Advanced Microeconomics: Lecture Note 8

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1 Signaling for job market

1 The key to (partially) resolve the adverse selection: some "mechanisms/procedures" to help distinguish among workers.

We have already seen the screening approach, which leads to second-best outcome and mitigates the efficiency loss.

2 Signaling is another mechanism, which was first investigated by Spence (1973, 1974).

Basic idea: The high-ability workers may have costly actions to distinguish themselves from low-ability workers.

3 The ideal case: Workers can take a costless test that reveals their types.

Then in any SPE, all workers except with lowest ability will take the test and the market will achieve the full-information/first-best outcome.

In general, no procedure exists that directly reveals a worker's type.

4 There are two types of workers with productivities θ_L and θ_H, where $0 < \theta_L < \theta_H$ and $\lambda = \text{Prob}(\theta = \theta_H) \in (0, 1)$.

5 Before entering the job market, a worker can get some education, and the amount of education that a worker receives is observable.

The cost of obtaining education level e for a type-θ worker is given by $c(e, \theta)$. We assume $c(e, \theta)$ is twice continuously differentiable and $c(0, \theta) = 0$, $c_e(e, \theta) > 0$, $c_{ee}(e, \theta) > 0$, $c_\theta(e, \theta) < 0$ for all $e > 0$, and $c_{e\theta}(e, \theta) < 0$.

Assumption: The education does nothing for a worker's productivity. This assumption can be relaxed.

6 Utility for a type-θ worker who chooses education level e and receives wage w is $w - c(e, \theta)$.

A type-θ worker can earn $r(\theta)$ by working at home.

7 For simplicity, assume $r(\theta) = 0$.

Thus, the unique equilibrium in the absence of the ability to signal: $w^* = \mathbf{E}[\theta]$.

8 Game: One worker and two firms.

- A random move of nature determines whether the worker is of high or low ability.
- Conditional her type, the worker chooses how much education level to obtain. After that, the worker enters the market.
- Conditional the observed education level, two firms simultaneously make wage offers.
- The worker decides whether to work for a firm and, if so, which one.

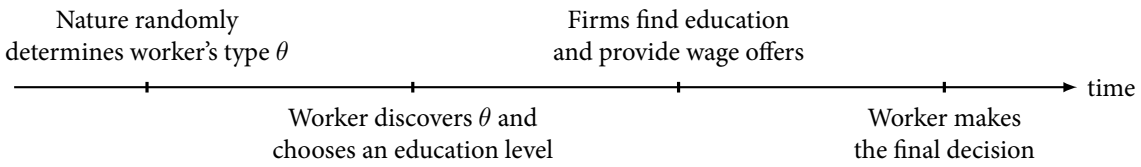


Figure 1: Timing

Why two firms? In this model, we indeed assume that workers have all the bargaining power. When the firm has all the bargaining power, the equilibrium wage is $w = 0$ no matter what the workers' productivity is. In this case, it is not in the workers' interest to acquire costly education so as to signal his productivity.

Here we model only a single worker of unknown type. The model with many workers can be thought of as simply having many of these single-worker games going on simultaneously, with the fraction of high-ability workers in the market being λ (it can be guaranteed by the law of large numbers).

2 PBE

9 A typical strategy of worker: $e(\theta) = \big(e(\theta_H), e(\theta_L)\big)$ (without considering the trivial decision process in the last step).

A typical strategy of firm: $w(e) \colon e \mapsto w(e)$.

10 Perfect Bayesian equilibrium: A pair of strategy profiles $\big(e^*(\theta), w_1^*(e), w_2^*(e)\big)$ and a belief function $\mu^*(e) \in [0, 1]$ giving the firms' common probability assessment that the worker is of high ability after observing education level e such that

- The worker's strategy $e^*(\theta)$ is optimal given the firms' strategies $w_1^*(e)$ and $w_2^*(e)$.
- The belief $\mu^*(e)$ is derived from the workers' strategies $e^*(\theta)$ via Bayes' rule when possible.
- Following each e (i.e., given each $\mu^*(e)$), the firms' wage offers $w_1^*(e)$ and $w_2^*(e)$ constitute a NE.

We focus on pure-strategy PBE.

11 At the end of the game:

(1) After seeing the education level e, the firms have belief $\mu(e)$ that the worker is type θ_H.

(2) The expected productivity is $\mu(e)\theta_H + (1 - \mu(e))\theta_L$.

(3) Like Bertrand pricing game, in any PBE, both firms offer wage $w(e) = \mu(e)\theta_H + (1 - \mu(e))\theta_L$.

For any e, $w(e) \in [\theta_L, \theta_H]$.

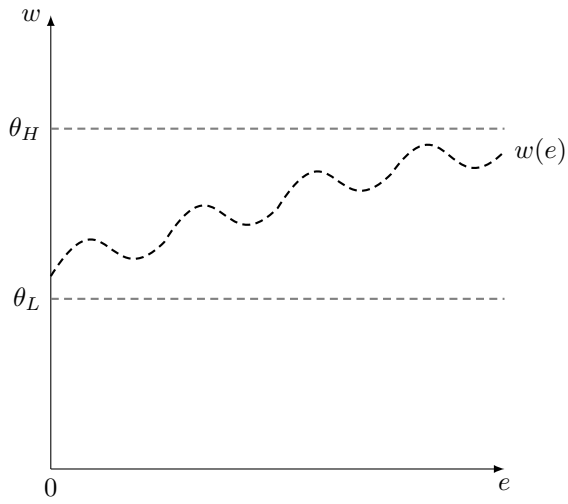


Figure 2: The range of wage

12 Single-crossing property: Due to the assumptions on $c(e, \theta)$, an indifference curve of type-θ_H worker and an indifference curve of type-θ_L worker cross at most once.

(a) A typical indifference curve of θ-worker is $w - c(e, \theta) =$ constant, i.e., $w = c(e, \theta) +$ constant. Then, at any (w, e), the marginal rate of substitution between wages and education is

$$\frac{\mathrm{d}w}{\mathrm{d}e} = c_e(e, \theta),$$

which describes the slope of the indifference curve.

(b) The slope $c_e(e, \theta)$ is decreasing in θ since $c_{e\theta}(e, \theta) < 0$. Thus, at a given point $(\hat{e}, \hat{w})$, for two indifference curves passing it,

$$\begin{aligned}\text{Slope of } \theta_H\text{-indifference curve} &= \left.\frac{\mathrm{d}w(e, \theta_H)}{\mathrm{d}e}\right|_{(\hat{e}, \hat{w})} = c_e(\hat{e}, \theta_H) \\ &< c_e(\hat{e}, \theta_L) = \left.\frac{\mathrm{d}w(e, \theta_L)}{\mathrm{d}e}\right|_{(\hat{e}, \hat{w})} = \text{Slope of } \theta_L\text{-indifference curve.}\end{aligned}$$

(c) For any $\hat{e}$, we also have $c_e(\hat{e}, \theta_H) < c_e(\hat{e}, \theta_L)$. Thus, after point $(\hat{e}, \hat{w})$, θ_L-indifference curve grows faster than θ_H-indifference curve. There is no longer an intersection. Similarly, there is not an intersection before point $(\hat{e}, \hat{w})$.

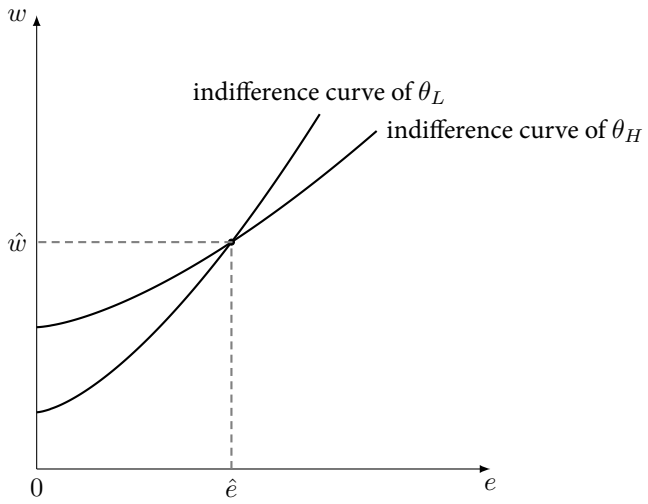


Figure 3: Single-crossing property

13 Preview of the result: The unique outcome of "good" PBE is the best separating PBE outcome:

- High-ability worker: $(\tilde{e}, \theta_H)$.
- Low-ability worker: $(0, \theta_L)$.

3 Separating PBE

14 In a separating PBE (if exists), two types of workers choose different education levels, i.e., $e^*(\theta_H) \neq e^*(\theta_L)$.

15 Lemma: In any separating PBE (if exists), $w^*\big(e^*(\theta_H)\big) = \theta_H$ and $w^*\big(e^*(\theta_L)\big) = \theta_L$.

Proof. (1) In a PBE, beliefs on the equilibrium path (i.e., education levels $e^*(\theta_H)$ and $e^*(\theta_L)$) must be correctly derived from the equilibrium strategy $e^*(\theta) = \big(e^*(\theta_H), e^*(\theta_L)\big)$ using Bayes' rule.

(2) After seeing $e^*(\theta_H)$, the firms should believe that the worker is of high ability θ_H, given worker's strategy $e^*(\theta)$.

(3) Similarly, after seeing $e^*(\theta_L)$, the firms should believe that the worker is of low ability θ_L.

(4) The resulting wages are θ_H and θ_L, respectively.

□

16 Lemma: In any separating PBE (if exists), $e^*(\theta_L) = 0$.

Proof. (1) The type-θ_L worker always receives wage θ_L.

(2) Thus, choosing $e = 0$ will save her cost of education, and is optimal.

□

17 Let $(\tilde{e}, \theta_H)$ be the intersection point of the curve $\theta_L = w - c(e, \theta_L)$ (θ_L-indifference curve passing $(0, \theta_L)$) and the curve $w = \theta_H$.

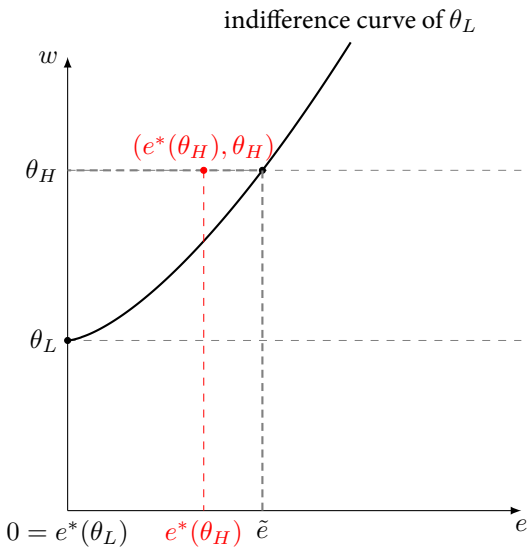


Figure 4: The lower bound of $e^*(\theta_H)$

Lemma: In any separating PBE (if exists), $e^*(\theta_H) \geq \tilde{e}$.

Proof. (1) Suppose $e^*(\theta_H) < \tilde{e}$.

(2) Then the type-θ_L worker will mimic the type-θ_H worker by choosing $e^*(\theta_H)$ (the red point):

$$\theta_L = \theta_H - c(\tilde{e}, \theta_L) < \theta_H - c(e^*(\theta_H), \theta_L).$$

(3) It is not an equilibrium. Contradiction.

□

18 Let (e_1, θ_H) be the intersection point of the curve $\theta_L = w - c(e, \theta_H)$ (θ_H-indifference curve passing $(0, \theta_L)$) and the curve $w = \theta_H$.

