Game Theory

Auction

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Introduction

- One of the most popular examples of static games of incomplete information is an auction.
- An auction is a mechanism of allocating goods.
  1. Analyze bidders' behaviors.
  2. Analyze auctioneer's "optimal" choice.
- Advantages of auctions:
  - a simple way of determining the market-based prices;
  - more flexible than setting a fixed price;
  - can usually achieve efficiency by allocating the resources to those who value them most highly.

Examples of auctions

- Consumption goods: antiques, art, wine.
- Government: treasury bills, mineral rights, assets.
- Stocks: IPOs.
- Procurement auctions/Subcontracting: automobiles.
- Internet: eBay, Amazon.

Types of auctions

- Number of objects
    - A single object or many?
- Open vs. sealed-bid
    - Do you know the bids of other participants?
- One-sided vs. two-sided
    - Do buyers and sellers both submit bids, or just buyers?
- Private value vs. common value
    - Do the bidders have the same valuation for the object?

Four classical auctions

- English auction (英氏拍卖)
  - ascending, open
- Dutch auction (荷氏拍卖)
  - descending, open
- First-price auction (一价拍卖)
  - simultaneous, sealed-bid
- Second-price auction (二价拍卖)
  - simultaneous, sealed-bid

A second-price auction

- Suppose there are n potential buyers (or bidders), with valuations $v_1, \ldots, v_n$ for an object.
- Let each v_i be drawn from some distribution $F_i(\cdot)$.
- Each bidder knows his own valuation but does not know other bidders' valuations.
- The bidders simultaneously submit bids $b_i \in [0, \infty)$.
- The highest bidder wins the object and pays the second highest bid, while the other bidders obtain nothing.
- If there are more than one winners, the object is allocated randomly among them.

Payoff

- Let r_i be the highest bid of all players other than player i:

$$r_i = \max_{j \neq i} b_j.$$

- The bidder i's payoff function is

$$u_i(b_i, b_{-i}; v_i) = \begin{cases} v_i - r_i, & \text{if } b_i > r_i, \\ \frac{v_i - r_i}{k}, & \text{if } b_i = r_i, \\ 0, & \text{if } b_i < r_i, \end{cases}$$

where k is the number of bids that equal b_i.

Truth telling

Proposition

For each player i, the strategy of bidding his valuation (i.e., $s_i^*(v_i) = v_i$) weakly dominates all other strategies.

- Compare bidder i's payoffs for different bids b_i.
- First, compare $b_i^* = s_i^*(v_i) = v_i$ with $b_i > v_i$,

$$\begin{array}{ccc} & u_i(b_i^*, b_{-i}; v_i) & u_i(b_i, b_{-i}; v_i) \\ r_i > b_i > v_i & 0 & 0 \\ r_i = b_i > v_i & 0 & \frac{v_i - r_i}{k} < 0 \\ b_i > r_i > v_i & 0 & v_i - r_i < 0 \\ b_i > r_i = v_i & \frac{v_i - r_i}{k} = 0 & v_i - r_i = 0 \\ b_i > v_i > r_i & v_i - r_i & v_i - r_i \end{array}$$

* Then $b_i^* = v_i$ weakly dominates $b_i > v_i$.

Truth telling (Cont.)

- Second, compare $b_i^* = v_i$ with $b_i < v_i$,

$$\begin{array}{lcc} & u_i(b_i^*, b_{-i}; v_i) & u_i(b_i, b_{-i}; v_i) \\ r_i > v_i > b_i & 0 & 0 \\ r_i = v_i > b_i & \frac{v_i - r_i}{k} = 0 & 0 \\ v_i > r_i > b_i & v_i - r_i > 0 & 0 \\ v_i > r_i = b_i & v_i - r_i > 0 & \frac{v_i - r_i}{k} \\ v_i > b_i > r_i & v_i - r_i > 0 & v_i - r_i \end{array}$$

* Then $b_i^* = v_i$ weakly dominates $b_i < v_i$.

Bayesian Nash equilibrium

- By the following proposition, $(s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium, where $s_i^*(v_i) = v_i$ for all i.

Proposition

Consider a strategy profile $(s_1^*, \ldots, s_n^*)$ in a Bayesian game. Suppose for any player i, any $t_i \in T_i$, $a_i \in A_i$, and $a_{-i} \in A_{-i}$,

$$u_i\big(s_i^*(t_i), a_{-i}; t_i\big) \geq u_i(a_i, a_{-i}; t_i),$$

(i.e., $s_i^*(t_i)$ weakly dominates every $a_i \in A_i$). Then $(s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium.

Bayesian Nash equilibrium: Proof

Proof.

- Because $s^*_{-i}(t_{-i}) \in A_{-i}$, the weak dominance implies

$$u_i\big(s_i^*(t_i), s^*_{-i}(t_{-i}); t_i\big) \geq u_i\big(a_i, s^*_{-i}(t_{-i}); t_i\big)$$

  for any $t_i \in T_i$ and $a_i \in A_i$.
- Then for each i and for each t_i, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i\big(a_i, s^*_{-i}(t_{-i}); t_i\big) p_i(t_{-i} | t_i).$$

- Therefore, $(s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium.

□

A first-price auction

- Suppose there are two bidders ($i = 1, 2$).
- The bidders' valuations for an object are v_1 and v_2, which are independently and uniformly distributed on $[0, 1]$.
- The valuation v_i is bidder i's private information, which is unknown to the other bidder.
- Bidders submit their bids b_1 and b_2 simultaneously.
- The higher bidder wins the object and pays the highest bid, while the other obtains nothing.
- If there is a tie, the winner is determined by a flip of a coin.

Normal-form representation

Normal-form representation of this static Bayesian game $G = \langle A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2 \rangle$:

- $A_1 = A_2 = [0, \infty)$, and each bid is $b_i \in A_i$;
- $T_1 = T_2 = [0, 1]$, and each valuation is $v_i \in T_i$;
- Player i believes that v_j is uniformly distributed on $[0, 1]$;
- The payoff $u_i(b_i, b_j; v_i)$ is

$$u_i(b_i, b_j; v_i) = \begin{cases} v_i - b_i, & \text{if } b_i > b_j; \\ \frac{1}{2}(v_i - b_i), & \text{if } b_i = b_j; \\ 0, & \text{if } b_i < b_j. \end{cases}$$

Strategy

- Bidder i's strategy is a function $b_i(v_i)$ from $[0, 1]$ to $[0, \infty)$.
- $\big(b_1^*(v_1), b_2^*(v_2)\big)$ is a Bayesian Nash equilibrium if and only if for each player i and each type $v_i \in [0, 1]$, $b_i^*(v_i)$ solves

$$\begin{aligned}
&\max_{b_i \geq 0} \mathbf{E}_{v_j} u_i\big(b_i, b_j^*(v_j); v_i\big) \\
= &\max_{b_i \geq 0} \Big\{ (v_i - b_i) \operatorname{Prob}\big[b_i > b_j^*(v_j)\big] + \tfrac{v_i - b_i}{2} \operatorname{Prob}\big[b_i = b_j^*(v_j)\big] \Big\}.
\end{aligned}$$

Linear Bayesian Nash equilibrium

- There may be many Bayesian Nash equilibria in this game.
- We just focus on equilibria in the form of linear functions:

$$b_1^*(v_1) = a_1 + c_1 v_1 \text{ and } b_2^*(v_2) = a_2 + c_2 v_2,$$

  where $c_i > 0$, and $0 \leq a_i < 1$ for $i = 1, 2$.
- To solve for the Bayesian Nash equilibria, we just need to find out a_i and c_i accordingly.

Assumptions

Rationale of the assumptions on a_i and c_i:

- $c_i > 0$: a bidder with higher valuation is willing to bid higher.
- $a_i \geq 0$: bids cannot be negative.
- $a_i < 1$: for $a_i > 1$, bidder i can never end up with a positive payoff given $v_i \in [0, 1]$.

Best response

- We need to determine each player's best response given the other's strategy.
- Suppose player j adopts a linear strategy $b_j^*(v_j) = a_j + c_j v_j$ in equilibrium.
- We have

$$\text{Prob}(b_i = a_j + c_j v_j) = \text{Prob}\left(v_j = \frac{b_i - a_j}{c_j}\right) = 0.$$

- For any $v_i \in [0, 1]$, player i's best response $b_i(v_i)$ maximizes

$$(v_i - b_i)\,\text{Prob}(b_i > a_j + c_j v_j) = (v_i - b_i)\,\text{Prob}\left(v_j < \frac{b_i - a_j}{c_j}\right).$$

Best response (Cont.)

- Since $b_j^*(v_j) = a_j + c_j v_j \in [a_j, a_j + c_j]$, we can restrict our attention to $b_i \in [a_j, a_j + c_j]$ (i.e., $b_i < a_j$ is pointless, while $b_i > a_j + c_j$ is not rational).
- Under the above restriction, we know

$$0 \leq \tfrac{b_i - a_j}{c_j} \leq 1.$$

- Player i's best response solves

$$\max_{a_j \leq b_i \leq a_j + c_j} (v_i - b_i) \tfrac{b_i - a_j}{c_j}.$$

Best response (Cont.)

- FOC implies the potentially optimal choice is $b_i^o = \frac{v_i + a_j}{2}$.
- It is the optimal choice iff $b_i^o = \frac{v_i + a_j}{2} \in [a_j, a_j + c_j]$ iff $a_j \leq v_i \leq a_j + 2c_j$.
- The best response function of bidder i is

$$b_i(v_i) = \begin{cases} a_j, & \text{if } v_i \leq a_j; \\ \frac{1}{2}(v_i + a_j), & \text{if } a_j < v_i \leq a_j + 2c_j; \\ a_j + c_j, & \text{if } v_i > a_j + 2c_j. \end{cases}$$

Best response (Cont.)

- We want the equilibrium bid $b_i(v_i)$ to be a linear function on $[0, 1]$.
- There are three cases:

$$[0,1] \subseteq \begin{cases} (-\infty, a_j] \\ [a_j, a_j + 2c_j] \\ [a_j + 2c_j, \infty) \end{cases}$$

Selection

- Case 1 violates the assumption $a_j < 1$.
- Case 3 violates the assumptions $a_j \geq 0$ and $c_j > 0$, which imply $a_j + 2c_j > 0$.
- Therefore, we have $[0, 1] \subseteq [a_j, a_j + 2c_j]$, and the best response is

$$b_i(v_i) = \frac{1}{2}(v_i + a_j).$$

Selection (Cont.)

- In a Bayesian Nash equilibrium,

$$b_i^*(v_i) = a_i + c_i v_i = \tfrac{1}{2}(v_i + a_j)$$

  for all $v_i \in [0, 1]$.
- Then we have

$$a_i = \tfrac{1}{2} a_j, \text{ and } c_i = \tfrac{1}{2}$$

  for $i, j = 1, 2$ and $i \neq j$.

Linear Bayesian Nash equilibrium

- Therefore,

$$a_1 = a_2 = 0, \text{ and } c_1 = c_2 = \tfrac{1}{2}.$$

- The unique linear Bayesian Nash equilibrium is

$$b_1^*(v_1) = \tfrac{1}{2}v_1, \text{ and } b_2^*(v_2) = \tfrac{1}{2}v_2.$$

Linear Bayesian Nash equilibrium (Cont.)

- Alternatively, if we can somehow guess that $\big(b_1^*(v_1), b_2^*(v_2)\big) = (\frac{v_1}{2}, \frac{v_2}{2})$ is a Bayesian Nash equilibrium, we can prove it directly.
- Suppose player j has adopted the strategy $b_j^*(v_j) = \frac{v_j}{2}$.
- Player i's best response solves

$$\max_{b_i \in [0, \frac{1}{2}]} (v_i - b_i) \operatorname{Prob}(b_i > \tfrac{v_j}{2}) = \max_{b_i \in [0, \frac{1}{2}]} (v_i - b_i) \times 2b_i.$$

- For any $v_i \in [0, 1]$, the unique maximizer is $b_i^*(v_i) = \frac{v_i}{2}$.
- Thus, $\big(b_1^*(v_1), b_2^*(v_2)\big) = (\frac{v_1}{2}, \frac{v_2}{2})$ is a Bayesian Nash equilibrium.

A double auction

- Consider the case in which a buyer and a seller each have private information about their valuations.
- For example, the buyer is a firm and the seller is a worker. The firm knows the worker's marginal product and the worker knows his or her outside opportunity (Hall and Lazear, 1984).
- Consider a trading game called a double auction.

A double auction

- The buyer's valuation for the good is v_b, and the seller's valuation is v_s.
- Both v_b and v_s are private information, and independently drawn from uniform distributions on $[0, 1]$.
- The seller names an asking price p_s, and the buyer simultaneously names an offer price p_b.
- If $p_b \geq p_s$, then trade occurs at a price $p = \frac{p_b + p_s}{2}$; otherwise no trade occurs.

Normal-form representation

Normal-form representation of this static Bayesian game:

- $A_b = A_s = [0, \infty)$, and $p_k \in A_k$ for $k = b, s$;
- $T_b = T_s = [0, 1]$, and $v_k \in T_k$ for $k = b, s$;
- The buyer believes that v_s is uniformly distributed on $[0, 1]$, and likewise for the seller;
- The buyer's payoff is

$$u_b(p_b, p_s; v_b) = \begin{cases} v_b - p, & \text{if } p_b \geq p_s, \\ 0, & \text{otherwise,} \end{cases}$$

  and the seller's payoff is

$$u_s(p_b, p_s; v_s) = \begin{cases} p - v_s, & \text{if } p_b \geq p_s, \\ 0, & \text{otherwise.} \end{cases}$$

Bayesian Nash equilibrium

- A strategy for the buyer is a function $p_b(v_b)$ specifying the price the buyer will offer for each of the buyer's possible valuations (likewise for the seller).
- A pair of strategies $\left(p_b^*(v_b), p_s^*(v_s)\right)$ is a Bayesian Nash equilibrium if the following two conditions hold.

Bayesian Nash equilibrium (Cont.)

- For buyer, for each $v_b \in [0, 1]$, $p_b^*(v_b)$ solves

$$\max_{p_b} \left[\int_{p_b \geq p_s^*(v_s)} \left[v_b - \tfrac{p_b + p_s^*(v_s)}{2} \right] \mathrm{d}v_s + \int_{p_b < p_s^*(v_s)} 0 \, \mathrm{d}v_s \right].$$

- For seller, for each $v_s \in [0, 1]$, $p_s^*(v_s)$ solves

$$\max_{p_s} \left[\int_{p_b^*(v_b) \geq p_s} \left[\tfrac{p_b^*(v_b) + p_s}{2} - v_s \right] \mathrm{d}v_b + \int_{p_b^*(v_b) < p_s} 0 \, \mathrm{d}v_b \right].$$

One-price equilibrium

- There are many Bayesian Nash equilibria.
- Consider the following one-price equilibrium in which trade occurs at a single price (if at all).
- The buyer's strategy is

$$p_b^*(v_b) = \begin{cases} x, & \text{if } v_b \geq x, \\ 0, & \text{otherwise,} \end{cases}$$

and the seller's strategy is

$$p_s^*(v_s) = \begin{cases} x, & \text{if } v_s \leq x, \\ 1, & \text{otherwise.} \end{cases}$$

One-price equilibrium (Cont.)

- Is it a Bayesian Nash equilibrium?
- Given the buyer's strategy $p_b^*(v_b)$, consider whether the seller with type v_s would want to deviate
  - For $v_s > x$, the seller would prefer "no trading" to "trading" at x.
  - For $v_s < x$, the seller would prefer "trading at x" to "no trading".
  - For $v_s = x$, the seller is indifferent between "trading" at x and "no trading", and will not deviate either.
- An analogous argument applies for the buyer.

One-price equilibrium: Trading area

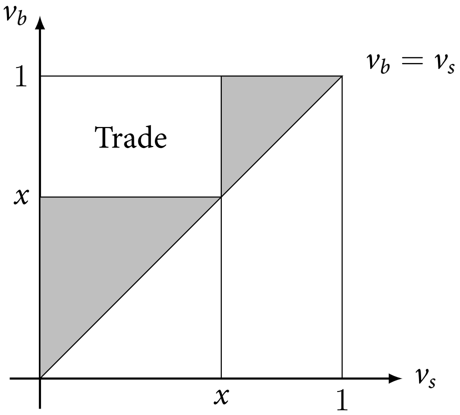

Figure: Outcome in one-price equilibrium

One-price equilibrium: Trading area (Cont.)

- In equilibrium, trade occurs when $v_s \leq x$ and $v_b \geq x$ (as shown in Figure 1).
- However, trade would be efficient for all pairs of (v_s, v_b) such that $v_b \geq v_s$, but does not occur in the two shaded regions.

Linear Bayesian Nash equilibrium

- Now we consider a linear Bayesian Nash equilibrium.
- Suppose the seller's strategy is $p_s(v_s) = a_s + c_s v_s$, and then p_s is uniformly distributed on $[a_s, a_s + c_s]$.
- Buyer's expected payoff, given his type v_b, is

$$
\begin{aligned}
&\mathbf{E}_{v_s}\pi_b\left[p_b, p_s(v_s) \mid v_b\right] \\
= &\int_{a_s \le p_s(v_s) \le p_b} \left[v_b - \tfrac{p_b + p_s(v_s)}{2}\right] \mathrm{d}v_s + \int_{p_b < p_s(v_s) \le a_s + c_s} 0 \, \mathrm{d}v_s \\
= &\int_{a_s \le u \le p_b} \left[v_b - \tfrac{p_b + u}{2}\right] \mathrm{d}\tfrac{u}{c_s} = \tfrac{p_b - a_s}{c_s}\left(v_b - \frac{3}{4}p_b - \frac{1}{4}a_s\right).
\end{aligned}
$$

Best response

- Maximizing $\mathbf{E}_{v_s}\pi_b\left[p_b, p_s(v_s) \mid v_b\right]$ yields buyer's best response

$$p_b(v_b) = \frac{2}{3}v_b + \frac{a_s}{3},$$

  which implies $c_b = \frac{2}{3}$ and $a_b = \frac{a_s}{3}$.
- Thus, if the seller plays a linear strategy, the buyer's best response is also linear.

Best response (Cont.)

- Analogously, suppose the buyer's strategy is $p_b(v_b) = a_b + c_b v_b$, and then p_b is uniformly distributed on $[a_b, a_b + c_b]$.
- Seller's expected payoff, given his type v_s, is

$$
\begin{aligned}
& \mathbf{E}_{v_b} \pi_s \left[p_s, p_b(v_b) \mid v_s \right] \\
= & \int_{a_b \le p_b(v_b) < p_s} 0 \, \mathrm{d} v_b + \int_{p_s \le p_b(v_b) \le a_b + c_b} \left[\tfrac{p_b(v_b) + p_s}{2} - v_s \right] \mathrm{d} v_b \\
= & \int_{p_s \le u \le a_b + c_b} \left[\tfrac{u + p_s}{2} - v_s \right] \mathrm{d} \tfrac{u}{c_b} = \tfrac{p_b - a_s}{c_s} \left(v_b - \tfrac{3}{4} p_b - \tfrac{1}{4} a_s \right) \\
= & \left(\tfrac{3}{2} p_s + \tfrac{a_b + c_b}{2} - 2 v_s \right) \tfrac{a_b + c_b - p_s}{2 c_b}
\end{aligned}
$$

- The best response function of the seller is

$$
p_s(v_s) = \tfrac{2}{3} v_s + \tfrac{1}{3} (a_b + c_b).
$$

Solving equilibrium

- In equilibrium, the buyer's strategy must be a best response to the seller's strategy, i.e.,

$$c_b = \tfrac{2}{3}, \text{ and } a_b = \tfrac{a_s}{3}.$$

- Analogously, the seller's strategy must be a best response to the buyer's strategy, i.e.,

$$c_s = \tfrac{2}{3}, \text{ and } a_s = \tfrac{a_b + c_b}{3}.$$

Linear Bayesian Nash equilibrium

- The linear equilibrium strategies are

$$p_b^*(v_b) = \frac{2}{3}v_b + \frac{1}{12},$$

and

$$p_s^*(v_s) = \frac{2}{3}v_s + \frac{1}{4}.$$

- Trade occurs if and only if

$$p_b^*(v_b) \geq p_s^*(v_s) \Longleftrightarrow v_b \geq v_s + \frac{1}{4}.$$

Linear Bayesian Nash equilibrium: Outcome

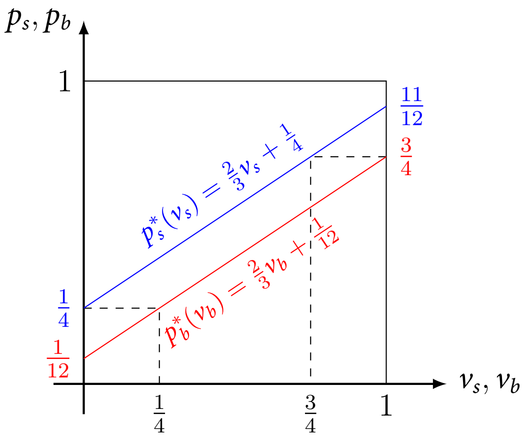

Figure: Strategies of players in the linear equilibrium

Linear Bayesian Nash equilibrium: Trading area

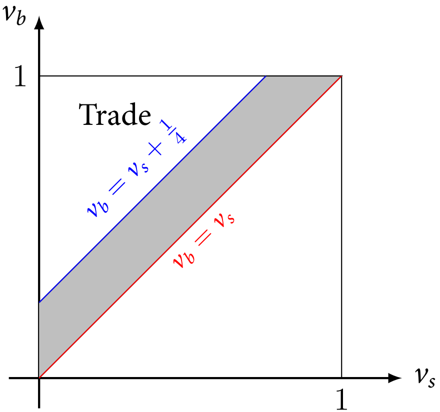

Figure: Outcome in the linear equilibrium

Linear Bayesian Nash equilibrium: Trading area (Cont.)

- In both equilibria, efficient trade does not always occur, but the most valuable trade (i.e., $v_b = 1$ and $v_s = 0$) does occur.
- The one-price equilibrium misses some valuable trades (such as $v_b = 1$ and $v_s = x - \epsilon$, where ϵ is small), and achieves some trades that are worth next to nothing (such as $v_b = x + \epsilon$ and $v_s = x - \epsilon$).
- The linear equilibrium misses all trades that are worth next to nothing, but achieves all trades worth at least $\frac{1}{4}$.
- The linear equilibrium dominates the one-price equilibrium in terms of the expected gains that the players receive.

Inefficiency

- Myerson and Satterwaite (1983) show that for the uniform distributions, the linear equilibrium yields the highest expected gains for the players among all possible Bayesian Nash equilibria in the double auction.
- This implies that there is no Bayesian Nash equilibrium of the double auction in which trade occurs if and only if it is efficient.
- The latter result can be quite general.