Game Theory

Social choice

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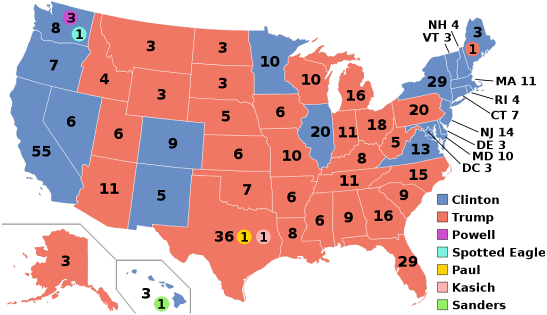
Voting

- Voting seems very simple.
- But sometimes things go wrong.
  - In 2000, the US presidential election came down to Florida.
  - George Bush won by 537 votes.
  - But Ralph Nader got 97,421 votes. Twice as many Nader voters would have chosen Gore over Bush.

Voting

2016 US presidential election:



	Donald Trump	Hillary Clinton
Electoral vote	304	227
Popular vote	62,984,828	65,853,514
Percentage	46.1%	48.2%

Setting

Our setting now:

- a set of outcomes or alternatives,
- agents have preferences over them,
- the 'goal': a social choice function: a mapping from profiles of preferences to a particular outcome.
    - Which such functions have desirable properties?

Preferences

- Given is a finite set of outcomes or alternatives O.
- Agents have (strict) preferences, $\succ$, over the outcomes: linear orders (or total orders).
- Linear orders $\mathcal{L}$: binary relations $\succ$ that are total and transitive:
  - total: for every pair of outcomes $a \neq b$ either $a \succ b$ or $b \succ a$ (but not both: so it is complete and antisymmetric).
  - transitive: $a \succ b$ and $b \succ c$ implies $a \succ c$.
- Weak preferences $\mathcal{L}_w$: binary relations $\succsim$ that are complete and transitive:
  - complete: for every a and b either $a \succsim b$ or $b \succsim a$ (both indicates indifference).
  - transitive: $a \succsim b$ and $b \succsim c$ implies $a \succsim c$.

Formal model

Given is a set of agents $N = \{1, 2, \ldots, n\}$, a finite set of outcomes (or alternatives, or candidates) O, and the set of preferences over outcomes, $\mathcal{L}$.

Definition (Social choice function)

A social choice function is a function $C\colon \mathcal{L}^n \to O$.

Definition (Social welfare function)

A social welfare function is a function $C\colon \mathcal{L}^n \to \mathcal{L}$.

Voting schemes: Scoring rules

- Plurality (多数决)
  - pick the outcome which is most-preferred by the most people.
- Cumulative voting (累积投票制)
  - distribute e.g., 5 votes each.
  - possible to vote for the same outcome multiple times.
- Approval voting (认可投票制)
  - vote for as many outcomes as you "like".

Voting schemes based on ranking

- Plurality with elimination (“instant runoff”, “transferable voting” 带消除的多数决)
  - if some outcome has a majority, it is the winner.
  - otherwise, the outcome with the fewest votes is eliminated (may need some tie-breaking procedure).
  - repeat until there is a winner.
- Borda Rule, Borda Count (波达计数法)
  - assign each outcome a number.
  - the most preferred outcome gets a score of $n-1$, the next most preferred gets $n-2$, down to the n-th outcome which gets 0.
  - sum scores for each outcome, and choose one with highest score.
- Successive elimination (连续淘汰制)
  - in advance, decide an ordering of alternatives.
  - everyone votes for the first or second, and the loser is eliminated.
  - then vote for winner vs third alternative, and loser is eliminated.
  - continue until the last alternative is considered.

Condorcet consistency

- If there is a candidate or outcome that is preferred to every other candidate in pairwise majority-rule comparisons, that candidate should be chosen.
- There is not always a Condorcet winner.
- Sometimes, there is a cycle where A defeats B, B defeats C, and C defeats A, known as a Condorcet cycle.

Condorcet example

$$499 \text{ agents: } A \succ B \succ C$$
$$3 \text{ agents: } B \succ C \succ A$$
$$498 \text{ agents: } C \succ B \succ A$$

- What is the Condorcet winner? B
- What would win under plurality voting? A
- What would win under plurality with elimination? C

Sensitivity to losing candidate

35 agents: $A \succ C \succ B$

33 agents: $B \succ A \succ C$

32 agents: $C \succ B \succ A$

- What candidate wins under plurality voting? A
- What candidate wins under Borda voting? A
- Now consider dropping C. Now what happens under both Borda and plurality? B wins.

Sensitivity to agenda setter

35 agents: $A \succ C \succ B$

33 agents: $B \succ A \succ C$

32 agents: $C \succ B \succ A$

- Who wins pairwise elimination, with the ordering A, B, C? C
- Who wins with the ordering A, C, B? B
- Who wins with the ordering B, C, A? A

Another pairwise elimination problem

$$1 \text{ agent: } B \succ D \succ C \succ A$$
$$1 \text{ agent: } A \succ B \succ D \succ C$$
$$1 \text{ agent: } C \succ A \succ B \succ D$$

- Who wins under pairwise elimination with the ordering A, B, C, D? D.
- What is the problem with this?
  - all of the agents prefer B to D—the selected candidate is Pareto-dominated!

Notation

- N is the set of agents.
- O is a finite set of outcomes with $|O| \geq 3$.
- $\mathcal{L}$ is the set of all possible strict preference orderings over O.
  - for ease of exposition we switch to strict orderings
  - we will end up showing that desirable SWFs cannot be found even if preferences are restricted to strict orderings
- $(\succ_i)_{i \in N}$ is an element of the set $\mathcal{L}^n$ (a preference ordering for every agent; the input to our social welfare function)
- $\succ_W$ (or simply $\succ$) is the preference ordering selected by the social welfare function W.

PE and IIA

Definition

W is Pareto efficient if for any $o_1, o_2 \in O$, for each $i \in N$, $o_1 \succ_i o_2$ implies that $o_1 \succ o_2$.

When all agents agree on the ordering of two outcomes, the social welfare function must select that ordering.

Definition

W is independent of irrelevant alternatives (IIA) if, for any $o_1, o_2 \in O$ and any two preference profiles $(\succ_i')_{i \in N}, (\succ_i'')_{i \in N} \in \mathcal{L}^n$, "for each i, $o_1 \succ_i' o_2$ if and only if $o_1 \succ_i'' o_2$" implies that "$o_1 \succ' o_2$ if and only if $o_1 \succ'' o_2$.

The selected ordering between two outcomes should depend only on the relative orderings they are given by the agents.

Nondictatorship

Definition

W does not have a dictator if there is no agent i such that for any o_1 and o_2, $o_1 \succ_i o_2$ implies $o_1 \succ o_2$.

- There does not exist a single agent whose preferences always determine the social ordering.
- We say that W is dictatorial if it fails to satisfy this property.

Theorem (Arrow, 1951)

Any social welfare function W that is Pareto efficient and independent of irrelevant alternatives is dictatorial.

Social choice functions

- Maybe Arrow's theorem held because we required a whole preference ordering.
- Idea: Social choice functions might be easier to find.
- We'll need to redefine our criteria for the social choice function setting; PE and IIA discussed the ordering.

Criteria

- A social choice function C is weakly Pareto efficient if it never selects an outcome o_2 when there exists another outcome o_1 such that for each i, $o_1 \succ_i o_2$.
    - A dominated outcome can't be chosen.
- C is monotonic if, for any $o \in O$ and any preference profile $(\succ_i) \in \mathcal{L}^n$ with $C\big((\succ_i)_i\big) = o$, then for any other preference profile $(\succ_i')$ with the property that for each $i \in N$, for each $o' \in O$, $o \succ_i' o'$ if $o \succ_i o'$, it must be that $C\big((\succ_i')_i\big) = o$.
    - An outcome o must remain the winner whenever the support for it is increased in a preference profile under which o was already winning.
- C is dictatorial if there exists an agent j such that C always selects the top choice in j's preference ordering.

Muller and Satterthwaite

Theorem (Muller and Satterthwaite, 1977)

Any social choice function that is weakly Pareto efficient and monotonic is dictatorial.

- Perhaps contrary to intuition, social choice functions are no simpler than social welfare functions after all.
- The proof repeatedly "probes" a social choice function to determine the relative social ordering between given pairs of outcomes.
- Because the function must be defined for all inputs, we can use this technique to construct a full social welfare ordering.

Plurality

- Plurality satisfies weak PE and ND, so it must not be monotonic.
- Consider the following preferences:

$$\begin{aligned}3 \text{ agents: } & a \succ b \succ c\\ 2 \text{ agents: } & b \succ c \succ a\\ 2 \text{ agents: } & c \succ b \succ a\end{aligned}$$

  Plurality chooses a.
- Increase support for a by moving c to the bottom:

$$\begin{aligned}3 \text{ agents: } & a \succ b \succ c\\ 2 \text{ agents: } & b \succ c \succ a\\ 2 \text{ agents: } & b \succ a \succ c\end{aligned}$$

  Now plurality chooses b.

Single-peaked preferences

- Sometimes voters' preferences have nicer properties.
- Prominent case: candidates can be ordered from left to right.
- Voters: have a most-preferred candidate and then candidates who are more extreme are less-preferred.