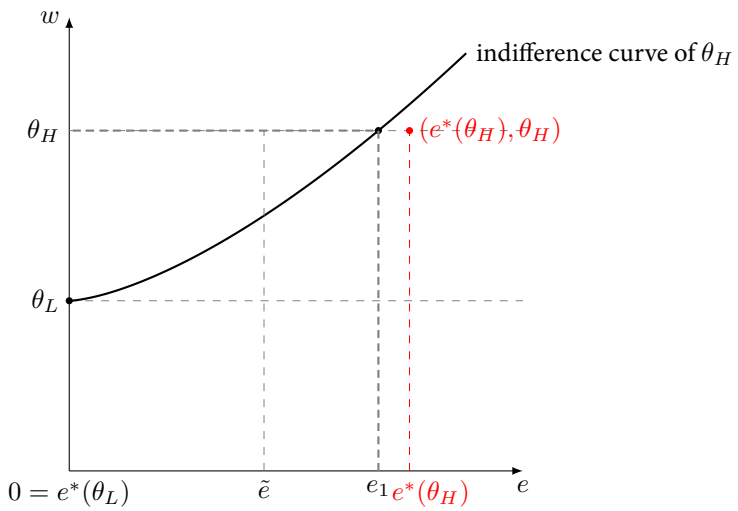


Figure 5: The upper bound of $e^*(\theta_H)$

Lemma: In any separating PBE (if exists), $e^*(\theta_H) \leq e_1$.

Proof. (1) Suppose $e^*(\theta_H) > e_1$.

(2) Then the type-θ_H worker (the red point) will mimic the type-θ_L worker by choosing 0:

$$\theta_L = \theta_H - c(e_1, \theta_H) > \theta_H - c(e^*(\theta_H), \theta_H).$$

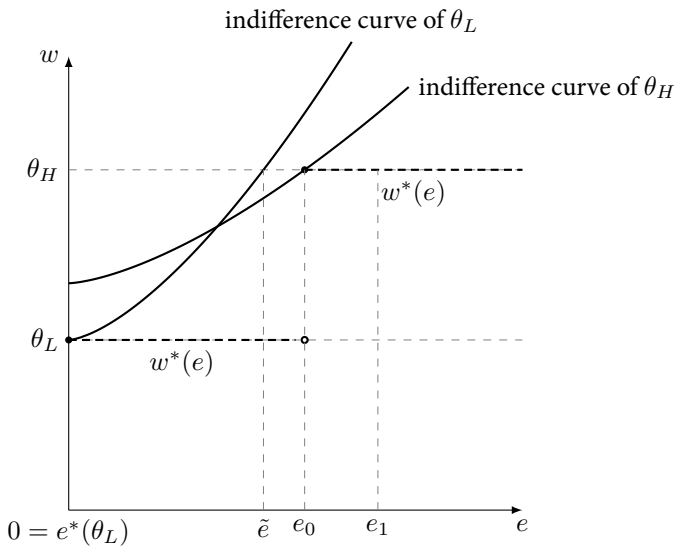
(3) It is not an equilibrium. Contradiction.

□

19 Remark: The above analysis is only heuristic, since we have not proved the existence of separating PBE.

20 Proposition: For each $e_0 \in [\tilde{e}, e_1]$, there is a separating PBE:

$$e^*(\theta_H) = e_0,\ e^*(\theta_L) = 0,\ \mu^*(e) = \begin{cases} 0, & \text{if } e = 0, \\ 0, & \text{if } 0 < e < e_0, \\ 1, & \text{if } e = e_0, \\ 1, & \text{if } e > e_0. \end{cases},\ w^*(e) = \begin{cases} \theta_L, & \text{if } e = 0, \\ \theta_L, & \text{if } 0 < e < e_0, \\ \theta_H, & \text{if } e = e_0, \\ \theta_H, & \text{if } e > e_0. \end{cases}.$$



Proof. There are two education levels on the equilibrium path: 0 and e_0.

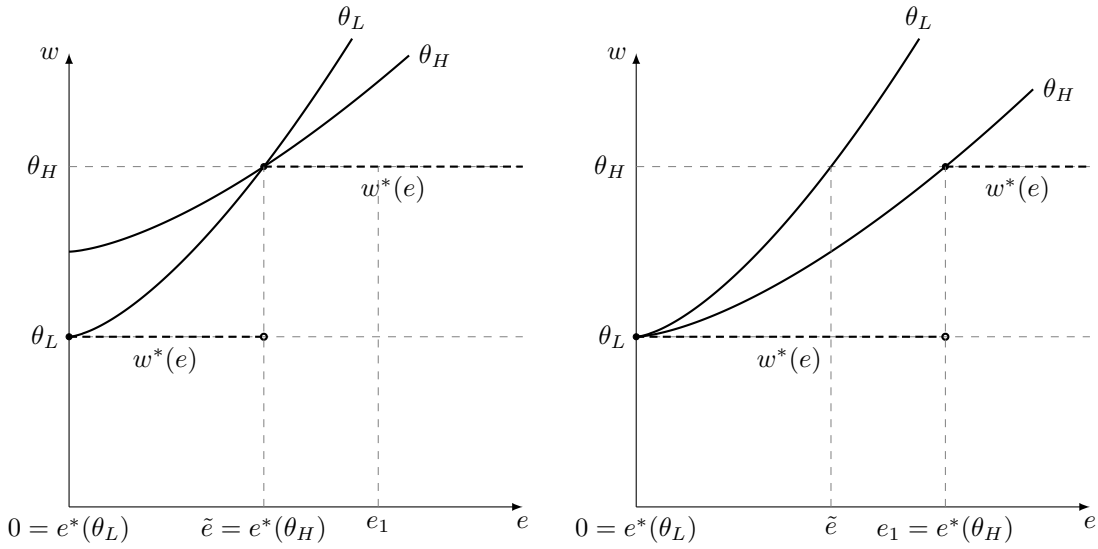
- Type-θ_L worker:
  - Deviation $e \in (0, e_0)$: worse off since $\theta_L - c(e, \theta_L) < \theta_L$.
  - Deviation $e \geq e_0$: not better off since $\theta_H - c(e, \theta_L) \leq \theta_H - c(\tilde{e}, \theta_L) = \theta_L$.
- Type-θ_H worker:
  - Deviation $e < e_0$: not better off since $\theta_L = \theta_H - c(e_1, \theta_H) \leq \theta_H - c(e_0, \theta_H)$.
  - Deviation $e > e_0$: worse off since $\theta_H - c(e, \theta_H) < \theta_H - c(e_0, \theta_H)$.
- Belief:
  - $\mu^*(0) = 0$ and $\mu^*(e_0) = 1$.

- For $e \notin \{0, e_0\}$, set $\mu^*(e)$ as in the statement: There is no restriction for believes on education levels off the equilibrium path. The off-path believes lead to corresponding off-path wages, which prevent profitable deviation of workers.

- Wage: Given the belief, it is optimal.

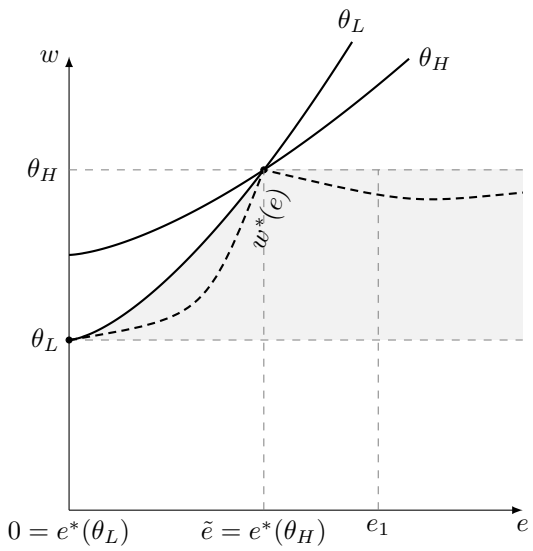
□

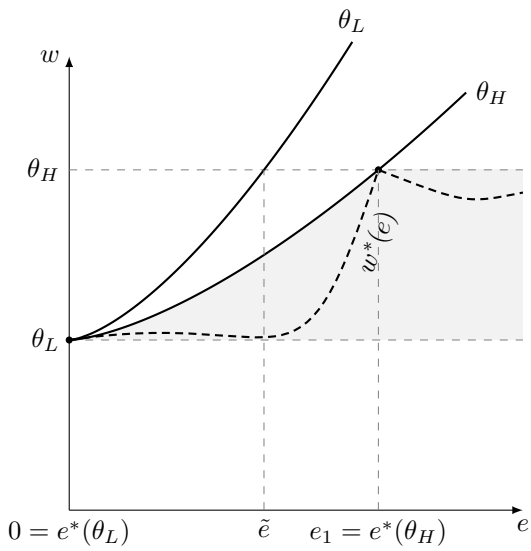
21 Two extreme separating PBE:

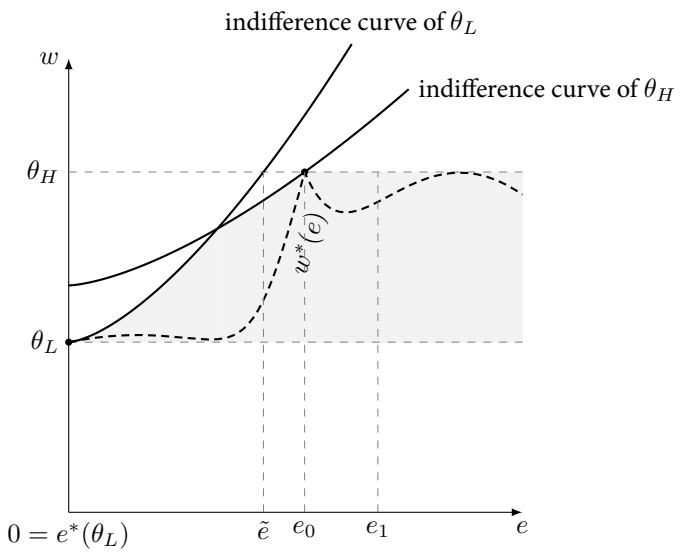


22 Notice:

- There is a one-to-one correspondence between $w^*(\cdot)$ and $\mu^*(\cdot)$:
  - $w^*(e) = \mu^*(e)\theta_H + \big(1 - \mu^*(e)\big)\theta_L$.
  - $\mu^*(e) = \begin{cases} \frac{\theta_H - \theta_L}{w^*(e) - \theta_L}, & \text{if } w^*(e) \neq \theta_L \\ 0, & \text{if } w^*(e) = \theta_L \end{cases}$.
- The Bayes' rule only requires that $\mu^*(0) = 0$ and $\mu^*(e_0) = 1$.
- However, after seeing $e \notin \{0, e_0\}$, the belief $\mu^*(e)$ could be arbitrary. It leads to multiple equilibria.
- The restriction of $w^*(e)$ (and $\mu^*(e)$):
  - It should be below the θ_L-indifference curve passing $(0, \theta_L)$, the θ_H-indifference curve passing (e_0, θ_H), and the line $w = \theta_H$.
  - It should be above the line $w = \theta_L$.







23 Key: The useless education serves as a signal because the marginal cost of education is higher for a low-ability worker.

- a type-θ_H worker may find it worthwhile to get some positive level of education to raise her wage by some amount,
- a type-θ_L worker may be unwilling to get this same level of education in return for the same wage increase.

24 Pareto efficiency among all the separating PBEs:

- Firms earn zero profits.
- A type-θ_L worker's utility is θ_L.
- A type-θ_H worker does strictly better in separating PBE where she gets a lower level of education.

Thus, the separating PBE in which the high-ability worker gets $\tilde{e}$ Pareto dominate all the others.

On the other hand, the Pareto dominated separating PBE are sustained because of the high-ability worker's fear: If she chooses a lower level of education than equilibrium education, firms will believe that she is not a high-ability

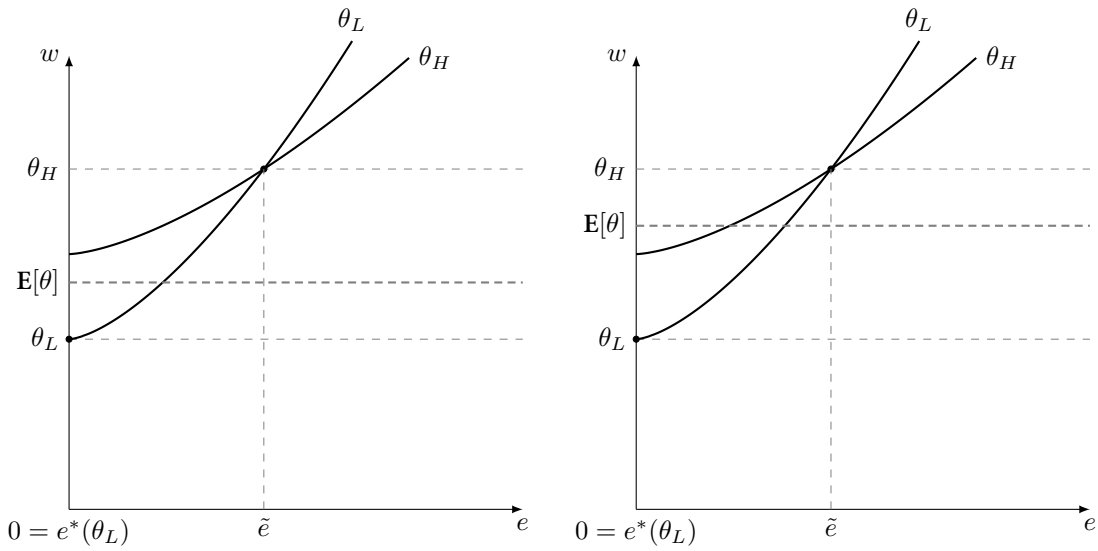
worker. These beliefs can be maintained because in PBE they are never disconfirmed (off-equilibrium path).

25 Welfare for type-θ_L workers: they are strictly worse off when signaling is possible, i.e., $\mathbf{E}[\theta] > \theta_L$.

26 Welfare for type-θ_H workers: they may be either better or worse off when signaling is possible.

- If $\mathbf{E}[\theta] < \theta_H - c(\tilde{e}, \theta_H)$, then the high-ability workers are better off because of the increase in their wages arising through signaling.
- If $\mathbf{E}[\theta] > \theta_H - c(\tilde{e}, \theta_H)$, then the high-ability workers are worse off than when signaling is impossible.

  In a separating PBE, the outcome $(0, \mathbf{E}[\theta])$ from no-signaling situation is no longer available to the high-ability workers.



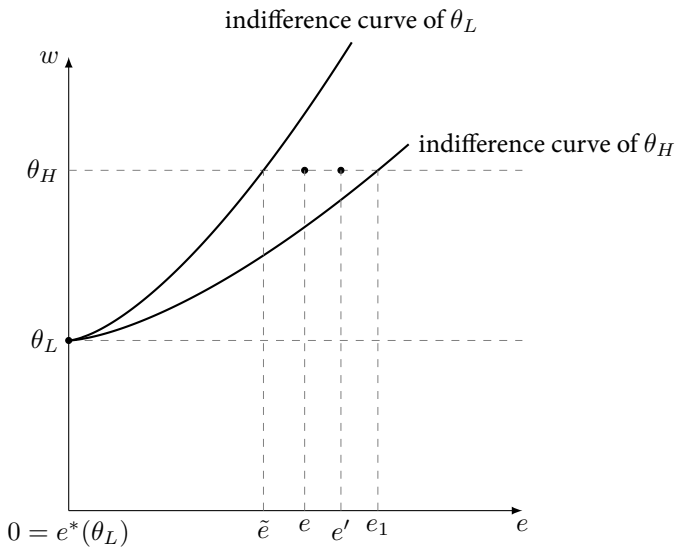
Summary:

- The set of separating PBE is completely unaffected by the fraction λ.
- As λ grows, it becomes more likely that the high-ability workers are worse off by the possibility of signaling.

27 Comparison with complete-information case:

- Complete-information case: $(0, \theta_L)$ for θ_L-worker and $(0, \theta_H)$ for θ_H-worker.
- Signaling: $(0, \theta_L)$ for θ_L-worker and $(\tilde{e}, \theta_H)$ for θ_H-worker.
- $\tilde{e}$ is the cost, paid by the beneficiary (i.e., θ_H-worker).

28 Refinement:



(1) For any $e' \in (\tilde{e}, e_1]$, consider the PBE: θ_L worker chooses education 0 and receives wage θ_L, and θ_H worker chooses education e' and receives wage θ_H.

(2) Pick any $e \in (\tilde{e}, e')$, a type-θ_L worker will never be better off by choosing e than 0 regardless of what firms believe about her as a result.

(3) Upon seeing $e \in (\tilde{e}, e')$, any belief other than $\mu(e) = 1$ seems unreasonable.

(4) Thus, $w^*(e) = \theta_H$.

(5) As a consequence, type-θ_H worker will deviate from e' to e. The given PBE is problematic.

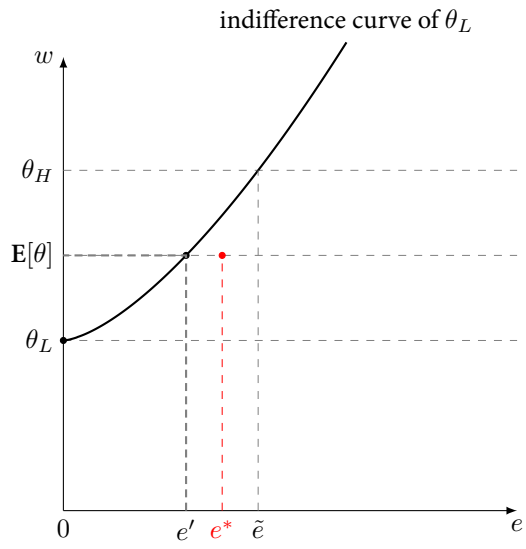
By this logic, the only reasonable separating PBE outcome is $(0, \theta_L)$ for θ_L workers and $(\tilde{e}, \theta_H)$ for θ_H workers.

4 Pooling PBE

29 In a pooling PBE, the two types of workers choose the same level of education, $e^*(\theta_L) = e^*(\theta_H) = e^*$.

30 After seeing e^* (on the equilibrium path), the firms should believe the worker is of high ability with probability λ. Thus, the wage $w^*(e^*) = \lambda\theta_H + (1-\lambda)\theta_L = \mathbf{E}[\theta]$.

31 Let $(e', \mathbf{E}[\theta])$ be the intersection point between the curve $\theta_L = w - c(e, \theta_L)$ and the curve $w = \mathbf{E}[\theta]$.



Lemma: In a pooling PBE (if exists), $e^* \leq e'$.

Proof. (1) Suppose $e^* > e'$.

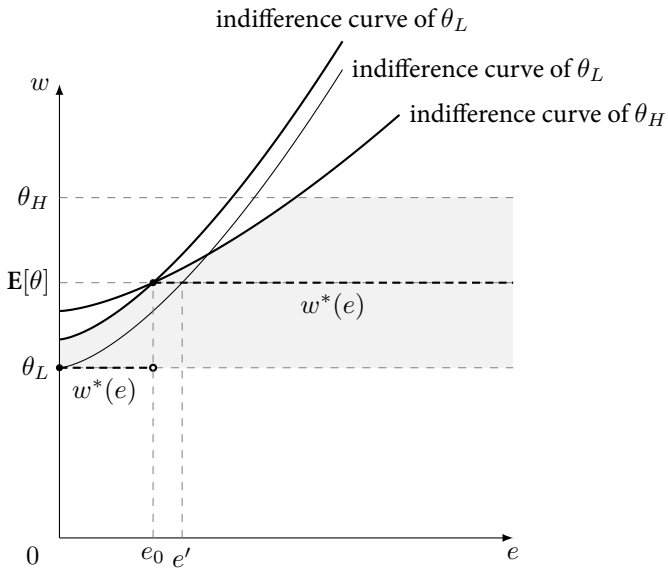
(2) Then the type-θ_L worker will deviate to 0: $\theta_L = \mathbf{E}[\theta] - c(e', \theta_L) > \mathbf{E}[\theta] - c(e^*, \theta_L)$.

(3) Thus, it is not an equilibrium. Contradiction.

□

32 Proposition: For any $e_0 \in [0, e']$, there is a pooling PBE:

$$e^*(\theta_L) = e^*(\theta_H) = e_0,\ \mu^*(e) = \begin{cases} 0, & \text{if } e < e_0, \\ \lambda, & \text{if } e = e_0, \\ \lambda, & \text{if } e > e_0. \end{cases},\ w^*(e) = \begin{cases} \theta_L, & \text{if } e < e_0, \\ \mathbf{E}[\theta], & \text{if } e = e_0, \\ \mathbf{E}[\theta], & \text{if } e > e_0. \end{cases}.$$



Proof. There is one education level on the equilibrium path: e_0.

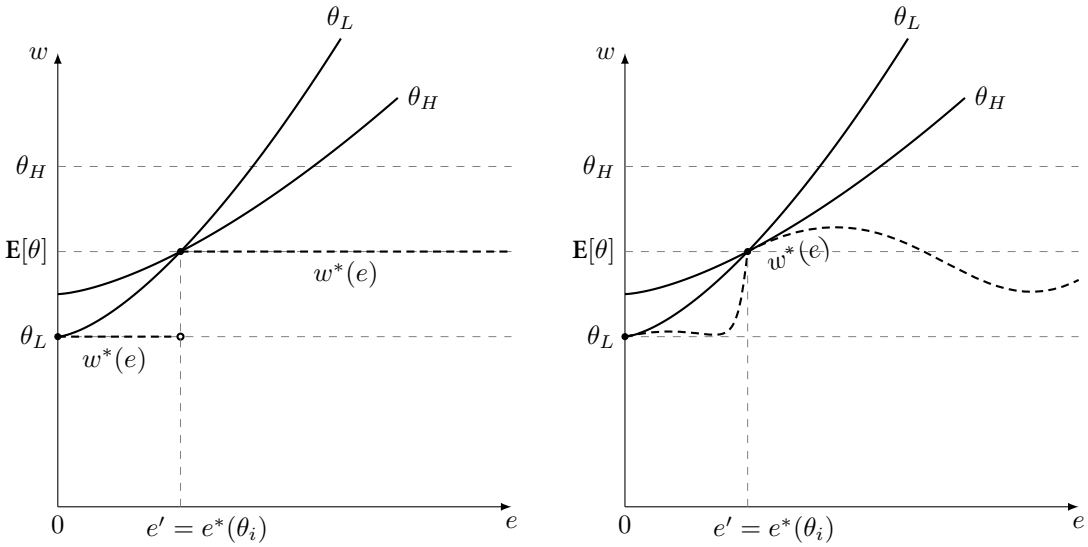
- For type-θ_L worker:
  - Deviation $e < e_0$: not better off since $\theta_L = \mathbf{E}[\theta] - c(e', \theta_L) \leq \mathbf{E}[\theta] - c(e_0, \theta_L)$.
  - Deviation $e > e_0$: worse off since $\mathbf{E}[\theta] - c(e, \theta_L) < \mathbf{E}[\theta] - c(e_0, \theta_L)$.
- For type-θ_H worker:
  - Deviation $e < e_0$: worse off since $\theta_L = \mathbf{E}[\theta] - c(e', \theta_L) < \mathbf{E}[\theta] - c(e_0, \theta_H)$.
  - Deviation $e > e_0$: worse off since $\mathbf{E}[\theta] - c(e, \theta_H) < \mathbf{E}[\theta] - c(e_0, \theta_H)$.
- Belief: $\mu^*(e_0) = \lambda$. For $e \neq e_0$, $\mu^*(e)$ could be arbitrary. We set $\mu^*(e)$ as in the statement.
- Wage: Given the belief, it is optimal.

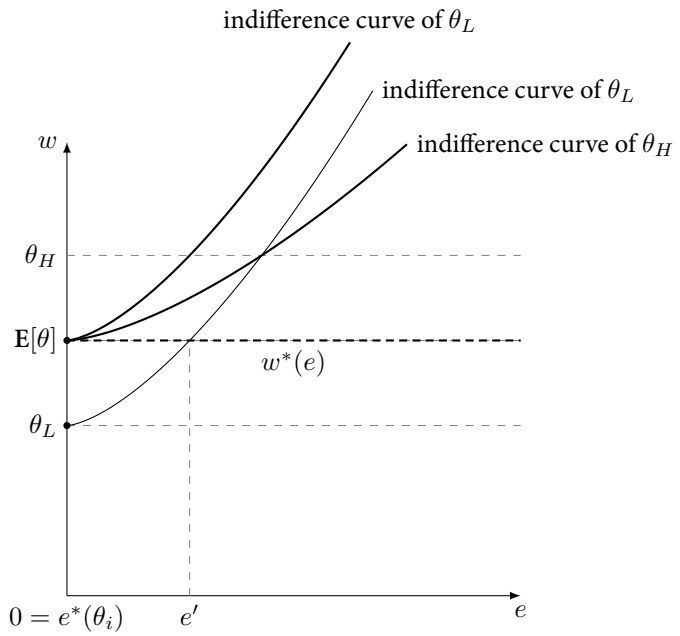
□

Note that

- $w^*(e_0) = \mathbf{E}[\theta]$.
- $w^*(e)$ should be below the θ_L-indifference curve passing $(e_0, \mathbf{E}[\theta])$, the θ_H-indifference curve passing $(e_0, \mathbf{E}[\theta])$, and the line $w = \theta_H$.
- $w^*(e)$ should be above the line $w = \theta_L$.

33 Two extreme pooling PBE:





34 Remark: $e' < \tilde{e} < e_1$.

35 Pareto efficiency:

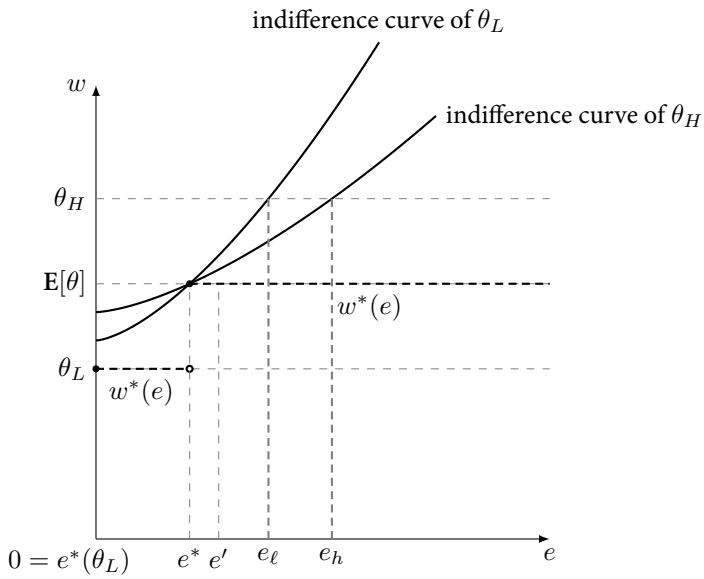
A pooling PBE in which both types of worker get no education Pareto dominates any pooling PBE with a positive education level.

The Pareto-dominated pooling PBE are sustained by the worker's fear: A deviation will lead firms to have an unfavorable impression of her ability.

36 For any pooling PBE (e^*, μ^*, w^*) where $e^* \in [0, e']$,

- let (e_ℓ, θ_H) be the intersection point between the curve $\mathbf{E}[\theta] - c(e^*, \theta_L) = w - c(e, \theta_L)$ and the curve $w = \theta_H$,
- let (e_h, θ_H) be the intersection point between the curve $\mathbf{E}[\theta] - c(e^*, \theta_H) = w - c(e, \theta_H)$ and the curve $w = \theta_H$.

37 Refinement (intuition criterion/forward induction):



- To support the education choice e^* as a pooling PBE outcome, we must have $\mu(e) < 1$ after seeing $e \in (e_\ell, e_h)$:
  - If $\mu(e) = 1$ for some $e \in (e_\ell, e_h)$, then the wage should be θ_H, and the type-θ_H worker will be better off by deviating to e:

$$\theta_H - c(e, \theta_H) > \theta_H - c(e_h, \theta_H) = \mathbf{E}[\theta] - c(e^*, \theta_H) \geq \mathbf{E}[\theta].$$

- Consider the off-equilibrium path: Suppose that a firm is confronted with a deviation to some education level $e \in (e_\ell, e_h)$ when it was expecting the equilibrium level of education e^* to be chosen.

  The firm will reason as follows:

  (1) a type-θ_L worker would be worse off deviating to e regardless of what beliefs firms have after that:

$$\mathbf{E}[\theta] - c(e^*, \theta_L) = \theta_H - c(e_\ell, \theta_L) > \theta_H - c(e, \theta_L).$$

  (2) a type-θ_H worker might be better off by doing this:

$$\mathbf{E}[\theta] - c(e^*, \theta_H) = \theta_H - c(e_h, \theta_H) < \theta_H - c(e, \theta_H).$$

  (3) Thus, this must not be a low-ability worker, that is, $\mu(e) = 1$.

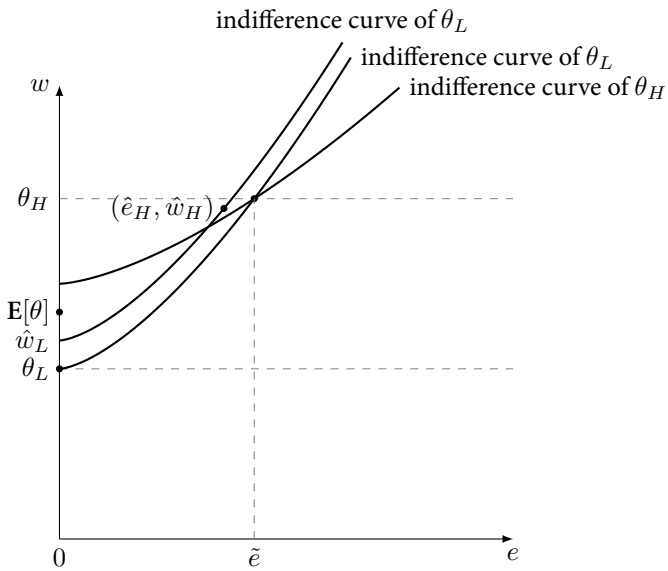
- Thus, e^* cannot be a pooling PBE education level. No pooling PBE survives.

5 Second-best intervention

38 In the presence of signaling, although the central planner cannot observe workers' types, it may be able to achieve a Pareto improvement relative to the market outcome.

39 Case 1: When the best separating PBE is Pareto dominated by the no-signaling outcome, a Pareto improvement can be achieved simply by banning the signaling activity.

40 Case 2: When the no-signaling outcome does not Pareto dominate the best separating PBE, a Pareto improvement can be achieved by "cross-subsidization":



The outcomes $(0, \hat{w}_L)$ and $(\hat{e}_H, \hat{w}_H)$ can be achieved by mandating

- workers with education levels below $\hat{e}_H$ receive wage $\hat{w}_L$,
- workers with education levels of at least $\hat{e}_H$ receive wage $\hat{w}_H$.

Thus, low-ability workers will choose $e = 0$ and high-ability workers will choose $e = \hat{e}_H$.

6 Pareto improvement for adverse selection

41 In the case with $r(\theta_H) = r(\theta_L) = 0$, the market outcome in the absence of signaling is Pareto optimal. So we just illustrate how the use of costly signaling can reduce welfare.

When the market outcome in the absence of signaling is not efficient, signaling's ability to reveal information about worker types may create a Pareto improvement by leading to a more efficient allocation of labor.

42 Suppose that we have $r = r(\theta_H) = r(\theta_L)$, with $\theta_L < r < \theta_H$ and $\mathbf{E}[\theta] < r$.

In this case, the equilibrium outcome without signaling has no workers employed. In contrast, any Pareto efficient outcome must have the high-ability workers employed by firms.

43 Lemma: Any pooling PBE must have both types choosing $e = 0$ and neither type accepting employment.

Proof. (1) Suppose that both types choose $\hat{e}$.

(2) Then $\mu^*(\hat{e}) = \lambda$ and $w^*(\hat{e}) = \mathbf{E}[\theta] < r$.

(3) So neither type accepts employment.

(4) Hence, if $\hat{e} > 0$, both types would be better off choosing $e = 0$ instead.

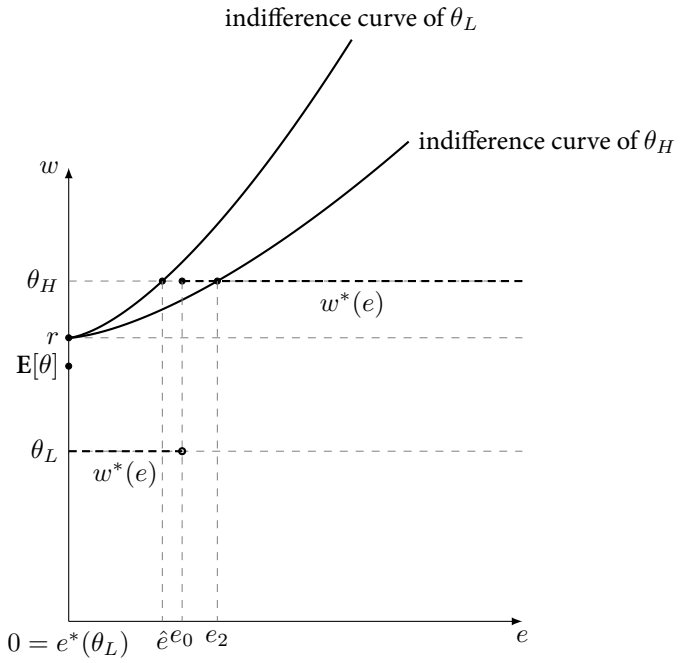
(5) Thus, only an education level of zero is possible in a pooling PBE.

□

It is easy to construct a zero-education pooling PBE. In this zero-education pooling PBE, the outcome is identical to the equilibrium outcome without signaling.

44 In any separating PBE,

- a low-ability worker sets $e = 0$, is offered a wage of θ_L, and chooses to work at home, thereby achieving a utility r.
- a high-ability worker selects an education level between $\hat{e}$ and e_2 in the figure, is offered a wage of θ_H, and accepts employment.



45 In all these PBE,

- the high-ability workers are weakly better off compared with the equilibrium arising without signaling and are strictly better off in separating PBE with $e^*(\theta) < e_2$.
- the low-ability workers are equally well off.
- the firms are also equally well off.

In the case with $\theta_L < r < \theta_H$ and $\mathbf{E}[\theta] < r$, any pooling or separating PBE weakly Pareto dominates the equilibrium outcome arising in the absence of signaling, and this Pareto dominance is strict for essentially all separating PBE.

Task

- Reading: 13.C in [MWG], 4.1–4.2 in [S].
- Understanding: