

Game Theory

Cooperative games

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Cooperative game

- In game theory, a cooperative game (or coalitional game 合作博弈) is a game with **competition between groups of players** (“coalitions”) due to the possibility of **external enforcement of cooperative behavior** (e.g. through contract law).
- Those are opposed to non-cooperative games in which there is either **no possibility to forge alliances** or **all agreements need to be self-enforcing** (e.g. through credible threats).

Cooperative game theory

- Given a set of agents, a coalitional game defines how well each group (or coalition) of agents can do for itself.
- We are not concerned with:
 - how the agents make individual choices within a coalition;
 - how they coordinate;

Instead, we take the payoffs to a coalition as given.

- Cooperative games are often analyzed through the framework of cooperative game theory, which focuses on predicting **which coalitions will form, the joint actions that groups take and the resulting collective payoffs**.
- It is opposed to the traditional non-cooperative game theory which focuses on predicting **individual players' actions and payoffs** and analyzing **equilibria**.

- 1 Setup
- 2 Shapley value
 - Examples
- 3 Core
 - Properties
 - Examples
 - Nonemptiness of core
- 4 Nash bargaining solution

Coalitional game

- A **coalitional game** (合作博弈) with transferable payoff (henceforth “coalitional game”) $\langle N, v \rangle$ consists of
 - a finite set N of **players**,
 - a **function** $v: 2^N \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$.
- Every subset S of N is called a **coalition** (联盟), and $v(S)$ is called the worth of the coalition S .

The function v is called the **characteristic function** (特征函数).

- Interpretation: $v(S)$ is a payoff that may be distributed in any way among the members of S .
- Convention: If $\{i_1, i_2, \dots, i_j\}$ is a coalition, we will write

$$v(i_1, i_2, \dots, i_j) = v(\{i_1, i_2, \dots, i_j\}).$$

Key questions

- Questions we use coalitional game theory to answer:
 - 1 Which coalition will form?
 - 2 How should that coalition divide its payoff among its members?
 - in order to be fair.
 - in order to be stable.
 - etc.
- The answer to (1) is often “the grand coalition” (all agents in N), though this can depend on having made the right choice about (2).

Super-additivity

- A coalitional game $\langle N, v \rangle$ is **super-additive** (超可加) if

$$S \cap T = \emptyset \text{ implies } v(S \cup T) \geq v(S) + v(T).$$

- Super-additivity is justified when coalitions can always work **without interfering with one another**.
 - The value of two coalitions will be no less than the sum of their individual values.
 - It implies that the **grand coalition** has the highest payoff.
- Assumption: The coalitional games are super-additive.

Super-additivity (Cont.)

- A coalitional game $\langle N, v \rangle$ is
 - **monotonic** (单调的) if $T \subseteq S$ implies $v(S) \geq v(T)$;
 - **cohesive** (内聚的) if

$$v(N) \geq \sum_{k=1}^K v(S_k) \text{ for every partition } \{S_1, S_2, \dots, S_K\} \text{ of } N;$$

- Super-additive \implies Monotonic and Cohesive.

- 1 Setup
- 2 Shapley value
 - Examples
- 3 Core
 - Properties
 - Examples
 - Nonemptiness of core
- 4 Nash bargaining solution

Axiomatic characterization

Question

What is a “fair” way for a coalition to divide its payoff?

- This depends on how we define “fairness.”
- One approach: Identify **axioms** that express properties of a fair payoff division.

Shapley value

- A coalition of players cooperates, and obtains a certain overall gain from that cooperation.
- Since some players may **contribute more** to the coalition than others or may possess **different bargaining power** (for example threatening to destroy the whole surplus), what final distribution of generated surplus among the players should arise in any particular game?
- Or phrased differently: **How important** is each player to the overall cooperation, and what payoff can he or she reasonably expect?
- Shapley's idea: Members should receive payments or shares **proportional to their marginal contributions** (边际贡献).

Substitutes and null player

- Two players i and j are **substitutes** (替代) in v if for all S containing neither i nor j ,

$$v(S \cup \{i\}) = v(S \cup \{j\}).$$

- Player $i \in N$ is called a **null player** if

$$v(S \cup \{i\}) = v(S) \text{ for all } S \subseteq N.$$

Shapley value

Given a coalitional game $\langle N, v \rangle$ where $N = \{1, 2, \dots, n\}$, the **Shapley value** (沙普利值) is an n -vector, denoted by

$$\phi(v) = (\phi_1(v), \phi_2(v), \dots, \phi_n(v)),$$

satisfying the following conditions:

- S1.** Symmetry condition: if i and j are substitutes in v , then $\phi_i(v) = \phi_j(v)$.
- S2.** Null player condition: if i is a null player, then $\phi_i(v) = 0$.
- S3.** Efficiency condition: $\sum_{i \in N} \phi_i(v) = v(N)$.
- S4.** Additivity condition: $\phi_i(v + w) = \phi_i(v) + \phi_i(w)$.

$\phi_i(v)$ is interpreted as the **power of player i** in the coalitional game $\langle N, v \rangle$, or **what it is worth to i** to participate in the game $\langle N, v \rangle$.

Interpretation

- S1. Substitutable agents should receive the same shares/payments.
- S2. Null players should receive nothing.
- S4. If we can separate a game into two parts $u = v + w$, then we should be able to decompose the payments.

Shapley theorem (Cont.)

Shapley theorem

Shapley value is uniquely determined:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|! \cdot (n - |S| - 1)!}{n!} \cdot \underbrace{[v(S \cup \{i\}) - v(S)]}_{\text{marginal contribution to } S}.$$

Interpretation

- Imagine the coalition being formed one player at a time, with each player demanding their contribution $v(S \cup \{i\}) - v(S)$ as a **fair compensation**.
- Then for each player take the average of this contribution **over the possible different permutations** in which the coalition can be formed.
 - Weight this quantity by the $|S|!$ ways the set S could have been formed prior i 's addition.
 - Weight this quantity by the $(n - |S| - 1)!$ ways the remaining players could be added.
 - Sum over all possible sets S .
 - Average by dividing by $n!$ —the number of possible orderings of all the players.
- Application: Shapley-Shubik power index.

Computation

$$\begin{aligned}
 \phi_i(v) &= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)] \\
 &= \frac{1}{n} \sum_{s=0}^{n-1} \frac{1}{\binom{n-1}{s}} \sum_{S \subseteq N \setminus \{i\}, |S|=s} [v(S \cup \{i\}) - v(S)].
 \end{aligned}$$

Let $\gamma(s) = \frac{s!(n-s-1)!}{n!}$. Then we have

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \gamma(|S|) \cdot [v(S \cup \{i\}) - v(S)].$$

- 1 Setup
- 2 Shapley value
 - Examples
- 3 Core
 - Properties
 - Examples
 - Nonemptiness of core
- 4 Nash bargaining solution

Example: Two-person bargaining game

$N = \{1, 2\}$, $v(1, 2) = 1$, $v(1) = v(2) = 0$.

- Since $n = 2$, we have $\gamma(0) = \gamma(1) = \frac{1}{2}$.
- For player 1, we have

S	\emptyset	$\{2\}$
$v(S \cup \{1\}) - v(S)$	0	1

Hence, $\phi_1(v) = 0\frac{1}{2} + 1\frac{1}{2} = \frac{1}{2}$.

- For player 2, we have

S	\emptyset	$\{1\}$
$v(S \cup \{1\}) - v(S)$	0	1

Hence, $\phi_2(v) = 0\frac{1}{2} + 1\frac{1}{2} = \frac{1}{2}$.

- For player 2, we can get $\phi_2(v) = \frac{1}{2}$ by efficiency condition directly.

Actually, 1 and 2 are substitutes, so $\phi_1(v) = \phi_2(v) = \frac{1}{2}$.

Example: Three-person majority game

$$N = \{1, 2, 3\}, v(1) = v(2) = v(3) = 0, \\ v(1, 2) = v(1, 3) = v(2, 3) = v(N) = 1.$$

- Since $n = 3$, we have $\gamma(0) = \gamma(2) = \frac{1}{3}$, and $\gamma(1) = \frac{1}{6}$.
- For player 1, we have

S	\emptyset	$\{2\}$	$\{3\}$	$\{2, 3\}$
$v(S \cup \{1\}) - v(S)$	0	1	1	0

Hence, $\phi_1(v) = 0\frac{1}{3} + 1\frac{1}{6} + 1\frac{1}{6} + 0\frac{1}{3} = \frac{1}{3}$.

Example: Three-person majority game (Cont.)

- For player 2, we have

S	\emptyset	$\{1\}$	$\{3\}$	$\{1, 3\}$
$v(S \cup \{2\}) - v(S)$	0	1	1	0

Hence, $\phi_2(v) = 0\frac{1}{3} + 1\frac{1}{6} + 1\frac{1}{6} + 0\frac{1}{3} = \frac{1}{3}$.

- For player 3, we can get $\phi_3(v) = \frac{1}{3}$ by efficiency condition directly.

Actually, 1, 2 and 3 are pairwise substitutes, so

$$\phi_1(v) = \phi_2(v) = \phi_3(v) = \frac{1}{3}.$$

Example: Market with two sellers and one buyer

$N = \{1, 2, 3\}$, $v(1, 2, 3) = v(1, 2) = v(1, 3) = 1$, and $v(S) = 0$ for all other $S \subseteq N$.

- Since $n = 3$, we have $\gamma(0) = \gamma(2) = \frac{1}{3}$, and $\gamma(1) = \frac{1}{6}$.
- For player 1, we have

S	\emptyset	$\{2\}$	$\{3\}$	$\{2, 3\}$
$v(S \cup \{1\}) - v(S)$	0	1	1	1

Hence, $\phi_1(v) = 0\frac{1}{3} + 1\frac{1}{6} + 1\frac{1}{6} + 1\frac{1}{3} = \frac{2}{3}$.

- For player 2, we have

S	\emptyset	$\{1\}$	$\{3\}$	$\{1, 3\}$
$v(S \cup \{2\}) - v(S)$	0	1	0	0

Hence, $\phi_2(v) = 0\frac{1}{3} + 1\frac{1}{6} + 0\frac{1}{6} + 0\frac{1}{3} = \frac{1}{6}$.

- For player 3, we can get $\phi_3(v) = \frac{1}{6}$ by efficiency condition directly.

- 1 Setup
- 2 Shapley value
 - Examples
- 3 Core
 - Properties
 - Examples
 - Nonemptiness of core
- 4 Nash bargaining solution

Stable payoff division

- The Shapley value defined a fair way of dividing the grand coalition's payment among its members.
- However, this analysis ignored questions of **stability**.
- Would the agents be willing to form the grand coalition given the way it will divide payments, or would some of them **prefer to form smaller coalitions**?
- Unfortunately, sometimes smaller coalitions can be **more attractive** for subsets of the agents, even if they lead to lower value overall.

Motivating example

Voting game

A parliament is made up of four political parties, A , B , C , and D , which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a \$100 million spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

- Shapley values: $(50, 16.67, 16.67, 16.67)$.
- Can a subcoalition gain by defecting?
- While A can't obtain more than 50 on its own, A and B have incentive to defect and divide the \$100 million between them (e.g., $(75, 25)$).

Core

- Under what payment divisions would the agents **want to form the grand coalition**?
- They would want to do so if and only if the payment profile is drawn from a set called the **core**.
- The **core** (核) is a solution concept for coalitional games that requires that no set of players be able to **break away** and take a **joint action** that makes all of them **better off**.

Core (Cont.)

- Let $\langle N, v \rangle$ be a coalitional game.
 - A vector $(x_i)_{i \in S}$ of real numbers is an **S-feasible payoff vector** if

$$v(S) = \sum_{i \in S} x_i.$$

- We refer to an **N-feasible** payoff vector as a **feasible** payoff profile.
- The **core** of the coalitional game $\langle N, v \rangle$ is the **set of feasible payoff profiles** $(x_i)_{i \in N}$ for which there is no coalition S and S -feasible payoff vector $(y_i)_{i \in S}$ for which $y_i > x_i$ for all $i \in S$.
- Analogous to Nash equilibrium, except that it allows deviations by groups of agents.

- 1 Setup
- 2 Shapley value
 - Examples
- 3 **Core**
 - **Properties**
 - Examples
 - Nonemptiness of core
- 4 Nash bargaining solution

Property

If x is in the core, then x satisfies

- (individual rational) $x_i \geq v(i)$ for all $i \in N$,
- (group rational) $\sum_{i \in N} x_i = v(N)$.

Proposition

The core is the set of feasible payoff profiles $(x_i)_{i \in N}$ for which

$$\sum_{i \in S} x(i) \geq v(S) \text{ for every coalition } S.$$

The core is the set of payoff profiles satisfying a system of weak linear inequalities and hence is closed and convex.

Proof of “ \Leftarrow ”

- ① Suppose that $x = (x_i)_{i \in N}$ satisfies

$$\sum_{i \in N} x_i = v(N), \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all coalition } S.$$

- ② Assume x is not in the core, that is, there exist a coalition S and $y = (y_i)_{i \in S}$, such that

$$\sum_{i \in S} y_i = v(S) \text{ and } y_i > x_i \text{ for all } i \in S.$$

- ③ Then we have

$$v(S) = \sum_{i \in S} y_i > \underbrace{\sum_{i \in S} x_i}_{\text{by assumption}} \geq v(S),$$

a contradiction.

Proof of “ \Rightarrow ”

- ① Suppose that $x = (x_i)_{i \in N}$ does not satisfy

$$\sum_{i \in N} x_i = v(N), \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all coalition } S.$$

- ② If $\sum_{i \in N} x_i \neq v(N)$, x cannot be in the core.

- ③ Suppose, then, that there is a coalition $S \neq \emptyset$ such that

$$\sum_{i \in S} x_i < v(S) \text{ or } \sum_{i \in S} x_i = v(S) - \epsilon,$$

where $\epsilon > 0$. For each $i \in S$, define $z_i = x_i + \frac{\epsilon}{|S|}$.

- ④ It is easily seen that $\sum_{i \in S} z_i = v(S)$ and $z_i > x_i$ for all $i \in S$. Hence x is not in the core.

- 1 Setup
- 2 Shapley value
 - Examples
- 3 **Core**
 - Properties
 - **Examples**
 - Nonemptiness of core
- 4 Nash bargaining solution

Example: Two-person bargaining game

$N = \{1, 2\}$, $v(N) = 1$, and $v(1) = v(2) = 0$.

Answer.

(x_1, x_2) is in the core if and only if

$$x_1 \geq 0, x_2 \geq 0, \text{ and } x_1 + x_2 = 1.$$



Example: Three-person bargaining game

$N = \{1, 2, 3\}$, $v(N) = 1$ and $v(S) = 0$ for all $S \subsetneq N$.

Answer.

- (x_1, x_2, x_3) is in the core if and only if

$$x_1 + x_2 + x_3 = v(N) = 1, \text{ and } \sum_{i \in S} x_i \geq v(S) = 0 \text{ for all } S \subsetneq N.$$

- The core is therefore the set

$$\{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \geq 0, x_1 + x_2 + x_3 = 1\}.$$



Example: Market with two sellers and a buyer

$N = \{1, 2, 3\}$, $v(N) = v(1, 2) = v(1, 3) = 1$, and $v(S) = 0$ for all other $S \subseteq N$.

Answer.

- x is in the core if and only if

$$x_1 + x_2 + x_3 = 1, x_1 + x_2 \geq 1, x_1 + x_3 \geq 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

- Hence the core is $\{(1, 0, 0)\}$.



- Note that the core allocation in the example above $(1, 0, 0)$ differs considerably from the Shapley value $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$.
- One can interpret that zero payoff to players 2 and 3 in the core allocation as the result of cutthroat competition between them.

Example: Three-person majority game

Suppose that three players can obtain one unit of payoff, any two of them can obtain 1 independently of the actions of the third, and each player alone can obtain nothing, independently of the actions of the remaining two players.

$N = \{1, 2, 3\}$, $v(N) = v(1, 2) = v(1, 3) = v(2, 3) = 1$ and $v(i) = 0$ for all $i \in N$.

Answer.

For x to be in the core, we need $x_1 + x_2 + x_3 = 1$, $x_i \geq 0$ for all $i \in N$, $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$ and $x_2 + x_3 \geq 1$. There exists no x satisfying these condition, so the core is empty. □

Example: A majority game

- A group of n players, where $n \geq 3$ is odd, has one unit to divide among its members.
- A coalition consisting of a majority of the players can divide the unit among its members as it wishes.
- This situation is modeled by the coalitional game $\langle N, v \rangle$ in which $|N| = n$ and

$$v(S) = \begin{cases} 1, & \text{if } |S| \geq \frac{n}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Example: A majority game (Cont.)

Answer.

The game has an empty core by the following argument.

- Assume that x is in the core. If $|S| = n - 1$ then $v(S) = 1$ so that $\sum_{i \in S} x_i \geq 1$. Since there are n coalitions of size $n - 1$, we have

$$\sum_{\{S: |S|=n-1\}} \sum_{i \in S} x_i \geq n.$$

- On the other hand, we have

$$\sum_{\{S: |S|=n-1\}} \sum_{i \in S} x_i = \sum_{i \in N} \sum_{\{S: |S|=n-1, S \ni i\}} x_i = \sum_{i \in N} (n-1)x_i = n-1,$$

a contradiction.



- 1 Setup
- 2 Shapley value
 - Examples
- 3 **Core**
 - Properties
 - Examples
 - **Nonemptiness of core**
- 4 Nash bargaining solution

Notation

- Denote by \mathcal{C} the set of all coalitions.
- Denote by $\mathbf{1}_S \in \mathbb{R}^{|N|}$ the **characteristic vector of S** given by

$$(\mathbf{1}_S)_i = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

- The function $S \mapsto \mathbf{1}_S$ is a bijection.

Balanced collection of weights

Balanced collection of weights

A collection $(\lambda_S)_{S \in \mathcal{C}}$ of numbers in $[0, 1]$ is a **balanced collection of weights** if for every player i the sum of λ_S over all the coalitions that contain i is 1:

$$\sum_{S \in \mathcal{C}} \lambda_S \mathbf{1}_S = \mathbf{1}_N.$$

- Example 1: the collection (λ_S) in which $\lambda_N = 1$ and $\lambda_S = 0$ for all other S is a balanced collection of weights.

Balanced collection of weights (Cont.)

Example 2: let $|N| = 3$.

- The collection $(\lambda_S)_{S \in \mathcal{C}}$ in which $\lambda_S = \frac{1}{2}$ if $|S| = 2$ and $\lambda_S = 0$ otherwise is a balanced collection of weights:

$$\sum_{S \in \mathcal{C}} \lambda_S \mathbf{1}_S = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{1}_N.$$

- So too is the collection $(\lambda_S)_{S \in \mathcal{C}}$ in which $\lambda_S = 1$ if $|S| = 1$ and $\lambda_S = 0$ otherwise:

$$\sum_{S \in \mathcal{C}} \lambda_S \mathbf{1}_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{1}_N.$$

Balanced game

Balanced game

A game $\langle N, v \rangle$ is **balanced** if

$$\sum_{S \in \mathcal{C}} \lambda_S v(S) \leq v(N) \text{ for every balanced collection of weights.}$$

Interpretation

- ① Each player has **one unit of time**, which he must **distribute** among all the coalitions of which he is a member.
 - * In order for a coalition S to be active for the fraction of time λ_S , all its members must be **active** in S for this fraction of time, in which case the coalition yields the payoff $\lambda_S v(S)$.
- ② The condition $\sum_{S \in \mathcal{C}} \lambda_S \mathbf{1}_S = \mathbf{1}_N$ is a **feasibility condition**:
For every individual, the sum of its amounts of his **time he spends** with each coalition must equal exactly the amount of **time he is endowed with**.
- ③ A game is **balanced** if there is no feasible allocation of time that yields the players more than $v(N)$.

Bondareva-Shapley theorem

Bondareva-Shapley theorem

A coalitional game has a non-empty core if and only if it is balanced.

Proof of “ \Rightarrow ”.

- 1 Let x be a payoff profile in the core of $\langle N, v \rangle$ and $(\lambda_S)_{S \in \mathcal{C}}$ a balanced collection of weights.
- 2 Then

$$\sum_{S \in \mathcal{C}} \lambda_S v(S) \leq \sum_{S \in \mathcal{C}} \lambda_S \sum_{i \in S} x_i = \sum_{i \in N} x_i \sum_{S \ni i} \lambda_S = \sum_{i \in N} x_i = v(N),$$

so that $\langle N, v \rangle$ is balanced.



Proof of “ \Leftarrow ”

- ① Assume that $\langle N, v \rangle$ is balanced. Then there is no balanced collection $(\lambda_S)_{S \in \mathcal{C}}$ of weights for which

$$\sum_{S \in \mathcal{C}} \lambda_S v(S) > v(N).$$

- ② Therefore the convex set $\{(\mathbf{1}_N, v(N) + \epsilon) \in \mathbb{R}^{|N|+1} : \epsilon > 0\}$ is disjoint from the convex cone

$$\left\{ y \in \mathbb{R}^{|N|+1} \mid y = \sum_{S \in \mathcal{C}} \lambda_S (\mathbf{1}_S, v(S)) \text{ where } \lambda_S \geq 0 \text{ for all } S \in \mathcal{C} \right\},$$

since if not then $\mathbf{1}_N = \sum_{S \in \mathcal{C}} \lambda_S \mathbf{1}_S$, so that $(\lambda_S)_{S \in \mathcal{C}}$ is a balanced collection of weights and $\sum_{S \in \mathcal{C}} \lambda_S v(S) > v(N)$.

Proof of “ \Leftarrow ” (Cont.)

- 3 Thus by hyperplane separating theorem there is a non-zero vector $(\alpha_N, \alpha) \in \mathbb{R}^{|N|} \times \mathbb{R}$ such that

$$(\alpha_N, \alpha) \cdot y \geq 0 > (\alpha_N, \alpha) \cdot (\mathbf{1}_N, v(N) + \epsilon)$$

for all y in the cone and all $\epsilon > 0$.

- 4 Since $(\mathbf{1}_N, v(N))$ is in the cone, we have $\alpha < 0$.
- 5 Now let $x = -\frac{\alpha_N}{\alpha}$.
- 6 Since $(\mathbf{1}_S, v(S))$ is in the cone for all $S \in \mathcal{C}$, we have $x(S) = x \cdot \mathbf{1}_S \geq v(S)$ for all $S \in \mathcal{C}$, and $v(N) \geq \mathbf{1}_N x = \sum_{i \in N} x_i$.
- 7 Thus $v(N) = \sum_{i \in N} x_i$, so that x is in the core of $\langle N, v \rangle$.

Core vs. Shapley value

- Core: based on coalitional threats—each coalition must get at least what it can generate alone.
- Shapley value: based on marginal contributions—what does each player contribute to each possible coalition.

- 1 Setup
- 2 Shapley value
 - Examples
- 3 Core
 - Properties
 - Examples
 - Nonemptiness of core
- 4 Nash bargaining solution

Bargaining problem

Bargaining problem

A two-person bargaining problem (议价问题), denoted by $\langle U, d \rangle$, consists of

- U is the set of possible agreements in terms of utilities that they yield to 1 and 2. An element of U is a pair $u = (u_1, u_2)$.
- d is a pair (d_1, d_2) , called the disagreement point or threat point.

