

# Game Theory

Dynamic games of incomplete information

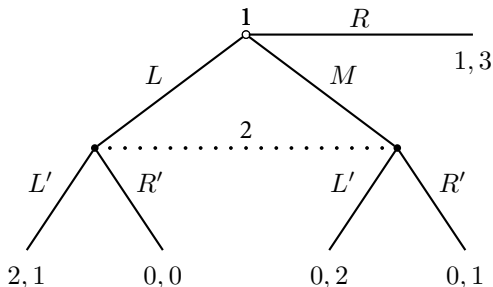
Xiang Sun

2021 Summer

- 1 Motivation
- 2 Perfect Bayesian equilibrium
  - Example
  - How to find PBE
- 3 Relationship between equilibrium concepts
- 4 Summary

# Motivating example

- Example 1:



- What are the pure-strategy Nash equilibria and subgame-perfect Nash equilibria in this game?

# NE and SPNE

- The normal-form representation of the game is

		Player 2	
		$L'$	$R'$
Player 1	$L$	2, 1	0, 0
	$M$	0, 2	0, 1
	$R$	1, 3	1, 3

- Two pure-strategy Nash equilibria:

$(L, L')$  and  $(R, R')$ .

- Since the above game has no subgames, both  $(L, L')$  and  $(R, R')$  are subgame-perfect Nash equilibria.

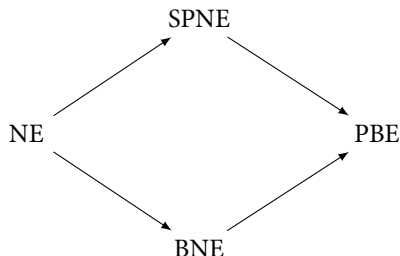
# Problem of SPNE

However,  $(R, R')$  is based on a **non-credible threat** (不可置信的威胁) from player 2.

- If player 1 believes player 2's threat of playing  $R'$ , then player 1 should choose  $R$  to end the game with payoff 1, which is larger than 0 by choosing  $L$  or  $M$ .
- If player 1 doesn't believe the threat and plays  $L$  or  $M$ , then when player 2 gets the move, he will indeed choose  $L'$ , since  $L'$  strictly dominates  $R'$  for player 2.
- Thus, the threat of playing  $R'$  by player 2 is not credible.

# Problem of SPNE (Cont.)

- In Example 1, the equilibrium  $(R, R')$  is **not reasonable** as it depends on a non-credible threat.
- We need to **strengthen** the equilibrium concept to rule out some subgame-perfect Nash equilibria like  $(R, R')$ .
- A stronger equilibrium concept  $\Rightarrow$  **perfect Bayesian equilibrium** (精炼贝叶斯均衡).



- 1 Motivation
- 2 Perfect Bayesian equilibrium
  - Example
  - How to find PBE
- 3 Relationship between equilibrium concepts
- 4 Summary

# Belief

## Requirement 1

At each information set, the player with the move must have a **belief** about which node in the **information set** has been reached by the play of the game.

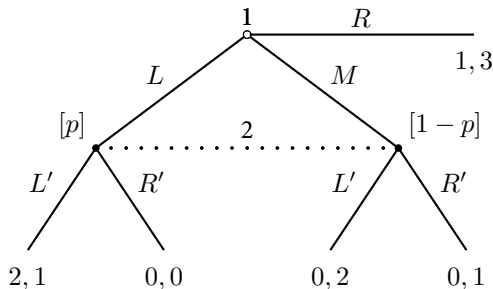
- For a **nonsingleton** information set, a belief is a **probability distribution** over the nodes in the information set;
- For a **singleton** information set, a belief puts **probability one** on the single decision node.



# Belief: Illustration

In Example 1, Requirement 1 implies that

- for player 2's nonsingleton information set, **player 2 must form a belief on which of the decision node has been reached;**
- i.e., player 2 believes that player 1 has chosen  $L$  with probability  $p$ , and  $M$  with probability  $1 - p$ , where  $p \in [0, 1]$ .



# Sequential rationality

## Requirement 2

Given their beliefs, the players' strategies must be **sequentially rational** (序贯理性).

That is, at each information set, the action taken by the player with the move (and the player's subsequent strategy) must be **optimal**, given the player's **belief** at that information set and the **other players' subsequent strategies** (where a “subsequent strategy” is a complete plan of actions covering every contingency that might arise after the given information set has been reached).

# Sequential rationality: Illustration

- Given this belief, player 2's expected payoffs are
    - playing  $L'$ :  $p \cdot 1 + (1 - p) \cdot 2 = 2 - p$ .
    - playing  $R'$ :  $p \cdot 0 + (1 - p) \cdot 1 = 1 - p$ .
  - Since  $R'$  is **never optimal** for any belief,  $(R, R')$  cannot satisfy Requirement 2.
  - Requirements 1 and 2 together can already eliminate the equilibrium  $(R, R')$  that relies on a non-credible threat.
  - Requirements 1 and 2 **allow for arbitrary beliefs**, including **unreasonable** ones.
- Further requirements on players' beliefs need to be introduced.

# Equilibrium path

## Definition

For a given **equilibrium** in a given extensive-form game, an information set

- is **on the equilibrium path** (在均衡路径上) if it will be reached with positive probability if the game is played according to the equilibrium strategies,
  - is **off the equilibrium path** (不在均衡路径上) if it is definitely not to be reached if the game is played according to the equilibrium strategies.
- 
- Here the “equilibrium” could refer to Nash equilibrium, subgame-perfect Nash equilibrium, Bayesian Nash equilibrium, or perfect Bayesian equilibrium.

## Equilibrium path (Cont.)

- In Example 1, consider player 2's nonsingleton information set.
- For the equilibrium  $(L, L')$ , the nonsingleton information set is **on the equilibrium path**, while there is no information set off the equilibrium path.
- For the equilibrium  $(R, R')$ , the nonsingleton information set is **off the equilibrium path**, and there is no information set on the equilibrium path.

# Belief is rational

## Requirement 3

At information sets **on the equilibrium path**, beliefs are determined by **Bayes' rule** and the **players' equilibrium strategies**.

- In Example 1, for the equilibrium  $(L, L')$ , Requirement 3 implies that player 2's belief must be  $p = 1$ .
- Consider a hypothetical situation: the game has a mixed-strategy equilibrium in which player 1 plays  $L$  with probability  $q_1$ ,  $M$  with probability  $q_2$ , and  $R$  with probability  $1 - q_1 - q_2$ .
- \* Requirement 3 would force player 2's belief to be

$$p = \text{Prob}(L \text{ is played} \mid L \text{ or } M \text{ is played}) = \frac{q_1}{q_1 + q_2}.$$

# Belief is rational+

## Requirement 4

At information sets **off the equilibrium path**, beliefs are determined by **Bayes' rule** and the players' equilibrium strategies **where possible**.

- In Example 1, for the equilibrium  $(R, R')$ , Requirement 4 does not put any restrictions on player 2's belief  $p$ .

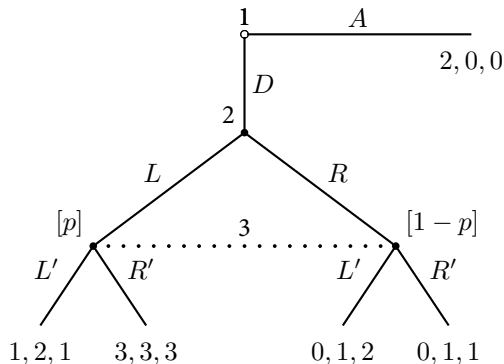
## Definition

A **perfect Bayesian equilibrium** (精炼贝叶斯均衡) consists of strategies and beliefs satisfying Requirements 1–4.

- 1 Motivation
- 2 Perfect Bayesian equilibrium
  - Example
  - How to find PBE
- 3 Relationship between equilibrium concepts
- 4 Summary



## Example 2



What are the (pure-strategy) Nash equilibria and subgame-perfect Nash equilibria of this game? Are they also perfect Bayesian equilibria?

# Nash Equilibrium

- The normal-form representation of the game:

	$L$	$R$
$A$	$\underline{2}, \underline{0}, \underline{0}$	$\underline{2}, \underline{0}, \underline{0}$
$D$	$1, \underline{2}, 1$	$0, 1, \underline{2}$

Player 3 chooses  $L'$

	$L$	$R$
$A$	$2, \underline{0}, \underline{0}$	$\underline{2}, \underline{0}, \underline{0}$
$D$	$\underline{3}, \underline{3}, \underline{3}$	$0, 1, 1$

Player 3 chooses  $R'$

- Player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix.
- Four pure-strategy Nash equilibria:

$$(A, L, L'), (A, R, L'), (A, R, R'), (D, L, R').$$

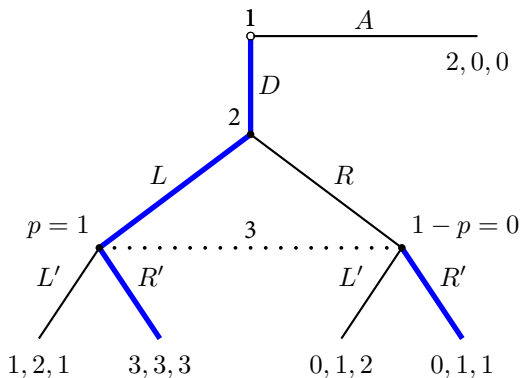
# Subgame-perfect Nash Equilibrium

- The game has a **unique subgame** (beginning at player 2's singleton information set), and the unique Nash equilibrium of this subgame is  $(L, R')$ .
- Hence, the unique subgame-perfect Nash equilibrium of the game is  $(D, L, R')$ .
- The other three Nash equilibria are not subgame-perfect.

# Perfect Bayesian Equilibrium

- Check whether each equilibrium is a perfect Bayesian equilibrium.
- Consider the subgame-perfect Nash equilibrium  $(D, L, R')$ .
- This strategy profile and the belief  $p = 1$  for player 3 satisfy Requirements 1–3.
- They also satisfy Requirement 4, since there is no information set off the equilibrium path.
- Then the strategy profile  $(D, L, R')$  and the belief  $p = 1$  indeed constitute a perfect Bayesian equilibrium.

# Perfect Bayesian Equilibrium

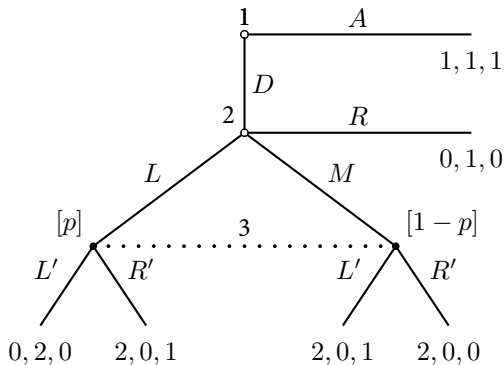


Perfect Bayesian equilibrium in Example 2:  $((D, L, R'); p = 1)$

# Perfect Bayesian Equilibrium

- The other three Nash equilibria do not satisfy all Requirements 1–4.
- For example, consider the Nash equilibrium  $(A, L, L')$ .
- Requirement 4 implies that for player 3's nonsingleton information set off the equilibrium path, player 3's belief must be  $p = 1$ .
- Requirement 2 then implies that for  $p = 1$ , player 3 must choose  $R'$  rather than  $L'$ .
- Therefore, the strategy profile  $(A, L, L')$  and the belief  $p = 1$  do not satisfy Requirements 1–4, and they are not a perfect Bayesian equilibrium.

# Example 3



# Perfect Bayesian Equilibrium

- Three pure-strategy Nash equilibria:

$$(A, L, L'), (A, R, L'), (A, R, R').$$

- Consider the strategy profile  $(A, L, L')$  and the belief  $p \leq \frac{1}{2}$ , which satisfy Requirements 1–3.
- Requirement 4 implies that for player 3's information set off the equilibrium path, the belief must be  $p = 1$ , which contradicts  $p \leq \frac{1}{2}$ .
- Therefore, there exists no belief together with the strategy profile  $(A, L, L')$  that constitutes a perfect Bayesian equilibrium.



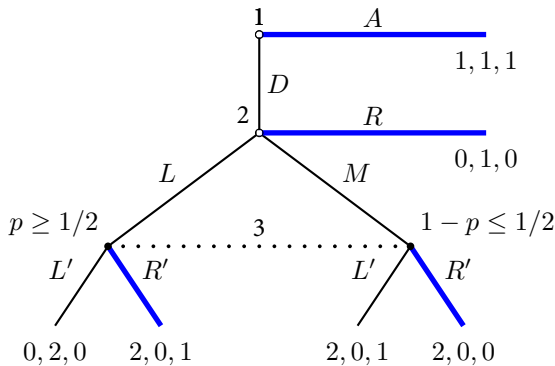
## Perfect Bayesian Equilibrium (Cont.)

- Consider strategy profile  $(A, R, L')$  and the belief  $p \leq \frac{1}{2}$ .
- They satisfy Requirement 4, which puts no restrictions on player 3's belief at the information set off the equilibrium path.
- They also satisfy Requirements 1 and 3.
- However, at player 2's singleton information set, player 2 should choose  $L$  rather than  $R$  given player 3's equilibrium strategy, which implies that Requirement 2 is violated.
- Thus, strategy profile  $(A, R, L')$  and the belief  $p \leq \frac{1}{2}$  do not constitute a perfect Bayesian equilibrium.

# Perfect Bayesian Equilibrium (Cont.)

- Consider the strategy profile  $(A, R, R')$  and the belief  $p \geq \frac{1}{2}$ .
- They satisfy all Requirements 1–4, and thus constitute a perfect Bayesian equilibrium.

