Advanced Microeconomics: Lecture Note 10

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1 Introduction to cheap talk

1. In the classic signaling example of Spence, there is a costly (and credible) signal—in the case of the worker the signal was education—then it can act as a credible way for the worker to signal his type and cause firms to believe him.

   Cheap talk games (空谈) are analogous to signaling games, but in cheap talk games the sender's messages are just talk—costless, nonbingding, nonverifiable claims.

2. Such talk cannot be informative in Spence's signaling game: a worker who simply announced "My ability is high" would not be believed.

   (a) All types have the same preferences over the receiver's possible actions: all workers prefer high wages, independent of ability.

   (b) Therefore, a situation when two types of sender send different messages and the receiver responds differently to these messages is impossible at equilibrium: the sender-type who gets a less favorable response is better off with changing his message to the one employed by the other type.

3. Real examples of cheap talk include:

   - Monetary mystique: a central bank is unwilling to make precise statements about its policy objectives.
   - Security analyst recommendations.

4. It turns out that in a variety of contexts cheap talk is informative. An example is an expert advising a politician. The politician, after hearing the opinion of the expert, makes a decision which affects the payoffs of both players.

   For cheap talk to be useful, the following conditions are necessary:

   - Different sender-types should have different preferences over the receiver's actions.
   - The receiver should prefer different actions depending on the sender's type.
   - The receiver's preferences over actions should not be completely opposed to the sender's. Otherwise, the sender is worse off revealing true information about his type. Therefore, cheap talk cannot be informative in this situation: the receiver will be mislead by the sender.

5. Model: A decision maker (receiver) must choose some decision a. Her payoff depends on a and on an unknown state of the world θ, which is distributed uniformly on Θ.

   The decision maker can base her decision on the costless message m sent by an expert (sender) who knows the precise value of θ.

The decision maker's payoff is

$$u_r(\theta, a) = -(a - \theta)^2,$$

and the expert's payoff is

$$u_s(\theta, a) = -\left[a - (\theta + b)\right]^2,$$

where $b \geq 0$ is a "bias" parameter that measures how nearly agents' interests coincide.

Notice that the signal m is irrelevant to the payoff functions, i.e., talk is cheap.

Although the message space M is independent of the state space Θ, we always let them be identical for sake of simplicity.

6. The sequence of play is as follows:

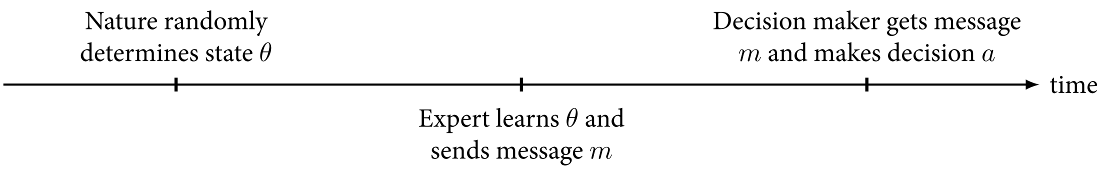


Figure 1: Timing

- The expert learns her type $\theta \in \Theta$;
- The expert sends a message $m \in \Theta$ to the decision maker; the message may be random, and can be viewed as a noisy estimate of θ;
- The decision maker processes the information in the expert's message and chooses an action $a \in \mathbb{R}$, which determines players' payoffs.

7. Because of the tractability of the "uniform-quadratic" specification, much of the cheap talk literature, restricts attention to this case.

   Quadratic loss means that the marginal cost is increasing in the distance between the state and action. It also means that the players are risk-averse; they prefer a constant gap to a varying gap (depending on θ) with the same mean.

8. Note that under this set up, the expert consistently prefers a bigger action than the decision maker (since $b \geq 0$). A more general case is to have

$$u_s(\theta, a) = -\left[a - (\lambda\theta + b)\right]^2,$$

   in which case the difference between the ideal action of the expert and decision maker depends on θ. When λ is greater than one and b is zero, the agent prefers a proportionally greater than action.

9. In this cheap talk game, a pure-strategy PBE of this game consists of

   - a strategy for the expert, denoted $m^*(\theta)\colon \Theta \to \Theta$,
   - a strategy for the decision maker, denoted $a^*(m)\colon \Theta \to \mathbb{R}$,
   - a belief system, denoted $\mu^*(\cdot \mid m) \in \Delta(\Theta)$,

   such that

   - Given the decision maker's strategy $a^*(m)$, the expert of type θ send a message $m^*(\theta)$ so that

$$m^*(\theta) \in \arg\max_{m \in \Theta} u_s\left(\theta, a^*(m)\right).$$

- Given the belief $\mu^*(\cdot \mid m)$, the decision maker with the message m chooses an action $a^*(m)$ satisfying

$$a^*(m) \in \arg\max_{a \in \mathbb{R}} \int_\Theta u_r(\theta, a)\, \mathrm{d}\mu^*(\theta \mid m).$$

- $\mu^*(\cdot \mid m)$ is derived from $m^*(\theta)$ via the Bayes' rule whenever possible.

10. Question: Can we find an informative PBE?

    - The expert conveys (partial) correct information to the decision maker—(partially) truthful telling.
    - The decision maker trusts those correct information (and then chooses correct actions).

    As we will see, the inherent conflict of interest between the expert and the decision maker will put limits on how much information the expert can credibly communicate to the decision maker in equilibrium.

2 The model with two types

11. We first consider the simplest case in which $\Theta = \{\theta_L, \theta_H\}$. The decision maker initially regards the two states as equally likely.

12. We now investigate the conditions under which communication can be informative, or when the strategy profile $m(\theta) = \theta$, $a(m) = m$ and the beliefs system $\mu(\theta \mid \theta) = 1$ consist of a PBE?

    The strategy profile and beliefs system can be rewritten as follows:

$$\begin{cases} m(\theta_H) = \theta_H \\ m(\theta_L) = \theta_L \end{cases}, \begin{cases} \mu(\theta_H \mid \theta_H) = 1 \\ \mu(\theta_L \mid \theta_L) = 1 \end{cases}, \begin{cases} a(\theta_H) = \theta_H \\ a(\theta_L) = \theta_L \end{cases}.$$

13. Given expert's strategy $m(\theta) = \theta$, every possible message (indeed two possible messages θ_H and θ_L) is on the path. Then Bayes' rule implies that the belief should be $\mu(\theta \mid \theta) = 1$. Thus, it is optimal to choose θ when receiving the message θ.

14. We then check the incentives facing the expert. Consider Figure 2, which shows the the expert's payoffs in different states.

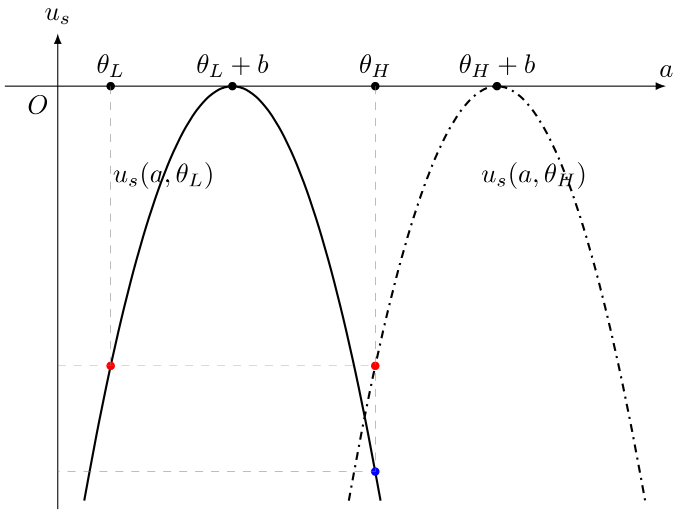


Figure 2: The expert's payoffs with two states

(1) Clearly, the expert has no incentive to misrepresent the facts when the state is θ_H. Reporting θ_H gives a payoff of $-[\theta_H - (\theta_H + b)]^2$. Reporting θ_L gives a payoff of $-[\theta_L - (\theta_H + b)]^2$. Clearly, the former is strictly larger than the latter. (See the dash-dotted line in Figure 2)

(2) If the state instead is θ_L, a truthful (and trusted) report by the expert induces a policy $a = \theta_L$. This is smaller than the expert's ideal policy of $\theta_L + b$ in state θ_L. His payoff is $-[\theta_L - (\theta_L + b)]^2$.

If the expert instead claims that the state is θ_H, the policy outcome will be $a = \theta_H$. The expert may prefer this larger policy, but it also might be too large even for her tastes. His payoff is $-[\theta_H - (\theta_L + b)]^2$.

The expert will report truthfully in state θ_L if and only if θ_L is closer to $\theta_L + b$ than θ_H, or

$$\theta_L \leq \theta_L + b \leq \theta_H \text{ and } (\theta_L + b) - \theta_L \leq \theta_H - (\theta_L + b).$$

Notice that this inequality is satisfied for the case depicted in Figure 2.

We can rewrite the inequalities as a limitation on the size of the divergence in preferences; that is,

$$b \leq \frac{\theta_H - \theta_L}{2}. \tag{1}$$

15. When Equation (1) is satisfied, there exists a pure-strategy PBE with informative communication.

    In such an equilibrium, the expert educates the decision maker about the state of the world. The equilibrium that results is fully revealing, because the decision maker learns the true state for all possible values of the random variable θ.

16. When $\frac{\theta_H - \theta_L}{2} \geq b \geq \frac{\theta_H - \theta_L}{4}$, there also exists a pure-strategy PBE with uninformative communication:

$$m(\theta) = \theta_H, \begin{cases} a(\theta_H) = \frac{\theta_L + \theta_H}{2} \\ a(\theta_L) = \theta_L \end{cases}, \begin{cases} \mu(\theta_H \mid \theta_H) = \mu(\theta_L \mid \theta_H) = \frac{1}{2} \\ \mu(\theta_L \mid \theta_L) = 1 \end{cases}.$$

When $\frac{\theta_H - \theta_L}{2} \geq b$, there exists a mixed-strategy PBE with uninformative communication:

$$m(\theta) = \tfrac{1}{2}\theta_H + \tfrac{1}{2}\theta_L,\ a(m) = \tfrac{1}{2}\theta_H + \tfrac{1}{2}\theta_L,\ \mu(\theta_H \mid \theta) = \mu(\theta_L \mid \theta) = \tfrac{1}{2}.$$

They are babbling equilibria, where no information is conveyed from the expert to the decision maker.

17. When $\frac{\theta_H-\theta_L}{2} \geq b > \frac{\theta_H-\theta_L}{4}$, there exists a mixed-strategy PBE with partially informative communication:

$$\begin{cases} m(\theta_H) = \theta_H \\ m(\theta_L) = \frac{1-2b}{2b}\theta_H + \frac{4b-1}{2b}\theta_L \end{cases}, \begin{cases} a(\theta_H) = 2b \\ a(\theta_L) = \theta_L \end{cases}, \begin{cases} \mu(\theta_H \mid \theta_H) = 2b \\ \mu(\theta_L \mid \theta_L) = 1 \end{cases}.$$

18. If, in contrast, Equation (1) is not satisfied, the expert's message lacks credibility. The decision maker would know in such circumstances that the expert had an incentive to announce the state as θ_H no matter what the true state happened to be.

    For this reason, the expert's message is uninformative, and the decision maker is well justified in ignoring its content. In the event, the decision maker sets the policy $a = \frac{\theta_H+\theta_L}{2}$ that matches her prior expectation about the mean value of θ.

    Evidently, the transmission of information via cheap talking requires a sufficient degree of alignment between the interests of the decision maker and the expert.

19. In summary:

    - When $\frac{\theta_H-\theta_L}{4} \geq b$, only babbling and fully revealing can be equilibrium.
    - When $\frac{\theta_H-\theta_L}{2} \geq b > \frac{\theta_H-\theta_L}{4}$, (mixed) babbling, fully revealing and (mixed) partially revealing can be equilibrium.
    - When $b > \frac{\theta_H-\theta_L}{2}$, the "unique" equilibrium is the babbling equilibrium.

20. Numerical example: Let $\theta_L = 0, \theta_H = 1, \frac{1}{2} \geq b > \frac{1}{4}$.

    The comparison of the decision maker's expected utility:

$$0 \text{ for fully revealing} > -\tfrac{1}{2}(1-2b) \text{ for partially revealing} > -\tfrac{1}{4} \text{ for babbling.}$$

    The comparison of the expert's expected utility:

$$-b^2 \text{ for fully revealing} > -\tfrac{1}{2}(1-b)^2 - \tfrac{1}{2}b^2 \text{ for partially revealing} > -\tfrac{1}{2}(\tfrac{1}{2}-b)^2 - \tfrac{1}{2}(\tfrac{1}{2}+b)^2 \text{ for babbling.}$$

3 The model with three types

21. We then consider the case that $\Theta = \{\theta_L, \theta_M, \theta_H\}$.

22. Similar with the two-state case, the θ_L-type expert will reveal his information truthfully if and only if

$$b \leq \frac{\theta_M - \theta_L}{2},$$

    and the θ_M-type expert will reveal his information truthfully if and only if

$$b \leq \frac{\theta_H - \theta_M}{2}.$$

    Therefore, truth-telling is a PBE strategy if and only if

$$b \leq \min\left\{\frac{\theta_M - \theta_L}{2}, \frac{\theta_H - \theta_M}{2}\right\}.$$

4 The model with a continuum of types

23. We finally turn to consider the case that $\Theta = [0, 1]$.

24. As the number of possible states grows, full revelation becomes ever more difficult to achieve.

    For a sender to be able to distinguish among all possible states, b must be smaller than one-half of the distance between any two of them. But as the number of states tends to infinity—as it must, for example, when θ represents a continuous variable—this requirement becomes impossible to fulfill.

25. Proposition: If the expert is even slightly biased, all equilibria entail some information loss.

    Proof. If the expert's message always revealed the true state and the decision maker believed him, then the expert would have the incentive to exaggerate the state: in some state θ, he would report $\theta + b$. □

26. Proposition: There always exists a "babbling equilibrium" in which the sender always send the same message and the message is always ignored.

    Proof. (1) Let the sender's strategy be to send a message $m_0 \in [0, 1]$ regardless of θ.

    (2) This means that the message is completely uninformative and receiver still believes that θ is distributed uniformly on $[0, 1]$.

    (3) Conditioning on receiving the message m_0, receiver maximizes his expected payoff

    $$\mathbf{E}_\theta u_r(\theta, a) = \int_0^1 -(a - \theta)^2 \, \mathrm{d}\theta = -\tfrac{1}{3} + a - a^2.$$

    This expected payoff is maximized when $a = \frac{1}{2}$.

    (4) Let receiver's off-equilibrium path beliefs be

    $$\text{Prob}(\theta = \tfrac{1}{2} \mid m \neq m_0) = 1$$

    so that his off-equilibrium path best response to any other message is $a = \frac{1}{2}$ as well.

    (5) It is easy to see that sender is indifferent between any of his message and hence choosing $m = m_0$ is a best response.

    □

27. The question then is, how much information can the expert credibly transmit to the decision maker?

28. We begin by constructing a PBE in which the expert uses one of two messages, m_1 and m_2, and the decision maker chooses a different action following each message, $a_1 < a_2$.

    A two-step ($n = 2$) equilibrium (m^*, a^*, μ^*):

    (1) The expert should use a threshold strategy as follows: all the types in the interval $[0, x_1)$ send one message m_1, while those in $[x_1, 1]$ send another message m_2.

    For any θ, the expert's payoffs from m_1 and m_2 are

    $$u_s(a_1, \theta) = -[a_1 - (\theta + b)]^2 \text{ and } u_s(a_2, \theta) = -[a_2 - (\theta + b)]^2,$$

which implies that the extra gain from choosing m_2 over m_1 is equal to

$$\Delta u_s(\theta) = -[a_2 - (\theta + b)]^2 + [a_1 - (\theta + b)]^2.$$

The derivative of $\Delta u_s(\theta)$ is $2(a_2 - a_1) > 0$. This implies that if type θ prefers to send message m_2 over m_1 then every type $\theta' > \theta$ will also prefer m_2. Similarly if type θ prefers to send message m_1 over m_2 then so will every type $\theta' < \theta$.

(2) After receiving the message m_1 from the types in $[0, x_1)$, the decision maker will believe that the expert's type is uniformly distributed on $[0, x_1)$, so the decision maker's optimal action will be $a_1 = \frac{x_1}{2}$.
Likewise, after receiving the message m_2 from the types in $[x_1, 1]$, the decision maker's optimal action will be $a_2 = \frac{x_1+1}{2}$.

(3) For the types in $[0, x_1)$ to be willing to send their message m_1, it must be that all these types prefer the action $\frac{x_1}{2}$ to the action $\frac{x_1+1}{2}$.
Likewise, all the types above x_1 must prefer $\frac{x_1+1}{2}$ to $\frac{x_1}{2}$.

(4) Since the expert's utility is symmetric around her optimal action $\theta + b$, the type-θ expert prefers $\frac{x_1}{2}$ to $\frac{x_1+1}{2}$ if the midpoint between these two actions exceeds that type's optimal action $\theta + b$, but prefers $\frac{x_1+1}{2}$ to $\frac{x_1}{2}$ if $\theta + b$ exceeds the midpoint.

(5) Thus, for each state $\theta_1 \in [0, x_1)$,

$$\theta_1 + b \leq \frac{1}{2}\left[\frac{x_1}{2} + \frac{x_1+1}{2}\right].$$

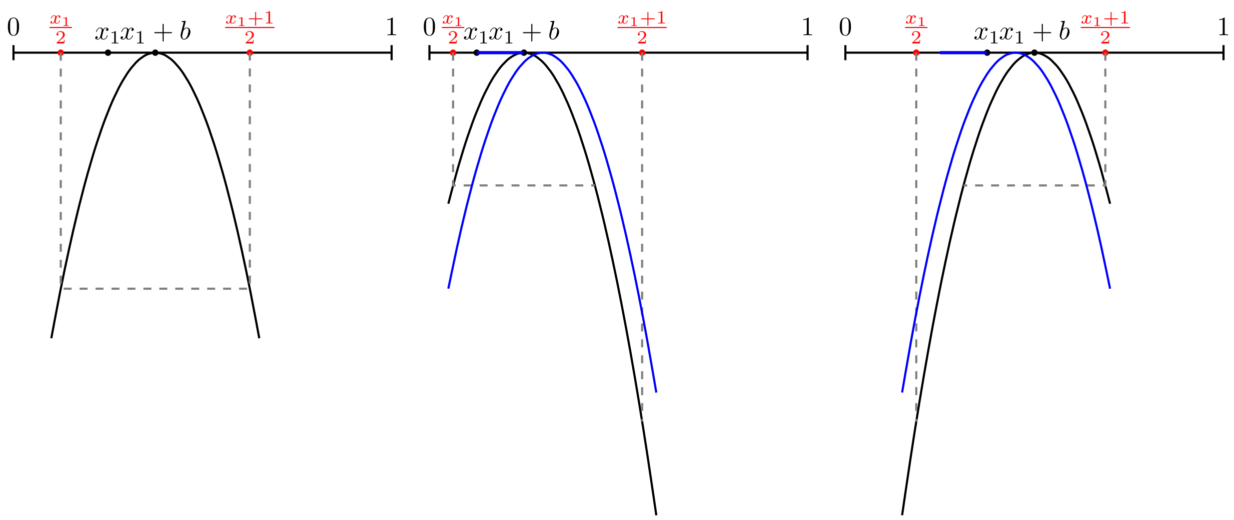
For each state $\theta_2 \in [x_1, 1]$,

$$\theta_2 + b \geq \frac{1}{2}\left[\frac{x_1}{2} + \frac{x_1+1}{2}\right].$$

(6) Therefore, for a two-step equilibrium to exist, the x_1-type expert must be indifferent between $\frac{x_1+1}{2}$ and $\frac{x_1}{2}$:

$$x_1 + b = \frac{1}{2}\left[\frac{x_1}{2} + \frac{x_1+1}{2}\right],$$

that is, $x_1 = \frac{1}{2} - 2b$.



In the left graph, the x_1-type expert is indifferent between $\frac{x_1}{2}$ and $\frac{x_1+1}{2}$. In the middle graph, the x_1-type expert prefers $\frac{x_1}{2}$ to $\frac{x_1+1}{2}$. In the right graph, the x_1-type expert prefers $\frac{x_1+1}{2}$ to $\frac{x_1}{2}$.

(7) Since the type space is $\Theta = [0,1]$, x_1 must be positive, so a two-step equilibrium exists only if $b < \frac{1}{4}$; for $b \geq \frac{1}{4}$ the players' preferences are too dissimilar to allow even the limited communication.

(8) To complete the characterization of this two-step equilibrium, we address the issue of messages that are off the equilibrium path.

For example, let the expert's strategy be that all types $\theta < x_1$ send the message $m_1 \in [0, x_1)$ and all types $\theta \geq x_1$ send the message $m_2 \in [x_1, 1]$. Then we may let the decision maker's off-path belief after observing any message from $[0, x_1) \setminus \{m_1\}$ be that θ is uniformly distributed on $[0, x_1)$, and after observing any message from $[x_1, 1] \setminus \{m_2\}$ be that θ is uniformly distributed on $[x_1, 1]$.

29. The steps we used to find the condition $b < \frac{1}{4}$ under which a two-message equilibrium exists can be used to find more informative equilibria.

An n-step equilibrium (m^*, a^*, μ^*). We will refer to the message sent when $\theta \in [x_{i-1}, x_i]$ as m_i for $i = 1, 2 \ldots, n$.

(1) By Bayes' rule, $\mu^*\big((a,b) \mid m_i\big) = \frac{|(a,b) \cap [x_{i-1}, x_i]|}{x_i - x_{i-1}}$, or the uniform distribution on $[x_{i-1}, x_i]$.

(2) Sequential rationality implies that

$$a^*(m_i) = \tfrac{x_{i-1}+x_i}{2}.$$

(3) In equilibrium, the x_i-type expert must be indifferent between m_i and m_{i+1} for $i = 1, 2, \ldots, n-1$. Given the quadratic-loss utility function, it must be that

$$(x_i + b) - \tfrac{x_{i-1}+x_i}{2} = \tfrac{x_i + x_{i+1}}{2} - (x_i + b),$$

equivalently,

$$(x_{i+1} - x_i) = (x_i - x_{i-1}) + 4b.$$

The width of each step increases by $4b$.

(4) If the first step is of length d, then the boundary condition must imply

$$d + (d + 4b) + \cdots + \big[d + (n-1)4b\big] = 1,$$

equivalently,

$$nd + n(n-1)2b = 1.$$

(5) Hence, given any n such that $n(n-1)2b < 1$, there exists a value of d such that $nd + n(n-1)2b = 1$. That is, there is an n-step equilibrium as long as $n(n-1)2b < 1$.

(6) Since the length of the first step must be positive, the largest possible number of steps in such an equilibrium, $n^*(b)$, is the largest value of n such that $n(n-1)2b < 1$, i.e., $n^*(b)$ is the largest integer less than

$$\tfrac{1}{2}\left[1 + \sqrt{1 + \tfrac{2}{b}}\right].$$

Imprecise messages can still be credible when the interests of the expert and the decision maker do not align completely.

30. Theorem: All the perfect Bayesian equilibria are equivalent to a partially pooling equilibrium of the following form: the type space is divided into the n intervals (steps)

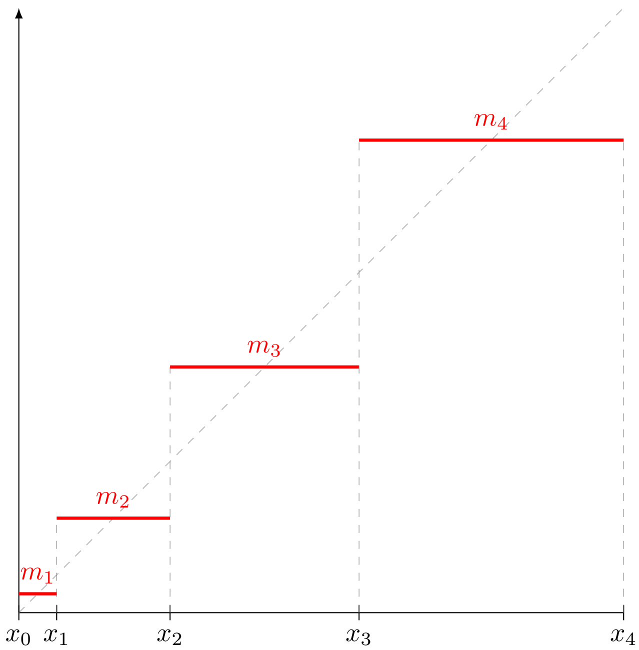
$$[x_0 = 0, x_1),\ [x_1, x_2),\ \ldots,\ [x_{n-1}, 1 = x_n];$$

all the types in a given interval send the same message, but types in different intervals send different messages.

Key of the proof: Since in equilibrium $a^*(m)$ is weakly increasing, every points in between must send the same message.

31. Remark: More communication can occur through cheap talk when the players' preferences are more closely aligned. But perfect communication cannot occur unless the players' preferences are perfectly aligned.

32. Numerical example: $b = \frac{1}{32}$. Then $n^*(b) = 4$, $d = \frac{1}{16}$, $x_1 = \frac{1}{16}$, $x_2 = \frac{4}{16}$, $x_3 = \frac{9}{16}$.



33. Remark: Theorem 30 crucially depends on the quadratic utility assumption. By this assumption, expert at different states has different points. This makes the expert has incentives to reveal some information.

    However, if we assume that the expert's utility function is monotonic in a, for example $u_s(\theta, a) = a - \theta$, then the unique equilibrium is the babbling equilibrium.

34. $n^*(b)$ decreases in b but approaches infinity only as b approaches zero: more communication can occur through cheap talk when the players' preferences are more closely aligned, but perfect communication cannot occur unless the players' preferences are perfectly aligned.

35. If there exists an equilibrium with n messages, there must be other equilibria with less than n messages. It always includes the babbling equilibrium in which the decision maker never listen the expert and the expert never convey the true information.

36. Expected welfare analysis: let us rank the equilibria by evaluating the expected welfare of the decision maker and the expert in each possible equilibrium.

    (1) Since the decision maker's utility function is $u_r(\theta, a) = -(\theta - a)^2$ and she sets $a = \frac{x_{i-1}+x_i}{2}$ if the decision maker heard θ is in $[x_{i-1}, x_i]$.

    (2) Hence, the decision maker's expected welfare is

$$U_r(n) = -\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left(\frac{x_{i-1}+x_i}{2} - \theta \right)^2 \mathrm{d}\theta = -\frac{1}{12} \sum_{i=1}^{n} (x_i - x_{i-1})^3 = \frac{1}{12n^2} + \frac{b^2(n^2-1)}{3}.$$

(3) Likewise, the expert's expected welfare is

$$U_s(n) = -\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left(\frac{x_{i-1} + x_i}{2} - \theta - b \right)^2 \mathrm{d}\theta = U_r(n) - b^2.$$

(4) Since $U_r(n)$ is an increasing function of n. Thus, we can conclude that in the *ex ante* sense, the more n we get, the better equilibrium we achieve.

5 Optimal communication mechanism (the commitment case)

37. Literature: Melumad and Shibano (1991).

38. We consider the case where decision maker can commit to a decision rule (a strategy for the decision maker) which is a mapping from the message space to the action space: $a\colon \Theta \to \mathbb{R}$.

39. We still assume that $\Theta = [0, 1]$ and θ is uniformly distributed on Θ.

40. By the revelation principle we need to consider truth-telling mechanism in which the sender's strategy is $m(\theta) = \theta$ for all θ. This requires the IC condition:

$$u_s(\theta, a(\theta)) \geq u_s(\theta, a(\theta'))$$

for any $\theta, \theta' \in \Theta$.

41. Lemma: $a\colon \Theta \to \mathbb{R}$ is increasing.

Proof. Consider any two distinct θ and θ'. The IC condition implies:

$$-\big(a(\theta) - \theta - b\big)^2 \geq -\big(a(\theta') - \theta - b\big)^2 \text{ and } -\big(a(\theta') - \theta' - b\big)^2 \geq -\big(a(\theta) - \theta' - b\big)^2.$$

These two inequalities yield

$$-\big(a(\theta) - \theta - b\big)^2 - \big(a(\theta') - \theta' - b\big)^2 \geq -\big(a(\theta') - \theta - b\big)^2 - \big(a(\theta) - \theta' - b\big)^2.$$

That is,

$$\big(a(\theta) - a(\theta')\big)\left(\theta - \theta'\right) \geq 0.$$

Therefore, a is increasing. □

42. Lemma: If a is continuous and strictly increasing on (θ_1, θ_2), then $a(\theta) = \theta + b$ on (θ_1, θ_2).

Proof. (1) Suppose that there exists θ such that $a(\theta) \neq \theta + b$.

(2) Without loss of generality, assume $a(\theta) > \theta + b$.

(3) Since a is continuous and strictly increasing, there exists $\theta' < \theta$ such that $\theta + b < a(\theta') < a(\theta)$.

(4) Then it is optimal for the sender under θ to misreport θ'. Contradiction.

□

43. Lemma: If a is discontinuous at θ_0, the discontinuity must be a jump discontinuity that satisfies

(i) $u_s(\theta_0, a^-(\theta_0)) = u_s(\theta_0, a^+(\theta_0))$, where $a^-(\theta_0) = \lim_{\theta \uparrow \theta_0} a(\theta)$ and $a^+(\theta_0) = \lim_{\theta \downarrow \theta_0} a(\theta)$.

(ii) $a(\theta) = \begin{cases} a^-(\theta_0), & \text{if } \theta \in [a^-(\theta_0) - b, \theta_0), \\ a^+(\theta_0), & \text{if } \theta \in (\theta_0, a^+(\theta_0) - b]. \end{cases}$

(iii) $a(\theta_0) \in \{a^-(\theta_0), a^+(\theta_0)\}$.