If agreement $u = (u_1, u_2) \in U$ is reached, then 1 gets utility u_1 and 2 gets utility u_2 . If no agreement is reached then 1 gets utility d_1 and 2 gets utility d_2 .

The set of two-person bargaining games is denoted by W .

Convention

Assume that

- U is compact and convex.
- U contains a point y for which $y_i > d_i$ for $i = 1, 2$, that is, bargaining is worthwhile for both the players.

Nash bargaining solution

The Nash bargaining solution (纳什议价解) is a mapping

$$f: W \rightarrow \mathbb{R}^2$$

that associates a unique element $f(U, d)$ with the game $\langle U, d \rangle$, satisfying the following axioms:

- N1.** Feasibility: $f(U, d) \in U$.
- N2.** Individual rationality: $f(U, d) \geq d$ for all $\langle U, d \rangle \in W$.
- N3.** Pareto optimality: $f(U, d)$ is Pareto optimal. That is, there does not exist a point $(u_1, u_2) \in U$ such that

$$u_1 \geq f_1(U, d), u_2 \geq f_2(U, d), (u_1, u_2) \neq f(U, d).$$

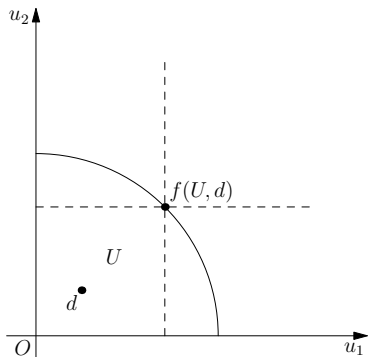
Nash bargaining solution (Cont.)

- N4.** Symmetry: If $\langle U, d \rangle \in W$ satisfies $d_1 = d_2$ and $(x_1, x_2) \in U$ implies $(x_2, x_1) \in U$, then $f_1(U, d) = f_2(U, d)$.
- N5.** Invariance under linear transformations: Let $a_1, a_2 > 0$, $b_1, b_2 \in \mathbb{R}$, and $\langle U, d \rangle, \langle U', d' \rangle \in W$ where $d'_i = a_i d_i + b_i$, $i = 1, 2$, and $U' = \{x \in \mathbb{R}^2 \mid x_i = a_i y_i + b_i, i = 1, 2, y \in U\}$. Then $f_i(U'_i, d'_i) = a_i f_i(U, d) + b_i$, $i = 1, 2$.
- N6.** Independence of irrelevant alternatives: If $\langle U, d \rangle, \langle U', d' \rangle \in W$, $d = d'$, $U \subseteq U'$, and $f(U', d') \in U$, then $f(U, d) = f(U', d')$.

The interpretation is that, given any bargaining problem $\langle U, d \rangle$, the solution function tells us that the agreement $u = f(U, d)$ will be reached.

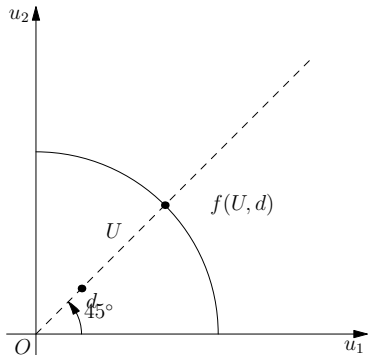
Pareto optimality

There are no points in U that are “North-East” of $f(U, d)$.



Symmetry

Suppose that $\langle U, d \rangle$ is such that U is symmetric around the 45-degree line and $d_1 = d_2$, then $f_1(U, d) = f_2(U, d)$, that is, when everything in $\langle U, d \rangle$ is symmetric, the point $f(U, d)$ is itself on the 45-degree line.



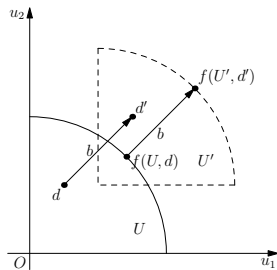
Invariance under linear transformations

Suppose we have two bargaining problems $\langle U, d \rangle$ and $\langle U', d' \rangle$ with the following property. For some vector $b = (b_1, b_2)$,

$$d' = d + b, \quad U' = U + b.$$

Then invariance under linear transformations imposes that

$$f(U', d') = f(U, d) + b,$$



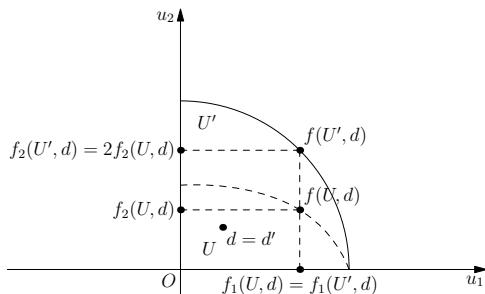
Invariance under linear transformations (Cont.)

Suppose we have two bargaining problems $\langle U, d \rangle$ and $\langle U', d' \rangle$ with $d = (0, 0)$ and the following property.

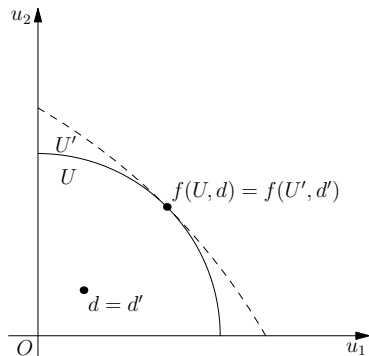
$$U'_1 = k_1 U_1, \quad U'_2 = k_2 U_2.$$

Then invariance under linear transformations imposes that

$$f_1(U', d) = k_1 f_1(U, d), \quad f_2(U', d) = k_2 f_2(U, d).$$



Independence of irrelevant alternatives



Nash bargaining solution

Nash's Theorem

A game $\langle U, d \rangle \in W$ has a **unique** Nash solution $u^* = f(U, d)$ satisfying Conditions N1 to N6. Furthermore, the solution u^* satisfies Conditions N1 to N6 if and only if

$$(u_1^* - d_1)(u_2^* - d_2) > (u_1 - d_1)(u_2 - d_2)$$

for all $u \in U$, $u \geq d$, and $u \neq u^*$.

- Existence of an optimal solution: Since the set U is compact and the objective function is continuous, there exists an optimal solution.
- Uniqueness: The objective function is strictly quasi-concave. Therefore, maximization problem has a unique optimal solution.

Example

Find the Shapley values of the game with $N = \{1, 2\}$ and the characteristic function v . Now consider the bargaining game where $U = \{(u_1, u_2) \mid u_1 + u_2 = v(N), u_1 \geq v(\{1\}), u_2 \geq v(\{2\})\}$ and $d = (v(\{1\}), v(\{2\}))$. Find the bargaining solution of the game (U, d) .

Answer.

Since $n = 2$, we have $\gamma(0) = \gamma(1) = \frac{1}{2}$. Denote $v = v(N)$, $v_1 = v(\{1\})$ and $v_2 = v(\{2\})$.

- Shapley value. For player i ,

$$\frac{S}{v(S \cup \{i\}) - v(S)} \quad \left| \quad \begin{array}{cc} \emptyset & \{j\} \\ v_i & v - v_j \end{array} \right.$$

Hence the Shapley value for player i is $\frac{v_i + v - v_j}{2}$.



Example (Cont.)

Answer (Cont.)

- To get the Nash bargaining solution, we solve the following problem

$$\max_{u_1+u_2=v, u_1 \geq v_1, u_2 \geq v_2} (u_1 - v_1)(u_2 - v_2).$$

The solution is $u_i^* = \frac{v_i + v - v_j}{2}$. Note that we need to check whether $u_i^* \geq v_i$.

- Hence, both Nash bargaining solution and the Shapley value give the same result.

