

Game Theory

Signaling games

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2021 Summer

- 1 Signaling games
 - Model
 - Perfect Bayesian equilibrium
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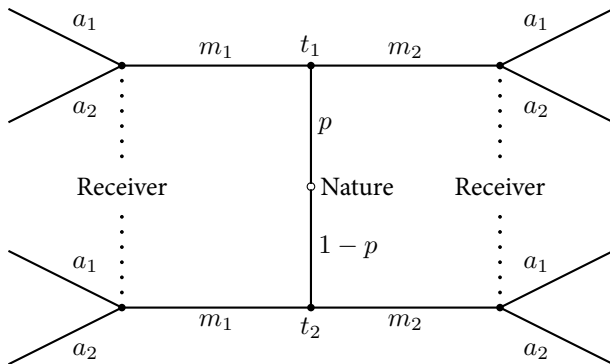
Introduction

- Signaling games (信号发送博弈) are a well-studied class of dynamic games of incomplete information.
- The concept of “signaling” (信号发送) refers to strategic models where **informed agents** (知情人) take some **observable actions** before **uninformed agents** (不知情的人) make their strategic decisions.
- Signaling games are a relatively simple setting in which to study
 - how players **update beliefs** based on observed actions (signals);
 - how players try to **strategically reveal or conceal private information** by their choice of actions.
- There are many applications of signaling games in economics (for example, Spence’s job-market signaling model).

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Illustration



Timing

- A simple signaling game is a dynamic game of incomplete information involving two players: a **Sender (S)** and a **Receiver (R)**.
- The timing of the game is as follows:
 - 1 **Nature draws** a type t_i for the Sender from a set of feasible types $T = \{t_1, \dots, t_I\}$ according to a probability distribution $P(t_i)$, where $P(t_i) > 0$ for every i and $P(t_1) + \dots + P(t_I) = 1$.
 - 2 The **Sender** observes t_i and then chooses a message m_j from a set of feasible messages $M = \{m_1, \dots, m_J\}$.
 - 3 The **Receiver** observes m_j (but not t_i) and then chooses an action a_k from a set of feasible actions $A = \{a_1, \dots, a_K\}$.
 - 4 **Payoffs** are given by $U_S(t_i, m_j, a_k)$ and $U_R(t_i, m_j, a_k)$.

Strategy

- Consider the following signaling game:

$$T = \{t_1, t_2\}, A = \{a_1, a_2\}, P(t_1) = p, M = \{m_1, m_2\}.$$

- The Sender has four pure strategies:

$$(m_1, m_1), (m_1, m_2), (m_2, m_1), (m_2, m_2).$$

- The strategy (m', m'') means the Sender of type t_1 chooses a message m' and type t_2 chooses a message m'' .

Strategy (Cont.)

- Similarly, the Receiver has four pure strategies:

$$(a_1, a_1), (a_1, a_2), (a_2, a_1), (a_2, a_2).$$

- The strategy (a', a'') means the Receiver plays a' if the Sender chooses m_1 and plays a'' if the Sender chooses m_2 .
- We call Sender's strategies
 - $(m_1, m_1), (m_2, m_2)$ to be **pooling (混同)** (because each type sends the same message);
 - $(m_1, m_2), (m_2, m_1)$ to be **separating (分离)** (because each type sends a different message).

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Signaling Requirement 1: Belief

We first translate the requirements for a perfect Bayesian equilibrium to the case of signaling games.

Signaling Requirement 1

After observing any message m_j from M , the Receiver must have a **belief** about which **types** could have sent m_j .

Denote this belief by the probability distribution $\mu(t_i|m_j)$, where $\mu(t_i|m_j) \geq 0$ for each $t_i \in T$, and $\sum_{t_i \in T} \mu(t_i|m_j) = 1$.

Signaling Requirement 2: Sequential rationality

Signaling Requirement 2R

For each $m_j \in M$, the Receiver's action $a^*(m_j)$ must **maximize** the Receiver's **expected utility**, given the belief $\mu(t_i|m_j)$ about which types could have sent m_j . That is, $a^*(m_j)$ solves

$$\max_{a_k \in A} \sum_{t_i \in T} \mu(t_i|m_j) U_R(t_i, m_j, a_k).$$

Signaling Requirement 2S

For each $t_i \in T$, the Sender's message $m^*(t_i)$ must **maximize** the Sender's **utility**, given the Receiver's strategy $a^*(m_j)$. That is, $m^*(t_i)$ solves

$$\max_{m_j \in M} U_S(t_i, m_j, a^*(m_j)).$$

Signaling Requirement 2: Sequential rationality (Cont.)

- These two requirements imply that both the Receiver and the Sender act in an **optimal way**.
- Given the Sender's optimal strategy $m^*(t_i)$, i.e., m^* is a function from T into M , let

$$T_j = \{t_i \in T : m^*(t_i) = m_j\}.$$

T_j is the set of all types sending the message m_j .

- The information set corresponding to m_j is
 - **on the equilibrium path** if $T_j \neq \emptyset$,
 - **off the equilibrium path** otherwise.

Signaling Requirement 3: Rational belief

Signaling Requirement 3

For each $m_j \in M$, if there exists $t_i \in T$ such that $m^*(t_i) = m_j$, i.e., $T_j \neq \emptyset$, then the Receiver's belief at the information set corresponding to m_j must follow from **Bayes' rule** and the Sender's strategy:

$$\mu(t_i|m_j) = \frac{P(t_i)}{\sum_{t \in T_j} P(t)}, \forall t_i \in T_j.$$

Perfect Bayesian Equilibria

Definition

A pure-strategy **perfect Bayesian equilibrium** in a signaling game is a pair of **strategies** $m^*(t_i)$ and $a^*(m_j)$ and a **belief** $\mu(t_i|m_j)$ satisfying Signaling Requirements (1), (2R), (2S), and (3).

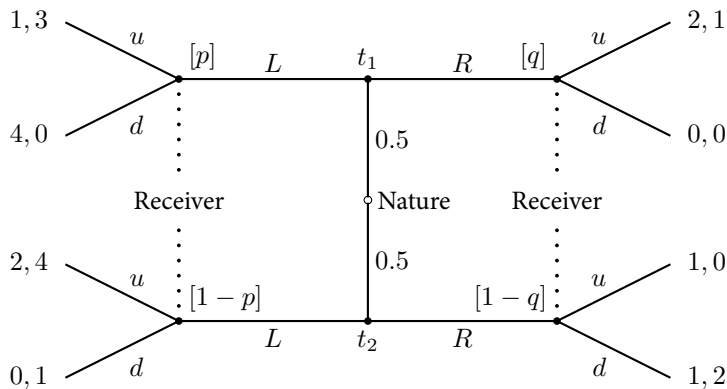
- A **strategy** for the Sender is a function from the type space T into the message space M .
- A **strategy** for the Receiver is a function from the message space M into the action space A .
- For a perfect Bayesian equilibrium of a signaling game, if the Sender's strategy is **pooling** 混同 (or **separating** 分离), then we call the equilibrium pooling (or separating), respectively.

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Example

Find all pure-strategy perfect Bayesian equilibria in the following signaling game.



The first (the second) number is the payoff to the Sender (the Receiver).

Formulation

- In this game,

$$T = \{t_1, t_2\}, P(t_1) = 0.5, M = \{L, R\}, A = \{u, d\}.$$

- The Sender's strategies are: (L, L) , (L, R) , (R, L) and (R, R) , where (m', m'') means that type t_1 chooses m' and type t_2 chooses m'' .
- The Receiver's strategies are: (u, u) , (u, d) , (d, u) , and (d, d) , where (a', a'') means that the Receiver plays a' following L and a'' following R .
- We analyze the possibility of the four Sender's strategies to constitute perfect Bayesian equilibria.

Case 1: PBE pooling on L

- Suppose the Sender adopts the strategy (L, L) .
- By Signaling Requirement 3, we have $p = 1 - p = 0.5$.
- * Given this belief (or any belief) of the Receiver, the Receiver's best response to message L is u , i.e., $a^*(L) = u$.
- For the message R , the Receiver's belief q cannot be determined by Sender's strategy, and thus we can choose **any** belief q .
- * Both $a^*(R) = u$ and $a^*(R) = d$ are possible for some q . Indeed $a^*(R) = u$ iff $q \geq \frac{2}{3}$; and $a^*(R) = d$ iff $q \leq \frac{2}{3}$.
- We only need to see if sending L is better than sending R for both types t_1 and t_2 .

Case 1: PBE pooling on L (Cont.)

- If $a^*(R) = u$, i.e., (u, u) is the Receiver's strategy, then for type t_1 , the Sender's payoff is 1 if L is sent and 2 if R is sent. Hence, sending L is not optimal.
- If $a^*(R) = d$, i.e., (u, d) is the Receiver's strategy, then for type t_1 , the Sender's payoff is 1 if L is sent and 0 if R is sent, choosing L is optimal; for type t_2 , choosing L is also optimal given $2 > 1$.
- Thus, (L, L) is the Sender's best response to the Receiver's strategy (u, d) .
- Moreover, (u, d) is also the Receiver's best response to the Sender's strategy (L, L) if $q \leq \frac{2}{3}$.
- Therefore, $((L, L), (u, d); p = \frac{1}{2}, q \leq \frac{2}{3})$ is a pooling equilibrium.

Case 1: PBE pooling on L (Cont.)

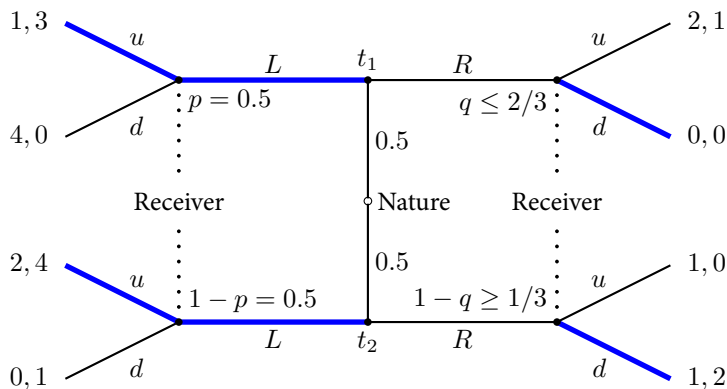


Figure: Pooling equilibrium: $\left((L, L), (u, d); p = \frac{1}{2}, q \leq \frac{2}{3}\right)$

Case 2: PBE pooling on R

- Suppose the Sender adopts the strategy (R, R) .
- Then Signaling Requirement 3 implies that $q = 1 - q = \frac{1}{2}$.
- * Given this belief, the Receiver's best response to R is d , i.e., $a^*(R) = d$, since $\frac{1}{2} < 1$.
- For the message L , we can choose **any** belief p .
- * But we know for any p , the Receiver's best response to L is u , i.e., $a^*(L) = u$.
- Given the Receiver's strategy (u, d) , for type t_1 , the Sender's payoff is 0 if R is sent and 1 if L is sent, and thus R is not optimal.
- Therefore, there is no equilibrium in which the Sender plays (R, R) .

Case 3: Separation with t_1 playing L

- Suppose the Sender adopts the separating strategy (L, R) .
- Then, Signaling Requirement 3 implies $p = 1$ and $q = 0$.
- ★ For these beliefs, we must have $a^*(L) = u$, and $a^*(R) = d$.
- Given the Receiver's strategy (u, d) , for type t_2 , the Sender's payoff is 4 if L is sent and 2 if R is sent. Hence R is not optimal.
- Therefore, there is no equilibrium in which the Sender plays (L, R) .

Case 4: Separation with t_1 playing R

- Suppose the Sender adopts the separating strategy (R, L) .
- Then, Signaling Requirement 3 implies $p = 0$ and $q = 1$.
- ★ For these beliefs, we have $a^*(L) = u$ and $a^*(R) = u$.
- Given the Receiver's strategy (u, u) , for type t_1 , the Sender's payoff is 1 if L is sent and 2 if R is sent. Hence R is optimal.
- For the Sender type t_2 , the payoff is 2 if L is sent and 1 if R is sent. Hence L is also optimal.
- Therefore, $((R, L), (u, u); p = 0, q = 1)$ is a separating perfect Bayesian equilibrium.

Case 4: Separation with t_1 playing R (Cont.)

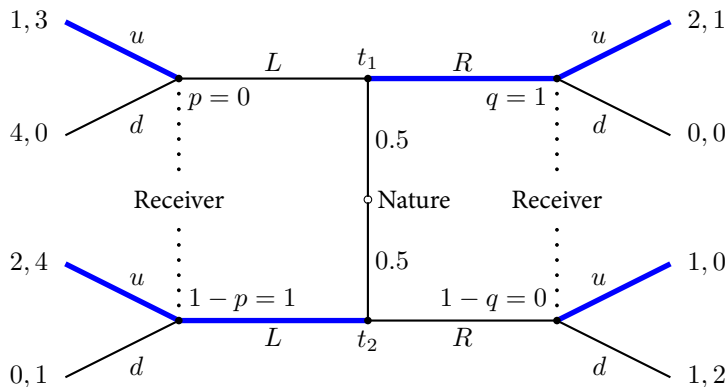


Figure: Separating equilibrium: $((R, L), (u, u); p = 0, q = 1)$

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How to find PBE

How to find (pure-strategy) perfect Bayesian equilibria in signaling games:

- 1 Start with a strategy of the Sender (pooling or separating);
- 2 If possible, calculate the beliefs of the Receiver using Bayes' rules. Otherwise, choose arbitrary beliefs;
- 3 Given the beliefs, find out the best response of the Receiver;
- 4 Check whether the Sender's strategy is a best response to the Receiver's strategy.

How to find PBE (Cont.)

- Consider an alternative way to find perfect Bayesian equilibria.
- We first find **Bayesian Nash equilibria**, and then check which equilibria are **perfect Bayesian equilibria**.
- Consider the following bi-matrix to represent the game:

		Receiver			
		(u, u)	(u, d)	(d, u)	(d, d)
Sender	(L, L)	$1, \underline{2}, \underline{3.5}$	$\underline{1}, \underline{2}, \underline{3.5}$	$\underline{4}, 0, 0.5$	$\underline{4}, 0, 0.5$
	(L, R)	$1, 1, 1.5$	$\underline{1}, 1, \underline{2.5}$	$\underline{4}, \underline{1}, 0$	$\underline{4}, \underline{1}, 1$
	(R, L)	$\underline{2}, \underline{2}, \underline{2.5}$	$0, \underline{2}, 2$	$2, 0, 1$	$0, 0, 0.5$
	(R, R)	$\underline{2}, 1, 0.5$	$0, 1, \underline{1}$	$2, \underline{1}, 0.5$	$0, \underline{1}, \underline{1}$

- Two (pure-strategy) Bayesian Nash equilibria: $((L, L), (u, d))$ and $((R, L), (u, u))$

How to find PBE (Cont.)

- To check whether they are perfect Bayesian equilibria, we only need to find beliefs, satisfying all four Signaling Requirements.
- For (L, L) , Bayes' rule requires $p = \frac{1}{2}$ and there is no requirement for q .
- ★ Given the belief, $a^*(L) = u$, and $a^*(R) = d$ iff $q \leq \frac{2}{3}$. Thus (u, d) is a best response to (L, L) iff $p = \frac{1}{2}$ and $q \leq \frac{2}{3}$.
- For (R, L) , Bayes' rule requires $p = 0$ and $q = 1$.
- ★ Given this belief, $a^*(L) = u$ and $a^*(R) = u$. Thus (u, u) is a best response to (R, L) .
- Therefore, $((L, L), (u, d); p = \frac{1}{2}, q \leq \frac{2}{3})$ and $((R, L), (u, u); p = 0, q = 1)$ are two perfect Bayesian equilibria.

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Cheap talk games

- Cheap talk games (空谈博弈) are analogous to signaling games, but the Sender's messages are just talk, i.e., **costless, non-binding, nonverifiable claims**.
- Cheap talk cannot be informative in some cases (for example, Spence's job-market signaling model).
- There are situations where cheap talk can convey some information (although may not be fully precise), for example, Stein (1989), Matthews (1989), Austen-Smith (1990).
- In general, cheap talk can be **informative** under certain conditions.

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Model

The timing of the simplest cheap talk game is identical to the timing of the simplest signaling game (only payoff functions differ):

- ① **Nature** draws a type t_i for the Sender from a set of feasible types $T = \{t_1, \dots, t_I\}$ according to a probability distribution $P(t_i)$, where $P(t_i) > 0$ for every i and $P(t_1) + \dots + P(t_I) = 1$.
- ② The **Sender** observes t_i and then chooses a message m_j from a set of feasible messages $M = \{m_1, \dots, m_J\}$.
- ③ The **Receiver** observes m_j (but not t_i) and then chooses an action a_k from a set of feasible actions $A = \{a_1, \dots, a_K\}$.
- ④ **Payoffs** are given by $U_S(t_i, a_k)$ and $U_R(t_i, a_k)$ (**independent of m_j**).

Preliminary

- The key feature of the cheap talk game is that the message has **no direct effect on the payoffs** of the Sender and the Receiver.
- The message can only be informative by changing the Receiver's belief about the Sender's type.
- Since anything can be said (i.e., M can be a very large set), it is typically assumed that $M = T$.
- The definition of perfect Bayesian equilibrium in a cheap talk game is identical to that in a signaling game.
- One key difference between these two games is that there always exists a pooling equilibrium in a cheap talk game.

Pooling equilibrium

- The following is a pooling equilibrium:

$$m^*(t_i) = t^*, \mu(t_i | m_j) = P(t_i), a^*(m_j) = a^*$$

for all $t_i \in T$ and $m_j \in M$, where t^* is any message, and a^* solves

$$\max_{a_k \in A} \sum_{t_i \in T} P(t_i) U_R(t_i, a_k).$$

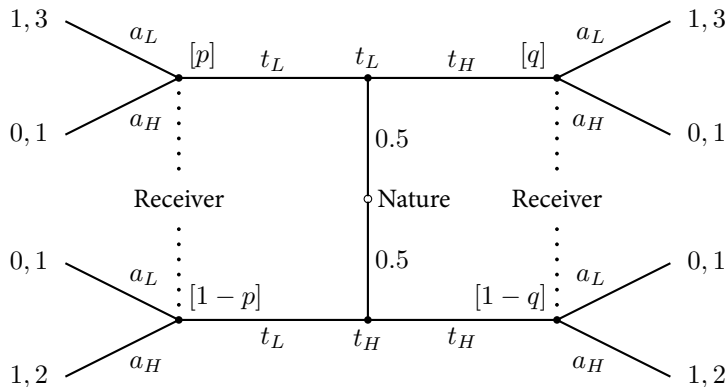
- In this pooling equilibrium, the Sender of all types sends the **same message t^*** , while the Receiver **keeps the prior belief** of all messages and takes an action optimally according to the belief.
- We call it a babbling equilibrium, which is not informative.
- An interesting question is whether there exists any **non-pooling equilibrium** in which **communication can be effective**.

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Example

Find all pure-strategy perfect Bayesian equilibria of the following game.



Example (Cont.)

- Note that the above signaling game is indeed a **cheap talk** game, since neither the Sender's payoff nor the Receiver's payoff depends on the signals.
- Clearly, there are two pooling equilibria:

$$[(t_L, t_L), (a_L, a_L); p = \frac{1}{2}, q \geq \frac{1}{3}],$$

and

$$[(t_H, t_H), (a_L, a_L); p \geq \frac{1}{3}, q = \frac{1}{2}].$$

- There also exists a separating equilibrium:

$$[(t_L, t_H), (a_L, a_H); p = 1, q = 0].$$

Cheap talk game with two types and two actions

- Consider a two-type, two-action example:

$$T = \{t_L, t_H\}, P(t_L) = p, A = \{a_L, a_H\}, M = T.$$

- We use the following matrix to represent the payoffs: the first (second) number is the payoff to the Sender (Receiver).

	t_L	t_H
a_L	$x, 1$	$y, 0$
a_H	$z, 0$	$w, 1$

It is independent of messages.

- Note that the above matrix differs from the normal-form representation of the game.

Cheap talk game with two types and two actions

(Cont.)

- Consider the following **separating equilibrium**:
 - the Sender's strategy: $[m^*(t_L) = t_L, m^*(t_H) = t_H]$;
 - the Receiver's beliefs: $\mu(t_L|t_L) = 1$ and $\mu(t_L|t_H) = 0$;
 - the Receiver's strategy: $[a^*(t_L) = a_L, a^*(t_H) = a_H]$.
- In the above equilibrium, each type of the Sender **tells the truth**.
- It can be shown that the separating equilibrium exists iff $x \geq z$ and $y \leq w$.
- In other words, the Sender's and the Receiver's interests **perfectly align**.
- In general, Crawford and Sobel (1982) have shown that more communication can occur through cheap talk when players' preferences are **more closely aligned**.