# Perfect Bayesian Equilibrium (Cont.)



Perfect Bayesian equilibrium in Example 3:  $((A, R, R'); p \geq \frac{1}{2})$

- 1 Motivation
- 2 **Perfect Bayesian equilibrium**
  - Example
  - **How to find PBE**
- 3 Relationship between equilibrium concepts
- 4 Summary

# How to find PBE

- The procedure to determine whether a given equilibrium is a perfect Bayesian equilibrium:
  - 1 Determine a **belief** for each information set by Bayes' rule;
  - 2 Check whether the equilibrium is **optimal** given each determined belief and the subsequent strategies.
- A perfect Bayesian equilibrium consists not only strategy profile but also beliefs of players, and it requires each player's strategy to be optimal given his or her **reasonable beliefs**.

- 1 Motivation
- 2 Perfect Bayesian equilibrium
  - Example
  - How to find PBE
- 3 Relationship between equilibrium concepts**
- 4 Summary

# Relationship between different equilibrium concepts

- Perfect Bayesian equilibrium is a **stronger** equilibrium concept that refines different types of equilibria.
- It refines Bayesian Nash equilibrium (in the same way as subgame-perfect Nash equilibrium refines Nash equilibrium).
- It strengthens subgame-perfect Nash equilibrium by explicitly analyzing beliefs.
- In addition, while a Nash equilibrium requires that no player chooses a strictly dominated strategy, a perfect Bayesian equilibrium requires no player's strategy to be strictly dominated beginning at any information set.

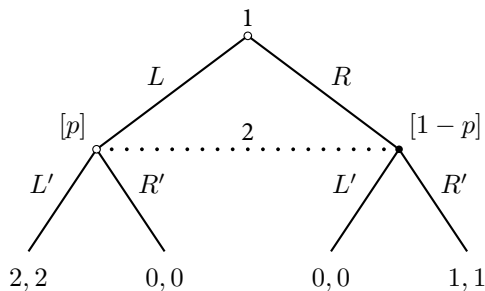
# Relationship between Different Equilibrium Concepts

Perfect Bayesian equilibrium corresponds to

- Nash equilibrium (with appropriate beliefs) in static games of complete information;
- Bayesian Nash equilibrium in static games of incomplete information;
- subgame-perfect Nash equilibrium (with appropriate beliefs) in dynamic games of complete and perfect information (and also many dynamic games of complete but imperfect information).



# Example 4



Three perfect Bayesian equilibria:

$$((L, L'); p = 1), ((R, R'); p = 0), \left( \left( \frac{1}{3}L + \frac{2}{3}R, \frac{1}{3}L' + \frac{2}{3}R' \right); p = \frac{1}{3} \right).$$

## Example 4 (Cont.)

- The normal-form representation of the game is

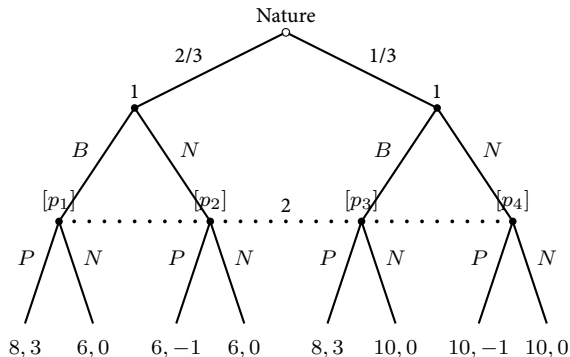
		Player 2	
		$L'$	$R'$
Player 1	$L$	2, 2	0, 0
	$M$	0, 0	1, 1

- Three Nash equilibria:

$$(L, L'), (R, R'), \left(\frac{1}{3}L + \frac{2}{3}R, \frac{1}{3}L' + \frac{2}{3}R'\right)$$

- Each Nash equilibrium (together with a correct belief) corresponds to a perfect Bayesian equilibrium in this static game of complete information.

# Example 5



Two Bayesian Nash equilibria:  $(BN, P)$  and  $(NN, N)$

## Example 5 (Cont.)

- Two perfect Bayesian equilibria:

$$((BN, P); p_1 = \frac{2}{3}, p_4 = \frac{1}{3}), ((NN, N); p_2 = \frac{2}{3}, p_4 = \frac{1}{3}).$$

- Consider the first equilibrium, for example.
- For the strategy  $BN$  chosen by player 1, Requirement 3 implies that the belief is  $p_1 = \frac{2}{3}$  and  $p_4 = \frac{1}{3}$ .
- Given this belief, it is optimal for player 2 to choose  $P$ .
- Given player 2's strategy  $P$ , it is optimal for player 1 type 1 to choose  $B$ , and type 2 to choose  $N$ .

- 1 Motivation
- 2 Perfect Bayesian equilibrium
  - Example
  - How to find PBE
- 3 Relationship between equilibrium concepts
- 4 **Summary**

# Summary

- We have considered dynamic games with incomplete information.
- Why to introduce PBE?
- PBE vs. SPNE/BNE