Game Theory
One-sided matching

Xiang Sun

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Key question

- How to allocate objects to agents, taking several reasonable requirements into consideration?
- The requirements and solutions may depend on particular situations.

Housing market

- Housing market model was introduced by Shapley and Scarf (1974).
- Each agent owns a house, and a housing market is an exchange (with indivisible objects) where agents have the opinion to trade their houses in order to get a better one.

Housing market (Cont.)

Formally, a housing market is a quadruple $\langle A, H, \succ, e\rangle$ such that

- $A = \{a_1, a_2, \ldots, a_n\}$ is a set of agents,
- H is a set of houses such that $|A| = |H|$,
- $\succ = (\succ_a)_{a\in A}$ is a strict preference profile such that for each agent $a \in A$, $\succ_a$ is a strict preference over houses.
  - Let $\mathcal{P}_a$ be the set of preferences of agent a.
  - The induced weak preference of agent a is denoted by $\succsim_a$ and for any $h, g \in H$, $h \succsim_a g$ if and only if $h \succ_a g$ or $h = g$.
- $e\colon A \to H$ is an initial endowment matching, that is, $h_i \triangleq h_{a_i} \triangleq e(a_i)$ is the initial endowment of agent i.

Example

In a housing market, agents can trade the houses among themselves according to certain rules and attempt to make themselves better off.

Let $A = \{a_1, a_2, a_3, a_4\}$ and let h_i be the occupied house of agent a_i. Let the preference profile $\succ$ be given as:

a_1	a_2	a_3	a_4
h_4	h_3	h_2	h_3
h_3	h_4	h_4	h_2
h_2	$\boldsymbol{h_2}$	h_1	h_1
$\boldsymbol{h_1}$	h_1	$\boldsymbol{h_3}$	$\boldsymbol{h_4}$

Example (Cont.)

- These four agents can trade the houses and get the following (Pareto) improved reallocation

$$\mu_1 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ h_4 & h_3 & h_1 & h_2 \end{bmatrix}.$$

- They also have the following (Pareto) improved reallocation

$$\mu_2 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ h_4 & h_3 & h_2 & h_1 \end{bmatrix}.$$

- What are desirable outcomes of such a reallocation process? What allocative mechanisms are appropriate for achieving desirable outcomes?

Matching and mechanism

- In a housing market, a matching (allocation) is a bijection $\mu\colon A \to H$.
- Here $\mu(a)$ is the assigned house of agent a under matching μ.
- Let $\mathcal{M}$ be the set of matchings.
- A mechanism is a procedure that assigns a matching for each housing market $\langle A, H, \succ, e\rangle$.
- For the fixed sets of agents A and houses H, a mechanism becomes a function

$$\varphi\colon \times_{a\in A} \mathcal{P}_a \to \mathcal{M}.$$

Individual rationality

- A matching μ is individually rational if for each agent $a \in A$,

$$\mu(a) \succsim_a h_a = e(a),$$

  that is, each agent is assigned a house at least as good as her own occupied house.
- A mechanism is individually rational if it always selects an individually rational matching for each housing market.
- In Example, the matchings μ_1 and μ_2 are individually rational.

Pareto efficiency

- A matching μ is Pareto efficient if there is no other matching ν such that
  - $\nu(a) \succsim_a \mu(a)$ for all $a \in A$, and
  - $\nu(a_0) \succ_{a_0} \mu(a_0)$ for some $a_0 \in A$.
- A mechanism is Pareto efficient if it always selects a Pareto efficient matching for each housing market.
- In Example, the matchings μ_1 and μ_2 are Pareto efficient.

More requirement

- In Example, if houses are assigned according to μ_1, then agents 2 and 3 will not attend this reallocation process.
- Instead, they will trade with each other; that is, agent 2 gets house 3 and agent 3 gets house 2.
- Clearly, this trade benefits agent 3 and does not hurt agent 2, compared with μ_1.
- In other words, matching μ_1 is blocked by the coalition $\{2, 3\}$ and the trade between them.
- Such a matching is not good enough, and a core matching, defined in the following paragraphs, is required to exclude such blocks.

Core

- A matching μ is blocked by a coalition C via another matching σ if
  - $\sigma(C) \subseteq \mu(C)$,
  - $\sigma(a) \succsim_a \mu(a)$ for all $a \in C$, and $\sigma(a_0) \succ_{a_0} \mu(a_0)$ for some $a_0 \in C$.
- The core is the collection of matchings such that no coalition could improve their assigned houses even if they traded their initially occupied houses only among each other.
- We shall use core$(\succ)$ or core to denote the core.
- A matching in the core is called a core matching.
- A mechanism is called a core mechanism if it always selects a core matching for each housing market, denoted by φ^{core}.
- It is clear that a core matching is Pareto efficient (take $C = A$) and individually rational (take $C = \{a\}$ for some $a \in A$).

Core is non-empty

Theorem (Shapley and Scarf, 1974)

The core of a housing market is non-empty.

Gale's top trading cycles algorithm

Step 1:

- Each agent points to the owner of his favorite house.
- Due to the finiteness of agents, there exists at least one cycle (including self-cycles). Moreover, cycles do not intersect.
- Each agent in a cycle is assigned the house of the agent he points to and removed from the market.
- If there is at least one remaining agent, proceed with the next step.

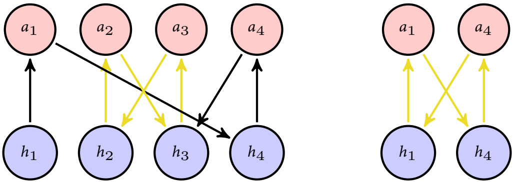
Gale's top trading cycles algorithm (Cont.)

Step k:
- Each remaining agent points to the owner of his favorite house among the remaining houses.
- Each agent in a cycle is assigned the house of the agent he points to and removed from the market.
- If there is at least one remaining agent, proceed with the next step.

End: No agents remain. It is clear that the algorithm will terminate within finite steps.

The mechanism determined by top trading cycles algorithm is denoted by TTC.

Revisit example



The outcome of TTC is

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ h_4 & h_3 & h_2 & h_1 \end{bmatrix}.$$

Proof: core

Let μ be the TTC assignment. Suppose instead μ is not in the core.

1. By definition of core, μ is blocked by some coalition C via matching σ.
2. Let i be the earliest matched agent in TTC among agents in C who are strictly better off under σ. Then $\sigma(i)$ must have been removed earlier than i in TTC.
3. Suppose $\sigma(i) = h_j$, the house of agent j. Then $j \in C$ and is matched earlier than i in TTC. By assumption on i, j is not strictly better off under σ, i.e., $\sigma(j) = \mu(j)$.
4. Likewise, $j \in C$ implies that the owner of $\sigma(j)$ is in C, is removed together with j in TTC, and is matched the same under σ and μ.
5. By induction, the agent who obtains h_j in TTC is in C, and since h_j is assigned to i at σ, this agent must be strictly improved at σ. Since she is removed earlier than i, we have a contradiction.

Uniqueness

Theorem (Roth and Postlewaite, 1977)

The matching produced by Gale's TTC is the unique core matching of housing market.

Proof: uniqueness

Suppose other than the TTC matching μ, σ is another matching in the core.

1. Then for some i, $\mu(i) \neq \sigma(i)$.
2. Note that first, i cannot be any agent who is matched in the first round of TTC, because such agents obtain their most favorite at μ and $\sigma(i) \neq \mu(i)$ implies that $\sigma(i)$ is worse.
3. Therefore, i can always block σ by forming a coalition with members in the same cycle in TTC.
4. By the same argument i cannot be any agent in the second round of TTC, and so on.

Strategy-proof mechanism

A mechanism φ is strategy-proof if for each housing market $\langle A, H, \succ, e\rangle$, for each $a \in A$, and for each $\succ'_a$, we have

$$\varphi[\succ](a) \succsim_a \varphi[\succ_{-a}, \succ'_a](a).$$

Theorem (Roth 1982)

TTC is strategy-proof.

Intuition: Once being pointed by others, an agent never loses the chain pointing to her, so she can get the house any later time if she wants.

Characterization

Theorem (Ma, 1994)

A mechanism is strategy-proof, Pareto efficient and individually rational if and only if it is TTC.

Suppose a mechanism φ satisfies all three axioms above. Let $\succ$ be any preference profile.

1. First, consider any agent i who trades in the first step of TTC. Suppose instead $\varphi[\succ](i) \neq \text{TTC}[\succ](i)$.
2. If i trades with herself in TTC, then since φ satisfies IR, $\varphi[\succ](i) = \text{TTC}[\succ](i) = h_i$.
3. Otherwise, i trades with others in TTC. For simplicity, suppose it is a two-way cycle $i \leftrightarrow j$.

Characterization (Cont.)

4. Then i top ranks h_j and $\mathrm{TTC}[\succ](i) = h_j$ is better than $\varphi[\succ](i)$: Consider an alternative preference $\succ_i'$: $h_j h_i \emptyset$ of i.
5. Since φ is strategy-proof, $\varphi[\succ_i', \succ_{-i}](i) \neq h_j$, because otherwise i has incentive to misreport $\succ_i'$ when her true preference is $\succ_i$; hence due to IR, $\varphi[\succ_i', \succ_{-i}](i) = h_i$.
6. Consequently, $\varphi[\succ_i', \succ_{-i}](j) \neq h_i$ and j is worse off than in TTC. Similarly, consider $\succ_j'$: $h_i h_j \emptyset$.
7. Since φ is strategy-proof, $\varphi[\succ_i', \succ_j', \succ_{-i-j}](j) \neq h_i$ and due to IR, $\varphi[\succ_i', \succ_j', \succ_{-i-j}](j) = h_j$.
8. At the preference profile $(\succ_i', \succ_j', \succ_{-i-j})$, φ is not Pareto efficient, a contradiction. The rest of the proof follows from induction.

House allocation

- The house allocation problem was introduced by Hylland and Zeckhauser (1979). In this problem, there is a group of agents and houses.
- Each agent shall be allocated a house by a central planner using preferences over the houses.
- A house allocation problem is a triple $\langle A, H, \succ \rangle$ such that
  - $A = \{a_1, a_2, \ldots, a_n\}$ is a set of agents,
  - $H = \{h_1, h_2, \ldots, h_n\}$ is a set of houses,
  - $\succ = (\succ_a)_{a \in A}$ is a strict preference profile such that for each agent $a \in A$, $\succ_a$ is a strict preference over houses.

Matching and mechanism

- In a house allocation problem $\langle A, H, \succ \rangle$, a matching (allocation) is a bijection $\mu \colon A \to H$.
- Here $\mu(a)$ is the assigned house of agent a under matching μ.
- Let $\mathcal{M}$ be the set of matchings.
- A mechanism is a procedure that assigns a matching for each house allocation problem $\langle A, H, \succ \rangle$.
- For the fixed sets of agents A and houses H, a mechanism becomes a function

$$\varphi \colon \times_{a \in A} \mathcal{P}_a \to \mathcal{M}.$$

Pareto efficiency

- A matching μ is Pareto efficient if there is no other matching σ such that
  - $\sigma(a) \succsim_a \sigma(a)$ for all $a \in A$, and
  - $\sigma(a_0) \succ_{a_0} \sigma(a_0)$ for some $a_0 \in A$.
- Let $\mathscr{E}$ denote the set of all Pareto efficient matchings.
- A mechanism is Pareto efficient if it always selects a Pareto efficient matching for each house allocation.

Simple serial dictatorship

- An ordering $f\colon \{1, 2, \ldots, n\} \to A$ is a one-to-one and onto function. Each ordering induces the following simple mechanism, which is especially plausible if there is a natural hierarchy of agents. Let $\mathcal{F}$ be the set of all orderings.
- Simple serial dictatorship induced by an ordering f, denoted by SD^f.

  Step 1: The highest priority agent $f(1)$ is assigned her top choice house under $\succ_{f(1)}$.

  Step k: The k-th highest priority agent $f(k)$ is assigned her top choice house under $\succ_{f(k)}$ among the remaining houses.

Simple serial dictatorship is Pareto efficient

Proposition

Simple serial dictatorship induced by an ordering f, SD^f, is Pareto efficient.

Proof.

1. Suppose that there is a matching σ that Pareto dominates $\mathrm{SD}^f[\succ]$.
2. Consider the agent $a = f(i)$ with the highest priority who obtains a strictly better house in σ than in $\mathrm{SD}^f[\succ]$.
3. Then $\sigma(a) = \mathrm{SD}^f[\succ](b)$ for some agent $b = f(j)$ with $j < i$.
4. By assumption, a is the agent with highest priority such that $\sigma(a) \succ_a \mathrm{SD}^f[\succ](a)$, so $\sigma(b) \succ_b \mathrm{SD}^f[\succ](b)$ is impossible.
5. Since σ Pareto dominates $\mathrm{SD}^f[\succ]$, $\sigma(b) \succsim_b \mathrm{SD}^f[\succ](b)$.
6. Therefore, $\sigma(b) = \mathrm{SD}^f[\succ](b)$, which leads to a contradiction.

Core from assigned endowments

Core from assigned endowments μ, denoted by TTC^{μ}:

- For any house allocation problem $\langle A, H, \succ \rangle$, select the unique element of the core of the housing market $\langle A, H, \succ, \mu \rangle$ where each agent a's initial house is $\mu(a)$. That is,

$$\mathrm{TTC}^{\mu}[\succ] = \mathrm{TTC}[\succ, \mu].$$

Efficient matching

Theorem (Abdulkadiroğlu and Sönmez, 1998)

- For any house allocation problem $\langle A, H, \succ \rangle$, for any ordering f, and for any matching μ, the simple serial dictatorship induced by f and the core from assigned endowments μ both yield Pareto efficient matchings.
- Moreover, for any Pareto efficient matching ν, there is a simple serial dictatorship and a core from assigned endowments that yield it.

House allocation with existing tenants

- Motivated by real-life on-campus housing practices, Abdulkadiroğlu and Sönmez (1998) introduced a house allocation problem with existing tenants.
- A set of houses shall be allocated to a set of agents by a centralized clearing house.
- Some of the agents are existing tenants, each of whom already occupies a house, referred to as an occupied house, and the rest of the agents are newcomers.
- Each agent has strict preferences over houses.
- In addition to occupied houses, there are vacant houses.
- Existing tenants are entitled not only to keep their current houses but also to apply for other houses.

House allocation with existing tenants (Cont.)

- A house allocation problem with existing tenants, denoted by $\langle A_E, A_N, H_O, H_V, \succ \rangle$, consists of
  - a finite set of existing tenants A_E,
  - a finite set of new applicants A_N,
  - a finite set of occupied houses $H_O = \{h_i \colon a_i \in A_E\}$,
  - a finite set of vacant houses H_V, and
  - a strict preference profile $\succ = (\succ_i)_{i \in A_E \cup A_N}$.
- Let $A = A_E \cup A_N$ denote the set of all agents and $H = H_O \cup H_V \cup \{h_0\}$ denote the set of all houses plus the null house.
- Agent i's strict preference $\succ_i$ is on H.
  - Let $\mathcal{P}$ be the set of all strict preferences on H.
  - Let $\succsim_i$ be agent i's induced weak preference. We assume that the null house h_0 is the last choice for each agent.

Top trading cycles algorithm, induced by a given ordering f of agents.

Step 1: Define the set of available houses for this step to be the set of vacant houses.

1. Each agent a points to her favorite house under her reported preference.
2. Each occupied house points to its occupant.
3. Each available house points to the agent with highest priority (*i.e.*, $f(1)$).

- Since the numbers of agents and houses are finite, there is at least one cycle, here a cycle is an ordered list of agents and houses $(j_1, j_2, \ldots, j_k)$ where j_1 points to j_2, j_2 points to j_3, …, j_k points to j_1.
- Every agent who participates in a cycle is assigned the house that she points to, and removed with her assignment.
- Whenever there is an available house in a cycle, the agent with the highest priority, $f(1)$, is also in the same cycle. If this agent is an existing tenant, then her house $h_{f(1)}$ can not be in any cycle and it becomes available for Step 2.
- All available houses that are not removed remain available.

Step k: The set of available houses for Step k is defined at the end of Step $(k-1)$.

1. Each remaining agent a points to her favorite house among the remaining houses under her reported preference.
2. Each remaining occupied house points to its occupant.
3. Each available house points to the agent with highest priority among the remaining agents.

- There is at least one cycle. Every agent in a cycle is assigned the house that she points to and removed with her assignment.
- If there is an available house in a cycle then the agent with the highest priority in this step is also in the same cycle. If this agent is an existing tenant, then her house can not be in any cycle and it becomes available for Step $(k+1)$.
- All available houses that are not removed remain available.

End: If there is at least one remaining agent and one remaining house, then the process continues.

We use TTC^f to denote the top trading cycles mechanism induced by the ordering f.

Properties of TTC^f

- For any ordering f, the induced top trading cycles mechanism TTC^f is Pareto efficient.
- For any ordering f, the induced top trading cycles mechanism TTC^f is individually rational.
- For any ordering f, the induced top trading cycles mechanism TTC^f is strategy-proof.

YRMH-IGYT

You request my house—I get your turn (YRMH-IGYT) algorithm, induced by a given ordering f:

Phase 1: Assign the first agent her top choice, the second agent her top choice among the remaining houses, and so on, until someone demands the house of an existing tenant.

Phase 2:

- If at that point the existing tenant whose house is requested is already assigned another house, then do not disturb the procedure.
- Otherwise, modify the remainder of the ordering by inserting this existing tenant before the requestor at the priority order and proceed with the Phase 1 through this existing tenant.
- Similarly, insert any existing tenant who is not already served just before the requestor in the priority order once her house is requested by an agent.

YRMH-IGYT (Cont.)

Phase 3:

- If at any point a cycle forms, it is formed by exclusively existing tenants and each of them requests the house of the tenant who is next in the cycle.
- A cycle is an ordered list $(h_1, a_1, \ldots, h_k, a_k)$ of occupied houses and existing tenants where agent a_1 demands the house a_2, h_2, agent a_2 demands the house of agent a_3, h_3, . . . , agent a_k demands the house of a_1, h_1.
- In such case, remove all agents in the cycle by assigning them the house they demand and proceed similarly.

YRMH-IGYT

The YRMH-IGYT algorithm generalizes simple serial dictatorship and TTC:

- The YRMH-IGYT algorithm coincides with simple serial dictatorship when there are no existing tenants: Without existing tenants, the "you request my house ..." contingency simply does not happen, so the mechanism coincides with simple serial dictatorship.
- The YRMH-IGYT algorithm coincides with TTC when all agents are existing tenants and there is no vacant house: In this case, an agent's request always points to a house owned by someone, and the assignment of a house happens if and only if there is a cycle made of existing tenants.

YRMH-IGYT vs. TTC^f

Theorem (Abdulkadiroğlu and Sönmez, 1998)

For a given ordering f, the YRMH-IGYT algorithm yields the same outcome as the top trading cycles algorithm.

Summary

By now:

- For markets with ownership (e.g., housing market): Trading mechanisms.
- For markets without ownership (e.g., house allocation): Serial dictatorship.
- Mixed ownership (e.g., allocation with existing tenants): Mixed mechanism.

For general allocation problems

- Endow: create a housing market;
- and then apply trading mechanism.

Kidney exchange

- Mathematically, the problem of kidney exchange is quite similar to house allocation with existing tenants.
  - Kidney patients want to obtain a kidney for transplantation.
  - There are kidneys from diseased donors as well as "good Samaritan donors" (similar to "vacant houses").
  - Some kidney patients have willing but incompatible donors (similar to "existing tenants").
- However, there are some medical and logistical constraints that may make a direct application of existing theories impossible.
- This fact motivates new theories to be explored.

Kidney exchange: Background

- Transplant is an important treatment of serious kidney diseases. Over 90,000 patients are on waiting lists for kidney in the US. In 2011, there were
  - 11,043 transplants from diseased donors,
  - 5,771 transplants from living donors, while
  - 4,697 patients died while on the waiting list (and 2,466 others were removed because they were "too sick to transplant").
- Buying and selling kidneys is illegal in the US as well as many other countries.
- Given that constraint, donation is the most important source of kidneys.

Donation

There are two sources of donation:

- Deceased donors: In the US and Europe a centralized priority mechanism is used for the allocation of deceased donor kidneys. The patients are ordered in a waiting list, and the first available donor kidney is given to the patient who best satisfies a metric based on the quality of the match, waiting time in the queue, age of the patient, and other medical and fairness criteria.
- Living donors: Living donors usually come from friends or relatives of a patient (because the monetary transaction is prohibited).

Donation (Cont.)

Live donation has been increasing recently.

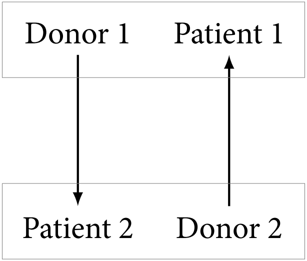
Donor types	2008	1998	1988
All donors	10,920	9,761	5,693
Deceased donors	5,992	5,339	3,876
Live donors	4,928	4,422	1,817

Compatibility issues

- For a successful transplant, the donor kidney needs to be compatible with the patient: blood-type incompatibility or antibodies.
- A problem with transplant from live donors: transplant is carried out if the donor kidney is compatible with the patient. Otherwise the willing donor goes home and the patient cannot get transplant.

Paired exchange

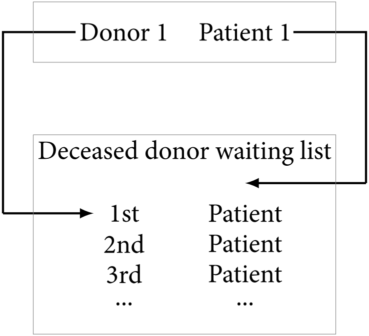
- A paired exchange (aka paired donation) involves two incompatible patient-donor pairs such that the patient in each pair feasibly receives a transplant from the donor in the other pair.
- This pair of patients exchange donated kidneys.
- The number of pairs in a paired exchange can be larger than two.



Take a look at the web page of Alliance for Paired Donation at `http://paireddonation.org/`.

List exchange

- A list exchange involves an exchange between one incompatible patient-donor pair and the deceased donor waiting list.
- The patient in the pair becomes the first priority person on the deceased donor waiting list in return for the donation of her donor's kidney to someone on the waiting list.



List exchange (Cont.)

- List exchanges can potentially harm O blood-type patients waiting on the deceased donor waiting list.
- Since the O blood type is the most common blood type, a patient with an incompatible donor is most likely to have O blood herself and a non-O bloodtype incompatible donor.
- Thus, after the list exchange, the blood type of the donor sent to the deceased donor waiting list has generally non-O blood, while the patient placed at the top of the list has O blood.
- Therefore, list exchanges are deemed ethically controversial.

Design

- In 2004, the Renal Transplant Oversight Committee of New England approved the establishment of a clearinghouse for kidney exchange.
- Roth, Sönmez and Unver as well as doctors design the clearinghouse.
- Potential issues include
    - Efficiency (Pareto efficiency; maximizing number of transplantation)
    - Fairness
    - Incentives (Strategy-proofness)

Incentive issues

Do patients and doctors behaves strategically? Here is one example indicating they do.

A news report by Reuters (2003-7-29)

Three Chicago hospitals were accused of fraud by prosecutors on Monday for manipulating diagnoses of transplant patients to get them new livers.

Two of the institutions paid fines to settle the charges.

"By falsely diagnosing patients and placing them in intensive care to make them appear more sick than they were, these three highly regarded medical centers made patients eligible for liver transplants ahead of others who were waiting for organs in the transplant region," said Patrick Fitzgerald, the U.S. attorney for the Northern District of Illinois.

Kidney exchange model

A kidney exchange model is composed of

- A set of donor-patient (kidney-transplant) pairs,
- A preference over all kidneys and "high priority in the waitlist" (in exchange of donating a kidney).

A matching is a function that specifies which patient obtains which kidney (or waitlist). We assume that the wait list can be matched with any number of patients.

Simplest design

Roth, Sonmez and Unver (2004) assume that

- There is no limit on the number of pairs participating in one exchange.
- Patients have strict preferences over compatible kidneys and the waitlist.

  Some justification by Opelz (1997). He shows that, in his data, increase in the number of HLA mismatch decreases the likelihood of kidney survival. Other characteristics such as body size and donor age affect kidney survival.

With the assumption of RSU (2004), the kidney exchange problem is mathematically very similar (almost identical!) to house allocation with existing tenants.