's net payoff = p_i .
 - Advertiser's net payoff = $\alpha_i \times v_A - p_i$.
 - Sum of payoffs = $p_i + (\alpha_i \times v_A - p_i) = \alpha_i \times v_A$.
 - \Rightarrow Any "deal" between Mr. i and advertiser A consist of a division of $\alpha_i \times v_A$.

Stable assignments (Cont.)

Definition

An assignment μ together with the prices p is **stable** if there does **not** exist an advertiser i and a position k such that

- $\mu(i) \neq k$ (k not assigned to i).
- $\alpha_k \times v_i > (i\text{'s net payoff under } \mu) + (k\text{'s net payoff under } \mu)$.
 - The right-hand side is the minimum amount needed for i and k to be better off.
 - The left-hand side is the available amount if assigned together.
- Stable assignments are known to be a fundamental property in real-life markets (that involve assignments).
- Markets that do not produce stable assignments tend to perform poorly or collapse.

Example

position	advertiser	price	click frequency	valuation
1st	i	6	200	15
2nd	j	5	150	13

- Value to be shared between 1st and $i = 200 \times 15 = 3,000$.
- 1st position's payoff = $6 \times 200 = 1,200$.
- Value to be shared between 2nd and $j = 150 \times 13 = 1,950$.
- j 's payoff = $150 \times (13 - 5) = 1,200$.

But j and 1st together can generate $13 \times 200 = 2,600$! For instance, with a price of 6.5 they get

- 1st: $200 \times 6.5 = 1,300 > 1,200$.
- j : $200 \times (13 - 6.5) = 1,300 > 1,200$.

Equilibrium vs. stable assignments

Long run equilibria (of the GSP) and stable assignments obey to two different logic:

- **Equilibria:**

Only the bidders deviate. If a bidder deviates and takes a new position she does not need to ask that position's permission.

- \Rightarrow Positions are mere objects, they don't have the right to an opinion.
- \Rightarrow A bidder deviates if she is better off. It does not matter if the position gets lower revenue.

- **Stability:**

If a bidder wants a different position, that will be possible only if the position agrees.

- It's not sufficient that the advertiser is better off at the new position.
- The new position also needs to be better off.

Equilibrium vs. stable assignments (Cont.)

Proposition

- The outcome of any locally envy-free equilibrium in the GSP auction is a stable assignment.
- Furthermore, if there are more advertisers than positions then any stable assignment is the outcome of a locally envy-free equilibrium.

Proof: envy-freeness \Rightarrow stability

- 1 Suppose we have an envy-free equilibrium and that p_1, \dots, p_n are that payments received by positions $1, \dots, n$.
- 2 For notational simplicity, assume that in equilibrium advertiser h at position h (for all $h \geq 1$).
- 3 Take an advertiser assigned to position k , looking at position h .

We want to show that:

Nash equilibrium + Envy-free \Rightarrow Stable assignment.

- 4 How do we prove that an assignment is stable?

Claim

Claim: If for any h and k the equilibrium is such that

$$\alpha_k v_k - p_k \geq \alpha_h v_k - p_h, \quad (1)$$

then it corresponds to a stable assignment.

Proof.

① p_h = Position h 's payoff.

$\alpha_h v_k$ = Size of the surplus to be shared between position h and advertiser k (if matched together).

② $\alpha_h v_k - p_h$ is advertiser's k **maximal payoff** she can hope if position h agrees to take her (instead of advertiser h):

③ But Eq. (1) implies

k 's payoff at position $k > \max.$ payoff can hope with pos. h .

④ \Rightarrow there is no way that h and k can be better off together. So the assignment is stable.

Proof strategy

- 1 We start with an equilibrium (Nash equilibrium + locally envy-freeness).
- 2 We take any advertiser k assigned to position k . **At the equilibrium** k does not want to **change the bid** and obtain position h :
 k would get a lower payoff if she changes her bid to get position h :

$$\underbrace{\alpha_k v_k - p_k}_{k\text{'s equilibrium payoff}} \geq \underbrace{\alpha_h v_k - \hat{p}_h}_{k\text{'s payoff if deviates}} . \quad (2)$$

(can do that bidding b such that $b_{h+1} < b < b_h$)

- 3 We'll show that (2) gives the stable assignment condition (1).

Case 1: advertiser k & position $h > k$

- We assumed that the equilibrium of repeated game is also a Nash equilibrium of the one-shot game, so we have

$$\underbrace{\alpha_k v_k - p_k}_{k\text{'s equilibrium payoff}} \geq \underbrace{\alpha_h v_k - \hat{p}_h}_{k\text{'s payoff if deviates}} .$$

- But payment to position h depends on bid of advertiser $h + 1$. Since advertiser h does not change her bid (only advertiser h),

$$\underbrace{\hat{p}_h}_{\text{what } k \text{ would pay if deviates}} = \underbrace{p_h}_{\text{what } h \text{ was paying}} .$$

- So we can rewrite we get:

$$\alpha_k v_k - p_k > \alpha_h v_k - p_h .$$

- That's the stability condition (1), what we wanted.

Case 2: advertiser k & position $k - 1$

- Let's rewrite the stability condition (1) replacing h by $k - 1$ to see what we need to show:

$$\alpha_k v_k - p_k \geq \alpha_{k-1} v_k - p_{k-1}. \quad (3)$$

So we need to show that the equilibrium conditions are such that (3) holds true.

- But that's the condition of envy-freeness! The definition of long run equilibrium requires envy-freeness. So we're done.

Now let's do the same for positions $k - 2, k - 3, \dots, 1$.

Case 3: advertiser k & position $m < k - 1$

Claim

Equilibrium \Rightarrow **Assortative match** (bidders ranked by their valuations),
i.e., $v_k \geq v_{k+1}$ for all k .

Proof.

- ① Nash Equilibrium condition: nobody wants to move one position down:

$$\alpha_k v_k - p_k \geq \alpha_{k+1} v_k - p_{k+1}. \quad (4)$$

- ② Envy-freeness: nobody wants to move one position up:

$$\alpha_{k+1} v_{k+1} - p_{k+1} \geq \alpha_k v_{k+1} - p_k. \quad (5)$$



Proof of Claim

Proof.

- ③ Add (4) and (5) and we get

$$\alpha_k v_k + \alpha_{k+1} v_{k+1} \geq \alpha_{k+1} v_k + \alpha_k v_{k+1}$$

iff

$$v_k(\alpha_k - \alpha_{k+1}) \geq v_{k+1}(\alpha_k - \alpha_{k+1}).$$

- ④ Since $\alpha_k > \alpha_{k+1}$ (higher position = more clicks), we have

$$v_k \geq v_{k+1},$$

claim is proved!



Case 3: advertiser k & position $m < k - 1$ (Cont.)

Now, suppose advertiser k and position $m < k - 1$ want to rematch.
Since equilibrium locally envy-free:

$$\begin{aligned}\alpha_k v_k - p_k &\geq \alpha_{k-1} v_k - p_{k-1}, \\ \alpha_{k-1} v_{k-1} - p_{k-1} &\geq \alpha_{k-2} v_{k-1} - p_{k-2}, \\ \alpha_{k-2} v_{k-2} - p_{k-2} &\geq \alpha_{k-3} v_{k-2} - p_{k-3}, \\ &\vdots \\ \alpha_{m+2} v_{m+2} - p_{m+2} &\geq \alpha_{m+1} v_{m+2} - p_{m+1}, \\ \alpha_{m+1} v_{m+1} - p_{m+1} &\geq \alpha_m v_{m+1} - p_m.\end{aligned}$$

Case 3: advertiser k & position $m < k - 1$ (Cont.)

Observe that for any $h > 1$,

$$\underbrace{\alpha_h v_h - p_h \geq \alpha_{h-1} v_h - p_{h-1}}_{\text{envy-freeness condition}} \Rightarrow v_h \leq \frac{p_{h-1} - p_h}{\alpha_{h-1} - \alpha_h}.$$

Since $\alpha_h < \alpha_{h-1}$ for any h , and since $v_k < v_h$ for any $h < k$, we can replace v_h by v_k in the second inequality and we get for any $h = 2, \dots, k - 1$,

$$v_k \leq \frac{p_{h-1} - p_h}{\alpha_{h-1} - \alpha_h} \text{ and } \alpha_h v_k - p_h \geq \alpha_{h-1} v_k - p_{h-1}.$$

So $j = m, \dots, k - 1$, from

$$\alpha_j v_j - p_j \geq \alpha_{j-1} v_j - p_{j-1},$$

one can obtain

$$\alpha_j v_k - p_j \geq \alpha_{j-1} v_k - p_{j-1}.$$

Case 3: advertiser k & position $m < k - 1$ (Cont.)

So we can rewrite the equations replacing v_j by v_k for $j = m, \dots, k - 1$.

$$\begin{aligned}
 \alpha_k v_k - p_k &\geq \alpha_{k-1} v_k - p_{k-1}, \\
 \alpha_{k-1} v_k - p_{k-1} &\geq \alpha_{k-2} v_k - p_{k-2}, \\
 \alpha_{k-2} v_k - p_{k-2} &\geq \alpha_{k-3} v_k - p_{k-3}, \\
 &\vdots \\
 \alpha_{m+2} v_k - p_{m+2} &\geq \alpha_{m+1} v_k - p_{m+1}, \\
 \alpha_{m+1} v_k - p_{m+1} &\geq \alpha_m v_k - p_m.
 \end{aligned}$$

Adding all these inequalities yields

$$\alpha_k v_k - p_k \geq \alpha_m v_k - p_m.$$

We're done! Envy-free \Rightarrow stable.

1 Introduction

2 Generalized second price auction (GSP)

- Payoff flows and quality score
- Pricing
- Equilibrium under GSP
- Generalized English auction (GEA)

3 VCG for internet ads

- GEA vs. VCG
- VCG vs. GSP

4 Summary

Generalized English Auction

- The Vickrey auction is a one-shot/simultaneous version of the English auction.
- Can we define a “generalized English auction” so that GSP would be its one-shot/simultaneous version?

Rules

- There is a clock showing the current price, which increases over time.
- Start: price = 0, all advertisers are in the auction.
- Advertiser can drop at any time. Their bid is the price on the clock at the time they drop out.
- Auction over when the next-to-last advertiser drops out.

Outcome

- Last advertiser ranked 1st, all others ranked according to the time they dropped out (the latest, the higher).
- Each advertiser pays the bid of the advertiser ranked just below him/her.

Generalized English Auction

Proposition

There is a unique (perfect Bayesian) equilibrium of the generalized English auction, where an advertiser with valuation v drops out at the price

$$p^* = v - \frac{\alpha_i}{\alpha_{i-1}}(v - b_{i+1}),$$

where $i = \#$ advertiser remaining (including him/her).

These prices imply that the payoffs in the generalized English auction are the same as in the VCG auction. (see later)

Intuition

- Suppose there are i bidders remaining (including me), and the next highest bid is b_{i+1} .
- The next bidder who drops out will pay b_{i+1} . If I'm the next to drop out, my payoff is

$$\alpha_i \times (v - b_{i+1}).$$

- If I wait a bidder to drop out (at price p) and drop out just after, my payoff is

$$\alpha_{i-1} \times (v - p).$$

Intuition (Cont.)

We have $\alpha_{i-1} > \alpha_i$, so if $p = b_{i+1}$ then

$$\alpha_i \times (v - b_{i+1}) < \alpha_{i-1} \times (v - p).$$

- But the right hand side decreases in p .
- So there will be a price p^* such that waiting and not waiting gives the same payoff:

$$\alpha_i \times (v - b_{i+1}) = \alpha_{i-1} \times (v - p^*)$$

iff

$$p^* = v - \frac{\alpha_i}{\alpha_{i-1}}(v - b_{i+1}).$$

- 1 Introduction
- 2 Generalized second price auction (GSP)
 - Payoff flows and quality score
 - Pricing
 - Equilibrium under GSP
 - Generalized English auction (GEA)
- 3 VCG for internet ads**
 - GEA vs. VCG
 - VCG vs. GSP
- 4 Summary

1 Introduction

2 Generalized second price auction (GSP)

- Payoff flows and quality score
- Pricing
- Equilibrium under GSP
- Generalized English auction (GEA)

3 VCG for internet ads

- GEA vs. VCG
- VCG vs. GSP

4 Summary

Comparing GEA and VCG

Proposition

Equilibrium payoffs of GEA = payoffs of VCG.

\Rightarrow GEA = sequential version of VCG.

Proof.

- Suppose $v_A > v_B > v_C > v_D$, 3 links at most.
 - 4 bidders \Rightarrow 3 links,
 - 3 bidders \Rightarrow 2 links.



Proof

- VCG \Rightarrow bidders truthful \Rightarrow assortative assignment.
If assignment not assortative then social value is not maximized.
- Take bidder B .
- Social welfare of others if bidder B present:

$$\alpha_1 v_A + \alpha_3 v_C.$$

- Social welfare if bidder B NOT present:

$$\alpha_1 v_A + \alpha_2 v_C.$$

- Price for B :

$$\alpha_1 v_A + \alpha_2 v_C - \alpha_1 v_A - \alpha_3 v_C = v_C(\alpha_2 - \alpha_3).$$

Proof (Cont.)

B 's price with GEA:

- Not difficult to show that C (D) drops out before B (D).
- Price of B is bid of C (the bidder who left just before).
- Bid of C depends on bid of D . Bid of $D = 0$.

$$bid_C = v_C - \frac{\alpha_3}{\alpha_2}(v_C - 0).$$

- To get the price **per period** for B we multiply by α_2 :

$$\text{price per period for } B = \alpha_2 \left(v_C - \frac{\alpha_3}{\alpha_2} v_C \right) = v_C (\alpha_2 - \alpha_3).$$

English auction \longrightarrow 2nd-price auction/Vickrey auction

Generalized English auction \longrightarrow VCG/generalized Vickrey auction

1 Introduction

2 Generalized second price auction (GSP)

- Payoff flows and quality score
- Pricing
- Equilibrium under GSP
- Generalized English auction (GEA)

3 VCG for internet ads

- GEA vs. VCG
- VCG vs. GSP

4 Summary

Comparing VCG and GSP

The envy-freeness condition written for advertiser $i + 1$ is

$$v_{i+1}\alpha_{i+1} - p_{i+1} \geq \alpha_i v_{i+1} - p_i,$$

which can be re-written as

$$p_i \geq v_{i+1}(\alpha_i - \alpha_{i+1}) + p_{i+1}.$$

Left-hand side is thus a **lower bound** (per period) of the **revenue of position i** .

- Take 4 bidders, 3 slots, $v_A > v_B > v_C > v_D$.

$$p_1 \geq v_B(\alpha_1 - \alpha_2) + p_2 \quad (6)$$

$$p_2 \geq v_C(\alpha_2 - \alpha_3) + p_3 \quad (7)$$

$$p_3 \geq v_D\alpha_3. \quad (8)$$

- Add (6), (7) and (8) and we get

$$p_1 \geq v_B(\alpha_1 - \alpha_2) + v_C(\alpha_2 - \alpha_3) + v_D\alpha_3.$$

Lower bound for revenue of position 1.

- Add (7) and (8) and we get

$$p_2 \geq v_C(\alpha_2 - \alpha_3) + v_D\alpha_3.$$

Lower bound for revenue of position 2.

- (8) is the lower bound for revenue of position 3.

Now take the VCG auction.

- Payment for bidder A (to get 1st position):

$$\begin{aligned}
 p_1 &= \underbrace{v_B\alpha_1 + v_C\alpha_2 + v_D\alpha_3}_{\text{max social value when } A \text{ not here}} - \underbrace{(v_B\alpha_2 + v_C\alpha_3)}_{\text{social value of others when } A \text{ here}} \\
 &= v_B(\alpha_1 - \alpha_2) + v_C(\alpha_2 - \alpha_3) + v_D\alpha_3.
 \end{aligned}$$

- Payment for bidder B (to get 2nd position):

$$p_2 = v_C(\alpha_2 - \alpha_3) + v_D\alpha_3.$$

- Payment for bidder C (to get 3rd position):

$$p_3 = v_D\alpha_3.$$

Comparing VCG and GSP

So, we have

Revenue with GSP \geq Revenue with VCG.

Why does Facebook uses VCG?

- On Facebook more uncertainty on CTR. Using VCG means advertisers spend more time figuring out the value of their valuations.
Life simpler for advertisers.
- Revenue lower for Facebook \Rightarrow revenue higher for advertisers.
In the long run, advertisers may prefer Facebook.
- Few ads per page, computational issue disappears.

- 1 Introduction
- 2 Generalized second price auction (GSP)
 - Payoff flows and quality score
 - Pricing
 - Equilibrium under GSP
 - Generalized English auction (GEA)
- 3 VCG for internet ads
 - GEA vs. VCG
 - VCG vs. GSP
- 4 Summary

Take-away

- Most ads on the internet are allocated through an auction.
- Bidders have a valuation per click, but make decisions taking into account the **click-through rate** (click frequency).
- A popular format is the **Generalized Second-Price** auction:
 - Advertisers place one bid for a spot on the webpage.
 - Advertisers allocated spots so that.
$$\text{bid rank} = \text{CTR rank of spot on the page.}$$
 - Bidders pay per click the bid of the bidder rank just below them.
- Truthful bidding is **not a dominant strategy** with the GSP.

Take-away (Cont.)

- Long run equilibrium of the repeated GSP game yield **stable assignments**.
- Website's revenue are higher with the GSP than with the VCG.
- In practice, websites multiply advertisers with a **quality score**: it maximizes the **payoff flows** for the website.
- Some websites (e.g., Facebook) use the VCG auction.