Proof. (i) Assume that $u_s\left(\theta_0, a^-(\theta_0)\right) < u_s\left(\theta_0, a^+(\theta_0)\right)$. Then the sender will strictly prefer $a^+(\theta_0)$ to $a^-(\theta_0) \approx a(\theta')$ at θ' slightly less than θ_0. Contradiction.

(ii) Since the sender's favorite action when $\theta = a^-(\theta_0) - b$ is $a^-(\theta_0)$, the IC condition requires that

$$a\left(a^-(\theta_0) - b\right) = a^-(\theta_0).$$

Since a is increasing, a must be flat between $[a^-(\theta_0) - b, \theta_0)$.

(iii) Assume that $a(\theta_0) \in \left(a^-(\theta_0), a^+(\theta_0)\right)$. Then the sender will strictly prefer $a(\theta_0)$ to either $a^+(\theta_0)$ or $a^-(\theta_0)$. So will the sender when θ' is near θ_0 since u_s is continuous in θ. Contradiction.

□

44. The following figure depicts a general IC decision rule according to the previous lemmas.

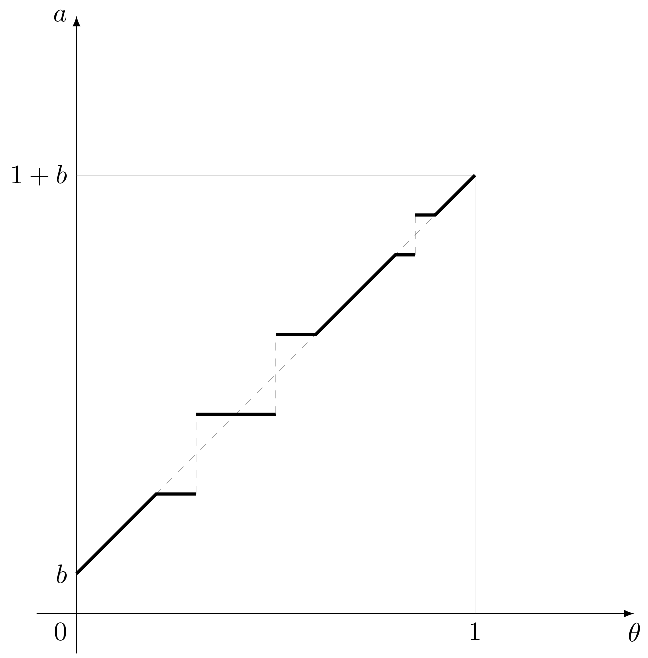


Figure 3

45. Lemma: The decision rule a is everywhere continuous.

Proof. (1) Suppose that a is not continuous at θ_0.

(2) Then we have

$$a(\theta) = \begin{cases} a^-(\theta_0), & \text{if } \theta \in [a^-(\theta_0) - b, \theta_0), \\ a^+(\theta_0), & \text{if } \theta \in (\theta_0, a^+(\theta_0) - b]. \end{cases}$$

(3) Note that replacing any segment of an IC decision rule by some other IC segment does not affect the IC property of the decision rule.

(4) We will argue that (i) it is incentive compatible to replace this part of the rule by $a'(\theta) = \theta + b$ (the sender's favorite action) and (ii) that the receiver will benefit from the change in contradiction to the assumed optimality of $a(\theta)$.

(5) When $\theta < a^-(\theta_0) - b$, we have $\theta + b < a^-(\theta_0)$, and hence

$$-(a^-(\theta_0) - \theta - b)^2 \geq -(a - \theta - b)^2$$

for any $a \in \left(a^-(\theta_0), a^+(\theta_0)\right)$.

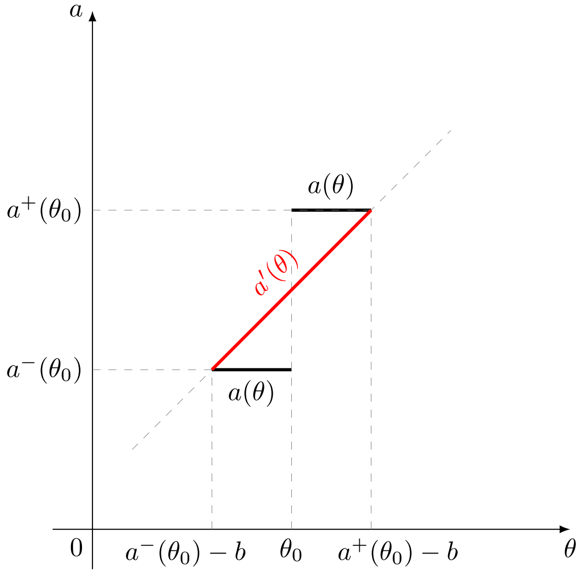


Figure 4

(6) Therefore, the sender prefers $a^-(\theta_0)$ to any $a \in \left(a^-(\theta_0), a^+(\theta_0)\right)$ when $\theta < a^-(\theta_0) - b$.

(7) Similarly, the sender prefers $a^+(\theta_0)$ to any $a \in \left(a^-(\theta_0), a^+(\theta_0)\right)$ when $\theta > a^+(\theta_0) - b$.

(8) So, the change will not create incentives for the sender to deviate when $\theta \notin \left(a^-(\theta_0), a^+(\theta_0)\right)$.

(9) When $\theta \in \left(a^-(\theta_0), a^+(\theta_0)\right)$, the sender will obtain his best action by telling the truth, so there is no incentive to deviate.

(10) The receiver's expected utility is, then,

$$-\int_{a^-(\theta_0)-b}^{a^+(\theta_0)-b} (a'(\theta) - \theta)^2 \,\mathrm{d}\theta = -\int_{a^-(\theta_0)-b}^{a^+(\theta_0)-b} (\theta + b - \theta)^2 \,\mathrm{d}\theta = -b^2 \left(a^+(\theta_0) - a^-(\theta_0)\right).$$

(11) The receiver's original expected utility is

$$-\int_{a^-(\theta_0)-b}^{\theta_0} (a^-(\theta_0) - \theta)^2 \,\mathrm{d}\theta - \int_{\theta_0}^{a^+(\theta_0)-b} (a^+(\theta_0) - \theta)^2 \,\mathrm{d}\theta = -\int_{\theta_0 - a^+(\theta_0)}^{\theta_0 - a^-(\theta_0)} x^2 \,\mathrm{d}x,$$

which is strictly less than $-b^2 \left(a^+(\theta_0) - a^-(\theta_0)\right)$ due to the Jensen's inequality.

□

46. Theorem: The optimal decision rule is

$$a^*(\theta) = \begin{cases} \theta + b, & \text{if } \theta \leq 1 - 2b, \\ 1 - b, & \text{if } \theta > 1 - 2b. \end{cases}$$

Proof. (1) We have already shown that the optimal a must be continuous and equal $\theta + b$ when it is strictly increasing. This means that the optimal rule must be of the form

$$a^*(\theta) = \begin{cases} \underline{\theta} + b, & \text{if } \theta \leq \underline{\theta}, \\ \theta + b, & \text{if } \theta \in (\underline{\theta}, \bar{\theta}), \\ \bar{\theta} + b, & \text{if } \theta \geq \bar{\theta}. \end{cases}$$

(2) The receiver chooses $\underline{\theta} \geq 0$ and $\bar{\theta} \in [\underline{\theta}, 1]$ to maximize:

$$U_r(\underline{\theta}, \bar{\theta}) = -\int_0^{\underline{\theta}} (\underline{\theta} + b - \theta)^2 \, \mathrm{d}\theta - \int_{\underline{\theta}}^{\bar{\theta}} b^2 \, \mathrm{d}\theta - \int_{\bar{\theta}}^1 (\bar{\theta} + b - \theta) \, \mathrm{d}\theta.$$

(3) It is easy to show that

$$\frac{\partial U_r}{\partial \underline{\theta}} = -2 \int_0^{\underline{\theta}} (\underline{\theta} + b - \theta) \, \mathrm{d}\theta < 0,$$

which implies that $\underline{\theta}^* = 0$.

(4) Similarly,

$$\frac{\partial U_r}{\partial \bar{\theta}} = -2 \int_{\bar{\theta}}^1 (\bar{\theta} + b - \theta) \, \mathrm{d}\theta.$$

(5) By letting $\frac{\partial U_r}{\partial \bar{\theta}} = 0$, we have $\bar{\theta} = 1 - 2b$ or $\bar{\theta} = 1$. However, the second derivative is negative at $1 - 2b$ but positive at 1. Therefore, U_r is maximized at $\bar{\theta}^* = 1 - 2b$.

□

47. The optimal rule is to set a limit on the highest action the sender (who is biased for higher actions) can take and let the sender picks the action.

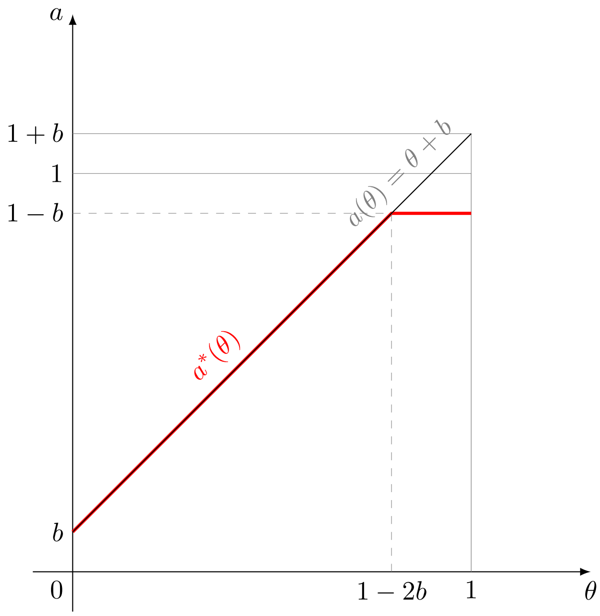


Figure 5

Task

- Reading: 4.3 in [S].
- Understanding: