Advanced Microeconomics: Lecture Note 4

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1 Three-type model

1 There are three types $\{\theta_L, \theta_M, \theta_H\}$ with $\theta_H - \theta_M = \theta_M - \theta_L = \Delta\theta$.

The respective probabilities are λ_L, λ_M, and λ_H with $\lambda_L + \lambda_M + \lambda_H = 1$.

2 As a benchmark, the first-best effort levels are respectively given by

$$S'(q_L^*) = \theta_L,\ S'(q_M^*) = \theta_M, S'(q_H^*) = \theta_H.$$

3 Principal would like to offer a menu of contracts $\{(q_L, t_L), (q_M, t_M), (q_H, t_H)\}$ hoping that

- θ_L agents will select (q_L, t_L),
- θ_M agents will select (q_M, t_M),
- and θ_H agents will select (q_H, t_H).

4 IC constraints for $\{(q_L, t_L), (q_M, t_M), (q_H, t_H)\}$:

$$t_L - \theta_L q_L \geq t_M - \theta_L q_M, \tag{IC$_{LM}$}$$
$$t_L - \theta_L q_L \geq t_H - \theta_L q_H, \tag{IC$_{LH}$}$$
$$t_M - \theta_M q_M \geq t_H - \theta_M q_H, \tag{IC$_{MH}$}$$
$$t_M - \theta_M q_M \geq t_L - \theta_M q_L, \tag{IC$_{ML}$}$$
$$t_H - \theta_H q_H \geq t_M - \theta_H q_M, \tag{IC$_{HM}$}$$
$$t_H - \theta_H q_H \geq t_L - \theta_H q_L. \tag{IC$_{HL}$}$$

- 4 local IC constraints: involving adjacent types.
- 2 global IC constraints: involving nonadjacent types.

5 Monotonicity condition (or implementability condition): Constraints (IC$_{LM}$) and (IC$_{ML}$) imply that $q_L \geq q_M$. Constraints (IC$_{MH}$) and (IC$_{HM}$) imply that $q_M \geq q_H$.

$$q_L \geq q_M \geq q_H. \tag{M}$$

6 Two local incentive constraints (IC$_{LM}$) and (IC$_{MH}$) lead to the global one (IC$_{LH}$) under $q_M \geq q_H$.

Similarly, two local incentive constraints (IC$_{ML}$) and (IC$_{HM}$) lead to the global one (IC$_{HL}$) under $q_L \geq q_M$.

In summary, global IC constraints are guaranteed by local IC constraints.

7 When (IC_{LM}) is binding, (IC_{ML}) and (M) are equivalent. Similarly, when (IC_{MH}) is binding, (IC_{HM}) and (M) are equivalent. Once we know (IC_{LM}) and (IC_{MH}) are binding, (IC_{ML}) and (IC_{HM}) can be replaced by (M).

So we consider only (IC_{LM}), (IC_{MH}) and (M).

8 IR constraints for $\{(q_L, t_L), (q_M, t_M), (q_H, t_H)\}$:

$$t_L - \theta_L q_L \geq 0, \tag{IR_L}$$

$$t_M - \theta_M q_M \geq 0, \tag{IR_M}$$

$$t_H - \theta_H q_H \geq 0. \tag{IR_H}$$

9 Clearly, (IR_H) and (IC_{MH}) imply (IR_M). Similarly, (IR_H) and (IC_{LH}) imply (IR_L).

That is, given that IC constraints hold, IR constraints of all 3 types are satisfied as long as (IR_H) holds.

10 The principal's problem is to solve

$$\begin{array}{ll} \underset{(q_L,t_L),(q_M,t_M),(q_H,t_H)}{\text{maximize}} & \lambda_L \left[S(q_L) - t_L\right] + \lambda_M \left[S(q_M) - t_M\right] + \lambda_H \left[S(q_H) - t_H\right] \\ \text{subject to} & \text{Constraints } (\text{IC}_{LM}), (\text{IC}_{MH}), (\text{M}) \text{ and } (\text{IR}_H). \end{array}$$

11 As usual, constraints (IC_{LM}), (IC_{MH}) and (IR_H) should be binding at the optimum:

$$t_L - \theta_L q_L = t_M - \theta_L q_M, \; t_M - \theta_M q_M = t_H - \theta_M q_H, \; t_H - \theta_H q_H = 0.$$

That is,

$$\begin{aligned} t_H &= \theta_H q_H, \\ t_M &= t_H + \theta_M q_M - \theta_M q_H = \theta_H q_H + \theta_M q_M - \theta_M q_H, \\ t_L &= t_M + \theta_L q_L - \theta_L q_M = \theta_H q_H + \theta_M q_M - \theta_M q_H + \theta_L q_L - \theta_L q_M. \end{aligned}$$

Hence, the information rents are

$$\begin{aligned} U_H &= t_H - \theta_H q_H = 0, \\ U_M &= t_M - \theta_M q_M = \theta_H q_H - \theta_M q_H = \Delta\theta q_H, \\ U_L &= t_L - \theta_L q_L = \Delta\theta q_H + \Delta\theta q_M. \end{aligned}$$

12 The principal's problem is rewritten as:

$$\begin{array}{ll} \underset{q_L,q_M,q_H}{\text{maximize}} & \lambda_L \big(S(q_L) - \theta_H q_H - \theta_M q_M + \theta_M q_H - \theta_L q_L + \theta_L q_M\big) \\ & + \lambda_M \big(S(q_M) - \theta_H q_H - \theta_M q_M + \theta_M q_H\big) + \lambda_H \big(S(q_H) - \theta_H q_H\big) \\ \text{subject to} & \text{Constraint (M).} \end{array}$$

13 Intuitively, more efficient types tend to claim to be less efficient. Momentarily, we ignore the incentive constraints (IC_{ML}) and (IC_{HM}), or (M). Ignore constraint (M) first.

First order condition for q_L:

$$S'(q_L^{\text{SB}}) = \theta_L.$$

First order condition for q_M:

$$S'(q_M^{\text{SB}}) = \theta_M + \frac{\lambda_L}{\lambda_M}(\theta_M - \theta_L) = \theta_M + \frac{\lambda_L}{\lambda_M}\Delta\theta.$$

First order condition for q_H:

$$S'(q_H^{\text{SB}}) = \theta_H + \frac{\lambda_M}{\lambda_H}(\theta_H - \theta_M) + \frac{\lambda_L}{\lambda_H}(\theta_H - \theta_M) = \theta_H + \frac{\lambda_M + \lambda_L}{\lambda_H}\Delta\theta.$$

14 Then check constraint (M):

- Clearly, $q_L^{\text{SB}} > q_M^{\text{SB}}$ automatically.
- $q_M^{\text{SB}} > q_H^{\text{SB}}$ iff $S'(q_M^{\text{SB}}) < S'(q_H^{\text{SB}})$ iff

$$\theta_M + \frac{\lambda_L}{\lambda_M}\Delta\theta < \theta_H + \frac{\lambda_M + \lambda_L}{\lambda_H}\Delta\theta,$$

  which is equivalent to

$$\lambda_M > \lambda_L\lambda_H.$$

In this case, the information rents are

$$\begin{aligned}
U_H^{\text{SB}} &= t_H^{\text{SB}} - \theta_H q_H^{\text{SB}} = 0,\\
U_M^{\text{SB}} &= t_M^{\text{SB}} - \theta_M q_M^{\text{SB}} = \theta_H q_H^{\text{SB}} - \theta_M q_H^{\text{SB}} = \Delta\theta q_H^{\text{SB}},\\
U_L^{\text{SB}} &= t_L^{\text{SB}} - \theta_L q_L^{\text{SB}} = \Delta\theta q_H^{\text{SB}} + \Delta\theta q_M^{\text{SB}}.
\end{aligned}$$

15 On the other hand (if $\lambda_M \le \lambda_L\lambda_H$), bunching (集束) result occurs:

(a) For a given λ_H, if λ_L is rather big and λ_M is small, then the information rent of θ_M agents is not too costly but that of θ_L is much more.

(b) Therefore, reducing rents calls for strongly reducing q_M, but a reduction in q_H is less necessary.

(c) However, due to the implementability condition, q_M cannot be reduced to be lower than q_H.

(d) We thus have $q_M = q_H$ at the optimum.

In this case, principal's problem is rewritten as:

$$\max_{q_L, q^P} \lambda_L\left[S(q_L) - \theta_H q^P - \theta_L q_L + \theta_L q^P\right] + \lambda_M\left[S(q^P) - \theta_H q^P\right] + \lambda_H\left[S(q^P) - \theta_H q^P\right].$$

First order condition for q^P:

$$(\lambda_M + \lambda_H)S'(q^P) = \lambda_M\theta_H + \lambda_H\theta_H + \lambda_L(\theta_H - \theta_L).$$

That is,

$$S'(q^P) = \theta_H + \frac{\lambda_L}{\lambda_M + \lambda_H}2\Delta\theta.$$

16 Theorem:

- Constraints (IC_{LM}), (IC_{MH}) and (IR_H) are all binding.

- When $\lambda_M > \lambda_H \lambda_L$, Constraint (M) is strictly satisfied. Optimal outputs are given by $q_L^{\text{SB}} = q_L^*$, $q_M^{\text{SB}} < q_M^*$ and $q_L^{\text{SB}} < q_L^*$ with

$$S'(q_M^{\text{SB}}) = \theta_M + \frac{\lambda_L}{\lambda_M}\Delta\theta,$$
$$S'(q_H^{\text{SB}}) = \theta_H + \frac{\lambda_M + \lambda_L}{\lambda_H}\Delta\theta.$$

- When $\lambda_M \leq \lambda_H \lambda_L$, some bunching emerges. We still have $q_L^{\text{SB}} = q_L^*$, but now $q_M^{\text{SB}} = q_H^{\text{SB}} = q^P < q_L^{\text{SB}}$, with

$$S'(q^P) = \theta_H + \frac{\lambda_L}{\lambda_M + \lambda_H} 2\Delta\theta.$$

17 To avoid bunching, modelers often chose to impose a sufficient condition on the distribution of types, the monotonicity of the hazard rate.

Definition: A distribution of types satisfies the monotone hazard rate property if and only if

$$\frac{\text{Prob}(\theta < \theta_M)}{\text{Prob}(\theta = \theta_M)} = \frac{\lambda_L}{\lambda_M} < \frac{\text{Prob}(\theta < \theta_H)}{\text{Prob}(\theta = \theta_H)} = \frac{\lambda_L + \lambda_M}{\lambda_H}.$$

18 The virtual costs of the different types, namely θ_L, $\theta_M + \frac{\lambda_L}{\lambda_M}\Delta\theta$ and $\theta_H + \frac{\lambda_M + \lambda_L}{\lambda_H}\Delta\theta$, are ranked exactly as the true physical costs.

The virtual surplus is maximized by a decreasing schedule of outputs ($q_L^{\text{SB}} > q_M^{\text{SB}} > q_H^{\text{SB}}$). Asymmetric information does not perturb the ranking of types.

2 Ex ante contract

19 So far, we have considered the case of contracts offered at the interim stage, i.e., once the agent already knows his type $\theta \in \{\theta_L, \theta_H\}$.

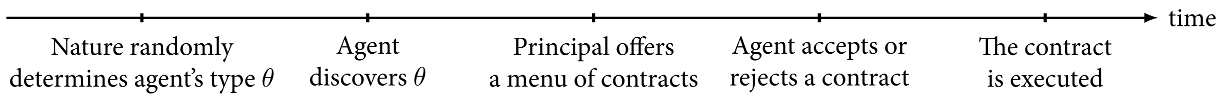

Figure 1: Timing

20 However, sometimes the principal and the agent can contract at the ex ante stage, i.e., before the agent discovers his type θ.

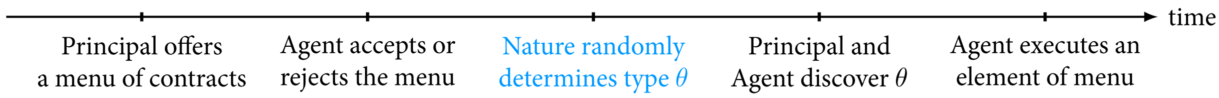

Figure 2: Timing

We now characterize the optimal contract for this alternative timing under various assumptions about the risk aversion of the two players.

21 The contract can only be written in terms of the verifiable variables. θ is not verifiable and cannot be written into a contract.

A menu $\{(q_L, t_L), (q_H, t_H)\}$ is a feasible instrument.

- When facing the menu $\{(q_L, t_L), (q_H, t_H)\}$, agent accepts the menu itself or not.
- In contrast, in the standard model, agent chooses (q_L, t_L), (q_H, t_H), or neither when he faces a menu $\{(q_L, t_L), (q_H, t_H)\}$.

When agent accepts such a contract $\{(q_L, t_L), (q_H, t_H)\}$, the agent anticipates that

- his choice of outputs q_L in state θ_L will satisfy the following interim constraint

$$t_L - \theta_L q_L \geq t_H - \theta_L q_H.$$

- his choice of outputs q_H in state θ_H will satisfy the following interim constraint

$$t_H - \theta_H q_H \geq t_L - \theta_H q_L.$$

These constraints are the same as the standard incentive compatibility constraints.

2.1 Risk-neutral agent and risk-neutral principal

22 If the agent is risk neutral, his ex ante participation constraint is now written as

$$\lambda U_L + (1-\lambda) U_H \geq 0,$$

where $U_L = t_L - \theta_L q_L$ and $U_H = t_H - \theta_H q_H$ respectively denote the information rents.

What matters now to ensure participation is that the agent's expected information rent remains non-negative.

This ex ante participation constraint replaces the two interim participation constraints.

23 Note that the principal's objective function is decreasing in the agent's expected information rent.

$$\lambda \left[S(q_L) - \theta_L q_L\right] + (1-\lambda) \left[S(q_H) - \theta_H q_H\right] - \left[\lambda U_L + (1-\lambda) U_H\right].$$

24 Ideally, the principal wants to impose a zero expected information rent to the agent and have the ex ante participation constraint, $\lambda U_L + (1-\lambda) U_H \geq 0$, be binding.

25 Moreover, the principal must structure the rents U_L and U_H to ensure that the wedge between those two levels is such that the incentive constraints remain satisfied.

$$U_L \geq U_H + \Delta\theta q_H,$$
$$U_H \geq U_L - \Delta\theta q_L.$$

26 An example of such a rent distribution that is both incentive compatible and satisfies the ex ante participation constraint with an equality is

$$U_L^* = (1-\lambda)\Delta\theta q_H^* > 0 \text{ and } U_H^* = -\lambda \Delta\theta q_H^* < 0.$$

(a) IC constraints imply

$$\Delta\theta q_L \geq U_L - U_H \geq \Delta\theta q_H.$$

(b) Let $U_L - U_H = \Delta\theta q_H$ (θ_L-IC is binding).

(c) Since the agent is risk-neutral, the principal can costlessly structure the information rents U_L and U_H such that the expected information rent will be zero. Therefore, $\lambda U_L + (1-\lambda)U_H = 0$ (ex ante IR is binding).

(d) The maximization problem can be simplified to an unconstrained optimization problem of two choice variables

$$\lambda \left[S(q_L) - \theta_L q_L\right] + (1-\lambda)\left[S(q_H) - \theta_H q_H\right].$$

(e) So the solutions are the first-best allocations q_L^* and q_H^*.

(f) Then given θ_L-IC and ex ante IR, the payoffs are

$$U_L^* = (1-\lambda)\Delta\theta q_H^* > 0 \text{ and } U_H^* = -\lambda\Delta\theta q_H^* < 0.$$

(g) Check θ_H-IC: $U_L^* - U_H^* = \Delta\theta q_H^* \leq \Delta\theta q_L^*$.

In this contract, the agent is rewarded when he is efficient and punished when he turns out to be inefficient.

Remember, we could have come up with other transfers leading to the same outputs as the optimal solution. We chose these transfers so to make the θ_L-IC binding.

27 There must be some risk in the distribution of information rents to induce information revelation, but this risk is costless for the principal because of the agent's risk neutrality.

28 Proposition: When the agent is risk neutral and contracting takes place ex ante, there exists an optimal incentive contract which implements the first-best outcome.

- First-best outcome is achieved;
- Expected information rent is zero.

29 Principal's utilities are

$$\begin{aligned} S(q_L^*) - U_L^* - \theta_L q_L^* &= S(q_L^*) - \theta_L q_L^* - (1-\lambda)\Delta\theta q_H^*, \\ S(q_H^*) - U_H^* - \theta_H q_H^* &= S(q_H^*) - \theta_H q_H^* + \lambda\Delta\theta q_H^*, \end{aligned}$$

which are not the same. That is, principal's utility has uncertainty.

30 We can also achieve this result by considering the following menu of contracts $\{(q_L^*, t_L^*), (q_H^*, t_H^*)\}$ which satisfy the incentive compatibility constraints as strict inequalities.

31 Let $t_L^* = S(q_L^*) - T^*$ and $t_H^* = S(q_H^*) - T^*$, with T^* being a lump-sum payment to be defined below.

32 This contract is incentive compatible since

$$\begin{aligned} t_L^* - \theta_L q_L^* &= S(q_L^*) - T^* - \theta_L q_L^* > S(q_H^*) - T^* - \theta_L q_H^* = t_H^* - \theta_L q_H^*, \\ t_H^* - \theta_H q_H^* &= S(q_H^*) - T^* - \theta_H q_H^* > S(q_L^*) - T^* - \theta_H q_L^* = t_L^* - \theta_H q_L^*. \end{aligned}$$

33 The fixed-fee T^* can be used to satisfy the agent's ex ante participation constraint with an equality by choosing

$$T^* = \lambda \left[S(q_L^*) - \theta_L q_L^*\right] + (1-\lambda)\left[S(q_H^*) - \theta_H q_H^*\right].$$

34 This implementation of the first-best outcome amounts to having the principal selling the benefit of the relationship to the risk-neutral agent for a fixed up-front payment T^*.

35 The agent benefits from the full value of the good and trades off the value of any production against its cost just as if he was an efficiency maximizer.

In this case, we say that the agent is residual claimant for the firm's profit.

36 Principal's utility is always T^* no matter the state is θ_L or θ_H.

2.2 Risk-neutral agent and risk-averse principal

37 Note that we have not identified whether the principal is risk-neutral or risk-averse.

In fact, this issue does not matter for the conclusion that the first-best can be implemented.

38 Consider now a risk-averse principal with a von Neumann-Morgenstern utility function $v(\cdot)$ defined on his monetary gains from trade $S(q) - t$ such that $v' > 0$, $v'' < 0$ and $v(0) = 0$.

39 Since the agent is risk-neutral, the principal can costlessly structure the information rents U_L and U_H such that the expected information rent will be zero. Thus, the optimal contract obviously calls for the first-best output q_L^* and q_H^* being produced.

40 It also calls for the principal to be fully insured between both states of nature and for the agent's ex ante participation constraint to be binding. This leads to the following two conditions that must be satisfied by agent's rents U_L^* and U_H^*.

$$S(q_L^*) - \theta_L q_L^* - U_L^* = S(q_H^*) - \theta_H q_H^* - U_H^* \text{ and } \lambda U_L^* + (1-\lambda)U_H^* = 0.$$

(前者降低委托人的风险溢价至零，后者降低信息租金至零)

So

$$U_L^* = (1-\lambda)\left[S(q_L^*) - \theta_L q_L^* - (S(q_H^*) - \theta_H q_H^*)\right],$$
$$U_H^* = -\lambda\left[S(q_L^*) - \theta_L q_L^* - (S(q_H^*) - \theta_H q_H^*)\right].$$

41 Note that the first-best profile of information rents satisfies both types' incentive compatibility constraints since

$$U_L^* - U_H^* = S(q_L^*) - \theta_L q_L^* - (S(q_H^*) - \theta_H q_H^*) > \Delta\theta q_H^*,$$
$$U_H^* - U_L^* = S(q_H^*) - \theta_H q_H^* - (S(q_L^*) - \theta_L q_L^*) > -\Delta\theta q_L^*.$$

42 Result: When the principal is risk-averse over the monetary gains $S(q) - t$, the agent is risk-neutral, and contracting takes place ex ante, the optimal incentive contract implements the first-best outcome.

- First-best outcome is achieved;
- Ex post full insurance for principal;
- Expected information rent is zero.

43 Take the lump-sum payment

$$T^* = \lambda\left[S(q_L^*) - \theta_L q_L^*\right] + (1-\lambda)\left[S(q_H^*) - \theta_H q_H^*\right],$$

which allows the principal to make the risk-neutral agent residual claimant for the hierarchy's profit, also provides full insurance to the principal.

By making the risk-neutral agent the residual claimant for the value of trade, ex ante contracting allows the risk-averse principal to get full insurance and implement the first-best outcome despite the informational problem.

2.3 Risk-averse agent

44 We have seen that when the agent is risk-neutral, then the first-best allocation can be implemented ex ante without any cost to the principle.

However, we will now see that when the agent is risk-averse, then the first-best is no longer implementable.

45 Consider now a risk-averse agent with a utility function $u(\cdot)$ defined on his monetary gains $t-\theta q$, such that $u' > 0$, $u'' < 0$ and $u(0) = 0$.

46 The incentive constraints are unchanged but the agent's ex ante participation constraint is now written as

$$\lambda u(U_L) + (1-\lambda)u(U_H) \geq 0.$$

47 As usual, we guess a solution such that IC-θ_H is slack at the optimum, and we check this ex post. The principal's program reduces now to

$$\begin{aligned} \underset{(q_L,U_L),(q_H,U_H)}{\text{maximize}} \quad & \lambda\left[S(q_L) - \theta_L q_L - U_L\right] + (1-\lambda)\left[S(q_H) - \theta_H q_H - U_H\right] \\ \text{subject to} \quad & U_L \geq U_H + \Delta\theta q_H, \\ & \lambda u(U_L) + (1-\lambda)u(U_H) \geq 0. \end{aligned}$$

48 With risk aversion, the principal can no longer costlessly structure the agent's information rents to ensure the efficient type's incentive compatibility constraint, contrary to previous part.

49 Creating a wedge between U_L and U_H to satisfy θ_L-IC makes the risk-averse agent bear some risk.

To guarantee the participation of the risk-averse agent, the principal must now pay a risk premium.

Reducing this premium calls for a downward reduction in the inefficient type's output (the difference between U_L and U_H is at least $\Delta\theta q_H$) so that the risk borne by the agent is lower.

50 Step 1: Form the following Lagrangian for the principal's problem

$$\begin{aligned} \mathcal{L}(q_L, q_H, U_L, U_H, \gamma, \mu) = {} & \lambda\left[S(q_L) - \theta_L q_L\right] + (1-\lambda)\left[S(q_H) - \theta_H q_H\right] - \left[\lambda U_L + (1-\lambda)U_H\right] \\ & + \gamma\left[U_L - U_H - \Delta\theta q_H\right] + \mu\left[\lambda u(U_L) + (1-\lambda)u(U_H)\right], \end{aligned}$$

where γ is the multiplier of efficient type's incentive compatibility constraint and μ is the multiplier of the ex ante participation constraint.

51 Optimizing with respect to U_L and U_H yields respectively

$$-\lambda + \gamma + \mu\lambda u'(U_L^{\text{SB}}) = 0,$$
$$-(1-\lambda) - \gamma + \mu(1-\lambda)u'(U_H^{\text{SB}}) = 0.$$

52 Summing these two equalities, we obtain

$$\mu\left[\lambda u'(U_L^{\text{SB}})+(1-\lambda)u'(U_H^{\text{SB}})\right]=1.$$

Then $\mu>0$, and hence ex ante IR is binding.

53 Using the expression of μ above yields

$$\gamma=\frac{\lambda(1-\lambda)\left[u'(U_H^{\text{SB}})-u'(U_L^{\text{SB}})\right]}{\lambda u'(U_L^{\text{SB}})+(1-\lambda)u'(U_H^{\text{SB}})}.$$

Moreover, θ_L-IC implies that $U_L^{\text{SB}}\geq U_H^{\text{SB}}$ and thus $\gamma\geq 0$, with $\gamma>0$ for a positive output q_H.

54 Optimizing with respect to outputs yields respectively

$$S'(q_L^{\text{SB}})=\theta_L,$$

and

$$S'(q_H^{\text{SB}})=\theta_H+\frac{\gamma}{1-\lambda}\Delta\theta=\theta_H+\Delta\theta\frac{\lambda\left[u'(U_H^{\text{SB}})-u'(U_L^{\text{SB}})\right]}{\lambda u'(U_L^{\text{SB}})+(1-\lambda)u'(U_H^{\text{SB}})}.$$

If $\gamma=0$, then the above equation implies that $S'(q_H^{\text{SB}})=\theta$. Hence, $q_H^{\text{SB}}=q^*>0$ and $\gamma>0$, which is a contradiction.

Therefore, $\gamma>0$ and θ_L-IC is binding.

55 Since θ_L-IC binds, $U_L^{\text{SB}}=U_H^{\text{SB}}-\Delta\theta q_H^{\text{SB}}>U_L-\Delta\theta q_L^{\text{SB}}$. That is, θ_H-IC holds automatically.

56 To solve q_H^{SB}, U_H^{SB} and U_L^{SB}, we need to solve the equations

$$\begin{aligned}S'(q_H^{\text{SB}})&=\theta_H+\Delta\theta\frac{\lambda\left[u'(U_H^{\text{SB}})-u'(U_L^{\text{SB}})\right]}{\lambda u'(U_L^{\text{SB}})+(1-\lambda)u'(U_H^{\text{SB}})},\\ U_L^{\text{SB}}&=U_H^{\text{SB}}+\Delta\theta q_H^{\text{SB}},\\ \lambda u(U_L^{\text{SB}})+(1-\lambda)u(U_H^{\text{SB}})&=0.\end{aligned}$$

57 Result: When the agent is risk-averse and contracting takes place ex ante, the optimal menu of contracts entails:

- No output distortion for the efficient type $q_L^{\text{SB}}=q_L^*$.
- A downward output distortion for the inefficient type $q_H^{\text{SB}}<q_H^*$, with

$$S'(q_H^{\text{SB}})=\theta_H+\Delta\theta\frac{\lambda\left[u'(U_H^{\text{SB}})-u'(U_L^{\text{SB}})\right]}{\lambda u'(U_L^{\text{SB}})+(1-\lambda)u'(U_H^{\text{SB}})}.$$

- Both θ_L-IC and ex ante IR are the only binding constraints.
- The efficient type gets a strictly positive ex post information rent, while the inefficient type gets a strictly negative ex post information rent; that is, $U_L^{\text{SB}}>0>U_H^{\text{SB}}$.

58 Numerical example: $S(q)=q^{\frac{1}{2}}$, $\theta_L=\frac{1}{3}$, $\theta_H=\frac{1}{2}$, $\lambda=\frac{1}{2}$, $v(t-\theta q)=\log(t-\theta q)$.

We have

$$q_L^*=\tfrac{9}{4},\ q_H^*=1,\ U_L^*=\tfrac{1}{12},\ t_L^*=\tfrac{5}{6},\ U_H^*=-\tfrac{1}{12},\ t_H^*=\tfrac{5}{12},$$

where $U_L^* - U_H^* = \frac{1}{6} = \Delta\theta q_H^*$.

And

$$q_L^{\text{SB}} = q_L^* = \frac{9}{4},\ q_H^{\text{SB}} = 0.949422 < 1 = q_H^*,$$
$$U_L^{\text{SB}} = 0.0822435 < \frac{1}{12} = U_L^*,\ U_H^{\text{SB}} = -0.0759935 > -\frac{1}{12} = U_H^*.$$

Clearly,

$$U_L^{\text{SB}} - U_H^{\text{SB}} = \Delta\theta q_H^{\text{SB}} < \Delta\theta q_H^* = U_L^* - U_H^*.$$

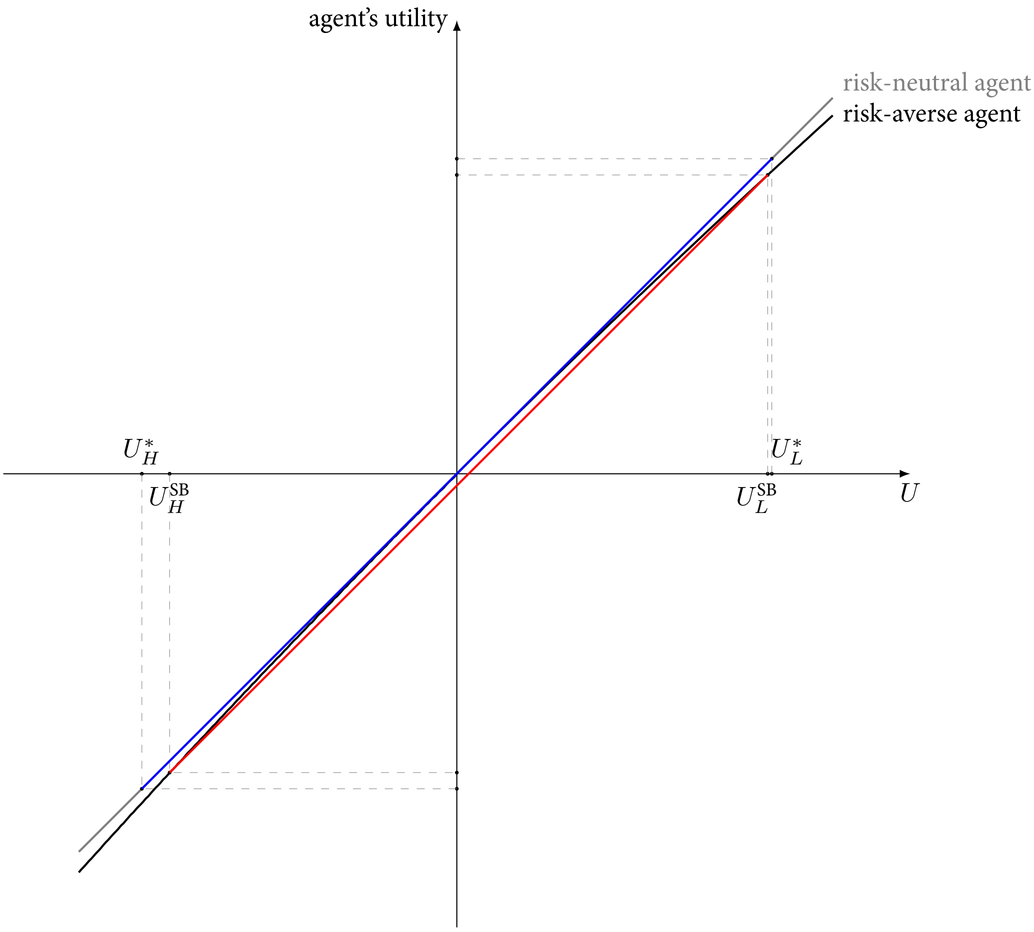

Figure 3

3 Limited liability

59 Sometimes the set of incentive-feasible contracts is constrained by some exogenous limits on the feasible transfers between the principal and the agent.

These exogenous financial constraints could reveal the existence of previous financial contracts that the agent might have already signed. Those constraints will of course affect the usual rent-efficiency trade-off.

60 A first possible limit is that the net transfer of the agent, taking into account his own asset holding l, should not be lower than zero.

This leads to the following limited liability constraints on transfers:

$$t_L \geq -l \text{ and } t_H \geq -l.$$

A possible motivation for this type of constraint is that the agent can use the transfer received from the principal to cover a debt of level $-l$.

The production cost θq being already sunk, it does not enter into the left-hand sides of limited liability constraints.

61 A second limit on transfers arises when the agent's information rent itself must be greater than an exogenous value $-l$. This leads to the following limited liability constraints on rents:

$$U_L \geq -l \text{ and } U_H \geq -l.$$

The production cost θq is now incurred when the transfer t takes place.

Again, the interpretation is that contracting with the principal may involve negative rents U_L or U_H as long as those losses can be covered by the agent's own liabilities l.

62 To assess the impact of these limited liability constraints, let us go back to the framework of ex ante contracting. When contracting takes place ex ante, we have seen that the first-best outcome can still be obtained provided that the inefficient risk-neutral agent receives a negative payoff, $U_H^* < 0$. Obviously this negative payoff may conflict with the constraint $U_H \geq -l$.

63 With ex ante contracting, we have already seen that the relevant incentive and participation constraints are, respectively,

$$U_L \geq U_H + \Delta\theta q_H \text{ and } \lambda U_L + (1-\lambda) U_H \geq 0.$$

64 Adding the limited liability constraints, the principal's program is written as

$$\begin{aligned}
\underset{(q_L U_L),(q_H,U_H)}{\text{maximize}} \quad & \lambda[S(q_L) - \theta_L q_L - U_L] + (1-\lambda)[S(q_H) - \theta_H q_H - U_H] \\
\text{subject to} \quad & U_L \geq U_H + \Delta\theta q_H, \\
& \lambda U_L + (1-\lambda) U_H \geq 0, \\
& \begin{cases} \text{either } t_L \geq -l, \ t_H \geq -l, \\ \text{or } U_L \geq -l, U_H \geq -l. \end{cases}
\end{aligned}$$

65 We first focus on limited liability constraints on rents.

(a) θ_L-LL constraint is redundant: $U_L \geq U_H + \Delta\theta q_H \geq U_H \geq -l$.

(b) Without θ_H-LL, the first-best allocations can be achieved, where IR always binds. In order to make θ_H-LL easier to meet, we need U_H to be as large as possible. Note that $U_L \geq U_H + \Delta\theta q_H$, the largest possible U_H is determined by $U_L = U_H + \Delta\theta q_H$. So

$$U_H^L = -\lambda\Delta\theta q_H^* \text{ and } U_L^L = (1-\lambda)\Delta\theta q_H^*.$$

(c) If $l > \lambda\Delta\theta q_H^*$, θ_H-LL is redundant. The principal implements the first-best outcome by fixing the above U_L^L and U_H^L.

(d) Suppose $l \leq \lambda\Delta\theta q_H^*$. Then the above first-best outcome cannot be achieved. To reduce the expected information rent, principal would first set $U_H^L = -l$. Principal then set $U_L^L = U_H^L + \Delta\theta q_H = -l + \Delta\theta q_H$. He can reduce q_H from q_H^* to reduce the information rent, but he has to satisfy ex ante IR.

(e) We ignore ex ante IR temporarily. Inserting $U_H^L = -l$ and $U_L^L = -l + \Delta\theta q_H$ into the principal's objective function and optimizing with respect q_L and q_H yields

$$S'(q_L^L) = \theta_L \text{ and } S'(q_H^L) = \theta_H + \frac{\lambda}{1-\lambda}\Delta\theta,$$

that is, $q_L^L = q_L^*$ and $q_H^L = q_H^{\mathrm{SB}}$.

(f) This solution is valid as long as the agent's ex ante IR constraint is strictly satisfied, i.e., $\lambda U_L^L + (1-\lambda)U_H^L = -l + \lambda\Delta\theta q_H^{\mathrm{SB}} > 0$, or $l < \lambda\Delta\theta q_H^{\mathrm{SB}}$.

(g) When $\lambda\Delta\theta q_H^{\mathrm{SB}} \leq l \leq \lambda\Delta\theta q_H^*$. Ex ante IR is not satisfied for the above solutions. We conjecture ex ante IR is also binding. In this case, we have $\lambda U_L^L + (1-\lambda)U_H^L = -l + \lambda\Delta\theta q_H^L = 0$, or $\lambda\Delta\theta q_H^L = l$. And $q_L^L = q_L^*$.

66 Result: Assume ex ante contracting and limited liability on rents. Then, the optimal contract entails

- For $l > \Delta\theta q_H^*$, the first-best allocations can be achieved, no output distortion. Ex ante IR is binding, θ_L-IC could be binding, and others are redundant.
- For $\lambda\Delta\theta q_H^{\mathrm{SB}} \leq l \leq \lambda\Delta\theta q_H^*$, θ_L-IC, ex ante IR, and θ_H-LL are binding.

$$q_L^L = q_L^* \text{ and } q_H^L = \tfrac{l}{\lambda\Delta\theta} \in [q_H^{\mathrm{SB}}, q_H^*].$$

- For $l < \lambda\Delta\theta q_H^{\mathrm{SB}}$, only θ_L-IC and θ_H-LL are binding.

$$q_L^L = q_L^* \text{ and } q_H^L = q_H^{\mathrm{SB}}.$$

67 A limited liability constraint on ex post rents may reduce the efficiency of ex ante contracting.

- If the limited liability constraint on the inefficient type is stringent enough, the principal must reduce the inefficient agent's output to keep the limited liability constraint satisfied. The agent is then subject to less risk on the allocation of ex post rents.
- When the limited liability constraint is even harder, the principal must give up his desire to hold the ex ante participation constraint binding. The limited liability constraint then implies an ex ante information rent.

68 The limited liability constraint on rents plays a similar role as the agent's risk aversion.

- In both cases, the principal finds it costly to create a wedge between U_L and U_H, and reducing this cost calls for incentives that are lower powered than one would find with risk neutrality and unlimited transfers.
- More precisely, with a limited liability constraint on rents, everything happens as if the agent has an infinite risk aversion below a wealth of $-l$.

69 Let us now turn to the case of limited liability constraints on transfers. Restricting the analysis to a few particular cases, we have the following characterization of the optimal contract.

Assume ex ante contracting and limited liability on transfers. Then the optimal contract entails:

- For $l > -[\lambda\theta_L + (1-\lambda)\theta_H]q_H^*$, only IR is binding and the first-best allocations can be achieved.

- For $-[\lambda\theta_L + (1-\lambda)\theta_H]q_L^* \leq l \leq -[\lambda\theta_L + (1-\lambda)\theta_H]q_H^*$, θ_H-LL, θ_L-IC, and IR are binding. The efficient agent produces effectively $q_L^L = q_L^*$, and the inefficient agent's production is distorted upwards from the first-best, with
$$q_H^L = -\frac{l}{\lambda\theta_L + (1-\lambda)\theta_H} \in (q_H^*, q_L^*).$$
- For $l < -[\lambda\theta_L + (1-\lambda)\theta_H]q_L^*$, there is bunching such that both types produce the same output q^L and θ_L-LL, θ_H-LL, θ_L-IC, and IR are all binding:
$$q^L = -\frac{l}{\lambda\theta_L + (1-\lambda)\theta_H}.$$

70 The limited liability constraints on transfers give rise to allocative distortions that are rather different from those highlighted in the proposition of rent.

- As the limited liability constraint $t_H \geq -l$ is more stringent, it becomes quite difficult to create the wedge between U_L and U_H that is necessary to ensure incentive compatibility.
- However, to relax the limited liability constraint $t_H \geq -l$, the principal now increases the inefficient type's output. Indeed, using the information rent to rewrite $t_H \geq -l$, we obtain
$$U_H \geq -l - \theta_H q_H.$$
- Therefore, distorting the inefficient type's output upward relaxes this limited liability constraint.
- A limited liability constraint on transfers implies higher-powered incentives for the agent. It is almost the same as what we would obtain by assuming that the agent is a risk lover.
- The limited liability constraint on transfers somewhat convexifies the agent's utility function.
- Of course, the principal cannot indefinitively raise the inefficient agent's output without conflicting with the implementability condition.
- Hence, some bunching emerges. In this case, the agent receives a fixed payment that covers their cost in expectation.

4 Summary

71 When it comes to solving the screening problem, it is useful to start from the benchmark problem without adverse selection, which involves maximizing the payoff of the principal subject to IR constraints. At the optimum, allocative efficiency is then achieved, because the principal can treat each type of agent separately and offer a type-specific package.

72 In the presence of adverse selection, however, the principal has to offer all types of agents the same menu of options. He has to anticipate that each type of agent will choose her favorite opinion. Without loss of generality, he can restrict the menu to the set of opinions actually chosen by at least one type of agent. It reduces the program of the principal to the maximization of his expected payoff subject to IC and IR constraints.

73 One can disregard the IC for low-ability agent and IR for high-ability agent. Contract then trades off optimally the allocative inefficiency of the low-ability agent with the information rent conceded to the high-ability agent.

In contrast, there is no allocative inefficiency for the high-ability agent and no rent for the low-ability agent.

74 For generalizations to more than two types, IC constraints can often be replaced by fewer local IC constraints and monotonicity condition. We have full separation under natural restrictions (monotone hazard rate).

75 In some cases, the distribution of types does not lead to full separation—for example, when there are intermediate types that the principal considers to be of low probability. There would then be an incentive for the principal to have severe allocative inefficiency for these types, in order to reduce the rents of adjacent types. But this incentive conflicts with the monotonicity condition. In this case, a procedure of "bunching and ironing" has been outlined to solved for the optimal contract. The monotonicity condition then binds for some types where bunching occurs.

Task

- Reading: 2.11, 3.1, 3.5 in [LM] (required), Appendix 3.1–3.2 in [LM] (optional).
- Understanding: